NMAI057 – Linear algebra 1

Tutorial 4

Date: October 27, 2020 TA: Pavel Hubáček

Problem 1. Establish sufficient conditions for a triangular matrix to be regular.

(Recall that an upper triangular matrix A has arbitrary values on and above the main diagonal, but it is all zero below the diagonal. Formally, for all $i > j$ it holds that $a_{ij} = 0$. Any lower triangular matrix A must satisfy the same condition in the reverse order w.r.t. the main diagonal or, in another words, A^T must be upper triangular.)

Problem 2. Consider the block matrix

$$A = \begin{pmatrix} \alpha & a^T \\ b & C \end{pmatrix},$$

where $\alpha \neq 0$, $a, b \in \mathbb{R}^{n-1}$ and $C \in \mathbb{R}^{(n-1)\times(n-1)}$. Apply to A one iteration of Gaussian elimination and use it to derive a recurrent test of regularity.

Problem 3. Find the inverse to the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 10 \end{pmatrix}.$$

Problem 4. Invert the matrices of the elementary row operations.

Recall that the matrices representing the elementary row operations are:

(a) Multiplying the i-th row with $\alpha \neq 0$:

$$E_i(\alpha) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \alpha & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

(b) Adding the α-multiple of the j-th row to the i-th row for $i \neq j$:

$$E_{ij}(\alpha) = \begin{matrix} \\ \\ \\ i \\ \\ \end{matrix} \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ & \ddots & \ddots & & \vdots \\ & & 1 & \ddots & \vdots \\ & \alpha & & \ddots & 0 \\ & j & & & 1 \end{pmatrix}.$$

(c) Swapping the i-th and j-th row:

$$E_{ij} = \begin{matrix} i \\ \\ j \\ \\ \end{matrix} \begin{pmatrix} 0 & 1 \\ & \\ 1 & 0 \\ i & j \end{pmatrix}.$$

Problem 5. For $n \in \mathbb{N}$, invert the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}.$$

Problem 6. Simplify the following expression assuming A, B are regular matrices of the same order:

$$(I - B^T A^{-1})A + (A^T B)^T A^{-1}.$$

Problem 7.

(a) Prove that for all $A, B \in \mathbb{R}^{n \times n}$, if A is regular then

$$(ABA^{-1})^k = AB^k A^{-1}.$$

(b) Let $A \in \mathbb{R}^{n \times n}$ be a regular matrix. Find the limit (in case you are not familiar with the formal definition, use the intuitive notion) for

$$\lim_{k \to \infty} AD^k A^{-1}, \quad \text{where} \quad D = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{n} \end{pmatrix},$$

and compute its rank.

(c) Apply the above result to compute the limit for any matrix A with the first column equal $e_1 = (1, 0, \dots, 0)^T$ and the first row equal $e_1^T = (1, 0, \dots, 0)$.