19/2 použijeme Heineovu větu $\qquad a, b > 0$

$$\lim_{x\to+\infty} \frac{\lg^a x}{x^b} = \lim_{x\to+\infty} \left(\frac{\lg^{\frac{a}{b}} x}{x}\right)^b = (+)$$

Počítáme pouze $\displaystyle \lim_{x\to+\infty} \frac{\lg^{\frac{a}{b}} x}{x} \overset{l'H}{=} \lim_{x\to+\infty} \frac{\left(\frac{a}{b}\lg^{\frac{a}{b}-1} x\right)\cdot\frac{1}{x}}{1}$

$$= \lim_{x\to+\infty} \frac{a}{b}\,\frac{\lg^{\frac{a}{b}-1} x}{x}$$

$$(+) = \lim_{x\to+\infty} \left(\frac{\lg x}{x^{b/a}}\right)^a = 0, \text{ protože}$$

$$\lim_{x\to+\infty} \frac{\lg x}{x^{b/a}} \overset{l'H}{=} \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{b}{a}x^{\frac{b}{a}-1}} = \lim_{x\to+\infty} \frac{1}{\frac{b}{a}x^{b/a}} = 0.$$

$$\lim_{x\to 0} \frac{\sin x - x}{[\sin(\sin x)]^3} = \lim_{x\to 0} \frac{1\,(\sin x - x)}{\left(\frac{\sin(\sin x)}{\sin x}\right)^3 \left(\frac{\sin x}{x}\right)^3 x^3}$$

$$= -\frac{1}{6}$$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3\sin(\sin x)^2 \underbrace{\cos(\sin x)}_{\to 1} \underbrace{\cos x}_{\to 1}} \overset{AL}{=} \frac{1}{3}\lim_{x\to 0} \frac{\cos x - 1}{(\sin \sin x)^2}$$

$$\overset{l'H}{=} \lim_{x\to 0} \cancel{\frac{-\sin x}{?}} \qquad \ldots$$

19/4

$$f(x) = \begin{cases} x^2\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\mathcal{D}(f) = \mathbb{R}$$

$$x \neq 0:\ f'(x) = 2x\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right) + \cancel{x^2}\left(\cos\left(\frac{1}{x}\right)\left(-\frac{1}{\cancel{x^2}}\right) - \sin\frac{1}{x}\left(-\frac{1}{\cancel{x^2}}\right)\right)$$

$$= 2x\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right) + \left(\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\cancel{2}}\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)}{\cancel{x}} = 0$$

Pro zajímavost: $\lim\limits_{x \to 0} f'(x)$ neex.!

Typické příklady na použití l'Hospitalova pravidla

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} \overset{l'H}{=} \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{2 \cdot 3} = -\frac{1}{6}$$

$$\frac{\sin x - x}{x^3} = \frac{1}{x^2}\left(\frac{\sin x}{x} - 1\right)$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \overset{l'H}{=} \lim_{x \to 0} \frac{\sin x}{2x} \overset{l'H}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

19/8

$$f(x) = \arcsin(\sin x)$$

$x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$: $$f'(x) = \frac{1}{\sqrt{\underbrace{1 - \sin^2 x}_{\cos^2 x}}} \cdot \cos x = \frac{\cos x}{|\cos x|}$$

19/5

$x \neq \frac{\pi}{2} + l\pi$, $l \in \mathbb{Z}$: $$f'(x) = \frac{1}{1 + \operatorname{tg}^4 x} \cdot (2 \operatorname{tg} x) \cdot \frac{1}{\cos^2 x}$$

$$= \frac{2 \operatorname{tg}^3 x}{1 + \operatorname{tg}^4 x} \cdot \frac{1}{\sin^2 x}$$

$$\lim_{x \to \frac{\pi}{2}} f'(x) = 0 \;; \left(f'(x) = \frac{1}{\operatorname{tg} x} \, \frac{2}{1 + 1/\operatorname{tg}^4 x} \cdot \frac{1}{\sin^2 x} \right)$$

20/8

$$\lim_{x \to 0} \frac{1}{x^2} \lg\left(\frac{\operatorname{arctg} x}{x}\right) = \lim_{x \to 0} \overbrace{\frac{\lg \frac{\operatorname{arctg} x}{x}}{\frac{\operatorname{arctg} x}{x} - 1}}^{\to 1\ (A,B,D)} \cdot \overbrace{\frac{\operatorname{arctg} x - x}{x^3}}^{\to -1/3\ (C)} \overset{AL}{=} -\frac{1}{3}$$

A) $$\lim_{x \to 0} \frac{\operatorname{arctg} x}{x} \overset{l'H}{=} \lim_{x \to 0} \frac{1}{1 + x^2} = 1$$

B) $$\lim_{y \to 1} \frac{\lg y}{y - 1} = 1$$

C) $$\lim_{x \to 0} \frac{\operatorname{arctg} x - x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} = \lim_{x \to 0} \frac{-x^2}{(1+x^2)(3x^2)} = -\frac{1}{3}$$

D) $\operatorname{arctg} x = x \Leftrightarrow x = 0$, protože $(\operatorname{arctg} x - x)' = \frac{1}{1+x^2} - 1 = \frac{-x^2}{1+x^2} < 0$ pro $x \neq 0$