Cvičení NMMA 101 – 27. 10.

8cv/7

$$\liminf_{n\to+\infty} a_n = \lim_{n\to+\infty} \inf\{a_k;\ k \geq n\}$$

Z vlastností inf a sup:

$$\underbrace{\inf\{a_k,\ k \geq n\}}_{b_n} \leq a_n \leq \underbrace{\sup\{a_k;\ k \geq n\}}_{c_n}$$

$$b_n \leq c_n$$

Věta 2.5
$\Rightarrow$

$$\lim_{n\to+\infty} b_n \leq \lim_{n\to+\infty} c_n \quad \text{(pokud ex.)}$$

$$\liminf_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n, \qquad \limsup_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n \qquad \perp.$$

$$(*)\ \inf\{a_k,\ k \geq n\} + \inf\{b_k;\ k \geq n\} \leq \inf\{a_k + b_k;\ k \geq n\}$$

Pokud $\liminf_{n\to+\infty} a_n,\ \liminf_{n\to+\infty} b_n \in \mathbb{R}$.

$$\overset{AL}{\Rightarrow} \liminf_{n\to+\infty} a_n + \liminf_{n\to+\infty} b_n \leq \liminf_{n\to+\infty} (a_n + b_n)$$

(dk (*)): $\forall l \geq n$:

$$\underbrace{\inf\{a_k,\ k \geq n\} + \inf\{b_k;\ k \geq n\}}_{\text{je dolní odhad } \{a_k + b_k;\ k \geq n\}} \leq a_l + b_l$$

$\Rightarrow (*)$.)

$$a_k = (-1)^k \ ; \quad b_k = (-1)^{k+1}$$

9cv/11

Př: $a_{n+1} = 6 - \frac{5}{a_n}$, $a_1 = 10$

Pokud ex. $\lim\limits_{n\to+\infty} a_n =: a \in \mathbb{R}$

$$a = 6 - \frac{5}{a}; \quad a^2 = 6a - 5; \quad a^2 - 6a + 5 = 0$$

$$(a-5)(a-1) = 0 \Rightarrow a = 5 \text{ nebo } a = 1$$

- Jak ukážeme, že lim ex.?

  1) V.2.9: monotónní posl $\Rightarrow$ ex. limity

  2) V 2.14: BC podmínka

- Ukážeme monotonii posl $\{a_n\}$

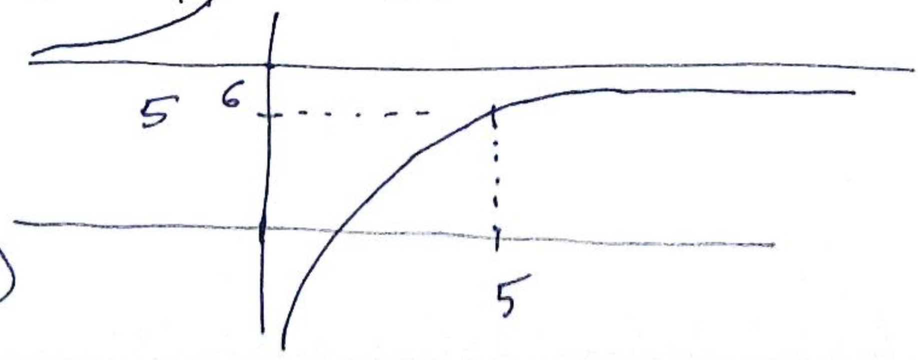
$$a_{n+1} - a_n = 6 - a_n - \frac{5}{a_n}$$

$$f(x) := 6 - x - \frac{5}{x} = \frac{x^2 - 6x + 5}{-x} = \frac{(x-5)(x-1)}{-x}$$

$f>0$		$f<0$		$f>0$		$f<0$
	0		1		5	

- Chceme: Pokud $a_n > 5$, pak $a_{n+1} > 5$.

$$g(x) := 6 - \frac{5}{x}$$



$\Rightarrow g > 5$ na $(5, +\infty)$

$\Rightarrow \quad a_n > 5 \Rightarrow a_{n+1} < a_n \, ; \, a_{n+1} > 5$

$\Rightarrow \{a_n\}$ je klesající a $\forall n \in \mathbb{N}; \, \underline{\underline{a_n > 5.}}$

$a_n \leq 10$

V2.9

$\Rightarrow \lim\limits_{n\to+\infty} a_n = a$ a ex. a $\quad a \in [5, 10]$

$\Rightarrow \quad a = 5$

9ar/2

2) $a_1 = \sqrt{2} \quad ; \quad a_{n+1} = \sqrt{2 + a_n}$

$$a = \sqrt{2+a}$$

Pokud ex. $\lim\limits_{n\to+\infty} a_n = a$

$; \; a^2 = 2 + a \; ; \; a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$\boxed{a = 2}$ nebo $a = -1$

$$\frac{a_{n+1}}{a_n} = \sqrt{\frac{2 + a_n}{a_n^2}} \; ; \; \cancel{f(x) \Rightarrow} \; ? \; 2 + a_n > a_n^2$$

$0 > a_n^2 - a_n - 2$

$0 > (a_n - 2)(a_n + 1)$

- $a_n \in (-1, 2) \Rightarrow \dfrac{a_{n+1}}{a_n} > 1$
- $a_n > 0$
- $a_n < 2 \; \forall n \in \mathbb{N}$: MI : 1) $a_1 < 2$

2) $a_n < 2 \Rightarrow 2 + a_n < 4$

$\Rightarrow a_{n+1} = \sqrt{2 + a_n} < \sqrt{4} = 2$

$\Rightarrow \{a_n\}$ je rostoucí; $\forall n \in \mathbb{N}$

$a_n \in (0, 2) \Rightarrow \lim\limits_{n\to+\infty} a_n = 2$

3) 9 cv.

$a_1 > 0$; $a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)$

pokud ex. lim $\downarrow$

$$a = \frac{1}{2}\left(a + \frac{1}{a}\right)$$

$$\frac{1}{2} a = \frac{1}{2}\frac{1}{a} \; ; \; a = \pm 1$$

• monotonie: $a_{n+1} - a_n = \frac{1}{2}\left(\frac{1}{a_n} - a_n\right) =: f(a_n)$

$$f(x) = \frac{1}{2}\left(\frac{1}{x} - x\right) = \frac{1}{2}\,\frac{1-x^2}{x}$$

$f>0$		$f<0$		$f>0$		$f<0$
	-1		0		1	

• $a_{n+1} = \frac{1}{2}\,\frac{a_n^2+1}{a_n} = \frac{a_n^2+1}{2a_n} \geq 1$ pro $a_n > 0$

$$0 \leq (a_n - 1)^2 = a_n^2 + 1 - 2a_n \Rightarrow 2a_n \leq a_n^2 + 1$$

$$a_n > 0 \Rightarrow 1 \leq \frac{a_n^2+1}{2a_n}$$

Tedy: $a_2 \geq 1$ • $a_2 = 1$ ... $a_n = 1$ $\forall n$ a $\lim_{n\to+\infty} a_n = 1$

• $a_2 > 1$ ... $\{a_n\}_{n=2}^{+\infty}$ je klesající

a $\forall n \in \{2, \dots\}: a_n > 1$

$\Rightarrow \lim_{n\to+\infty} a_n$ ex. a $\lim_{n\to+\infty} a_n = 1$