The order of convergence of polynomial approximations (Jackson's theorems)

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· Recall what is $\mathcal{T}_n$

• Best minimax approx. from $\mathcal{T}_n$ to $f \in C_{2\pi}$,

$$E_n(f) = \min_{q\in\mathcal{T}_n} \|f-q\|_\infty .$$

How to bound $E_n(f)$ using a ~~constant~~ term (depends on n) and a quantity that depends on the smoothness of f? (Lipschitz constant, $\|f'\|_\infty$, modulus of continuity)

Lemmas

• Any $f \in C^{(1)}_{2\pi}$ can be written as

$$(*) \qquad f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \theta f'(\theta+x+\pi)\,d\theta}_{\text{integration by parts}} .$$

• Let $g \in C_{2\pi}$, $h \in \mathcal{T}_n$. Then

$$(**) \qquad \psi(x) = \int_{-\pi}^{\pi} \underbrace{h(\theta)g(\theta+x)}\,d\theta \in \mathcal{T}_n .$$

separate variables

$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

use trigonometric identities

$\cos(\alpha\mp\beta)$

Proof. $\psi(x)$ is 2π-periodic, → integral does not change if $\theta \to \theta - x$ shift

$$\psi(x) = \int_{-\pi}^{\pi} \underbrace{h(\theta-x)}\,g(\theta)\,d\theta$$

↳ $\in \mathcal{T}_n$ → lin. combinations of $\cos(jx-j\theta)$ and $\sin(jx-j\theta)$

substitute, integrate with respect to θ

$$h(\theta-x) = \frac{1}{2}a_0(\theta) + \sum_{j=1}^{n}\left(a_j(\theta)\cos(jx) + b_j(\theta)\sin(jx)\right). \quad ☒$$

→ factor out $\cos(jx)$, $\sin(jx)$

• Powell, p 177-179.

$$(***) \qquad \min_{b_1,\dots,b_n} \int_0^{\pi} \left| x - \sum_{k=1}^{n} b_k \sin(kx) \right| dx = \frac{\pi^2}{2(n+1)}$$

↳ best L_1 approximation of $f(x) = x$ in $\mathcal{A} = \operatorname{span}\{\sin(x), \dots, \sin(nx)\}$.

Theorem (Jackson I)

Let $f \in C^{(1)}_{2\pi}$, $n \geq 0$. Then $E_n(f) \leq \frac{\pi}{2(n+1)}\|f'\|_\infty$.

Proof. Use f in the form $(*)$,

$$E_n(f) = \min_{q\in\mathcal{T}_n}\|f-q\|_\infty \overset{(*)}{\leq} \min_{q\in\mathcal{T}_n}\max_x \left| \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}_{D \text{ constant } \in \mathcal{T}_n} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta f'(\theta+x+\pi)\,d\theta - q \right|$$

any $q \in \mathcal{T}_n$ also $q + D$

$$= \min_{q\in\mathcal{T}_n}\max_x \left| \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta f'(\theta+x+\pi)\,d\theta - q \right|$$

use only special q in the form $(**)$

$$\overset{(**)}{\leq} \min_{q\in\mathcal{T}_n}\max_x \left| \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta f'(\theta+x+\pi)\,d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} q(\theta)\underbrace{f'(\theta+x+\pi)}_{g(\theta+x)}\,d\theta \right|$$

$$= \frac{1}{2\pi}\min_{q\in\mathcal{T}_n}\max_x \left| \int_{-\pi}^{\pi}(\theta - q(\theta))\, f'(\theta+x+\pi)\,d\theta \right|$$

$$\leq \frac{1}{2\pi}\min_{q\in\mathcal{T}_n}\int_{-\pi}^{\pi}|\theta - q(\theta)|\,d\theta \cdot \|f'\|_\infty$$

→ use only sines and the previous lemma $(***)$

$$\leq \frac{1}{2\pi}\,\frac{\pi^2}{n+1}\,\|f'\|_\infty = \frac{\pi}{2(n+1)}\|f'\|_\infty . \quad ☒$$

Remark. The constant $\frac{\pi}{2(n+1)}$ cannot be improved in general.

$\rightarrow$ theorem can be generalised for Lipschitz continuous functions.

Use the following construction:

- $f \in C_{2\pi} \Rightarrow F(x) \equiv \int_a^x f(\theta)\,dx$ is continuous and differentiable

  $\downarrow$ fundamental theorem of calculus $F'(x) = f(x)$

- The same holds for $\phi_\delta(x) \equiv \dfrac{F(x+\delta) - F(x-\delta)}{2\delta} = \dfrac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(\theta)\,d\theta$, $\delta > 0$ given.

  $\phi_\delta(x) \rightarrow f(x)$ as $\delta \rightarrow 0$

<u>Theorem (Jackson II)</u>. Let $f \in C_{2\pi}$ be Lipschitz continues with the constant L. Then $\forall n \geq 0$ it holds that

$$E_n(f) \leq \frac{\pi}{2(n+1)} L.$$

<u>Proof.</u>

- $\forall \phi \in C_{2\pi}$ and $q \in \mathcal{T}_n$: $E_n(f) \leq \|f-q\|_\infty \leq \|f-\phi\|_\infty + \|\phi - q\|_\infty$.
- Given $\delta > 0$, consider $\phi_\delta \in C_{2\pi}$ and $q \in \mathcal{T}_n$ to be the best approx. from $\mathcal{T}_n$ to ϕ_δ. Then

$$\boxed{E_n(f) \leq \|f - \phi_\delta\|_\infty + E_n(\phi_\delta).}$$

we will bound both

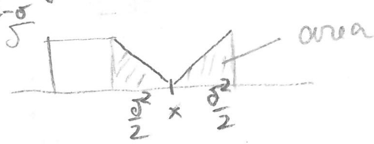
$\dfrac{f(x+\delta) - f(x-\delta)}{2\delta} \leq L\,|(x+\delta) - (x-\delta)|$

$\leq \int_{x-\delta}^{x} \omega(\delta) + \underbrace{\int_x^{x+\delta} \omega(\delta)}_{\delta\,\omega(\delta)}$

- Jackson I: $E_n(\phi_\delta) \leq \dfrac{\pi}{2(n+1)} \|\phi_\delta'\|_\infty \leq \dfrac{\pi}{2(n+1)} L$.

- $\|f - \phi_\delta\|_\infty = \max_x \left| f(x) - \dfrac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(\theta)\,d\theta \right| = \dfrac{1}{2\delta} \max_x \left| \int_{x-\delta}^{x+\delta} f(x) - f(\theta)\,d\theta \right|$

  $\leq \dfrac{L}{2\delta} \max_x \int_{x-\delta}^{x+\delta} |x - \theta|\,d\theta = \dfrac{L\delta}{2}$.



$\Rightarrow E_n(f) \leq L\left(\frac{\delta}{2} + \frac{\pi}{2(n+1)}\right)$ holds $\forall \delta > 0$. ∎

does not depend on δ

Can be generalised using the ~~modulus of continuity~~ $\omega(\delta)$ $\rightarrow$ defined $\forall \delta \geq 0$ by

$$\omega(\delta) \equiv \sup_{\substack{y,x \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|$$

$\omega(2\delta) \leq 2\omega(\delta)$

increasing (nondecreasing)

- $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ for continuous functions
- $\omega(\delta) \leq L\delta$ for Lipschitz

trivial for uniform but how fast? linear $\rightarrow$? Pich, Hemel

continuous f. On a compact subset of the real line: uniformly cont = continuous.

Theorem (Jackson III) $f \in C_{2\pi}$, $E_n(f) \le \frac{3}{2}\,\omega\left(\frac{\pi}{n+1}\right)$.

Proof. As in the previous theorem,

$$E_n(f) \le \|f - \phi_\delta\|_\infty + E_n(\phi_\delta)$$

- It holds that (from the previous proof)

$$\|f - \phi_\delta\|_\infty \le \omega(\delta).$$

$\left|\frac{f(x+\delta)-f(x-\delta)}{2\delta}\right|$

from the definition
$\omega(2\delta) \le \omega(\delta) + \omega(\delta)$
$\| \sup_{\substack{x<a<y\\ |x-a|\le\delta\\ |a-y|\le\delta}} |f(x)-f(a)+f(a)-f(y)|$

- Next (see Jackson II)

$$|\phi_\delta'(x)| \le \frac{\omega(2\delta)}{2\delta} \le \frac{\omega(\delta)}{\delta} \Rightarrow \|\phi'\|_\infty \le \frac{\omega(\delta)}{\delta}.$$

- Finally

$$E_n(f) \le \omega(\delta)\left[1 + \frac{\pi}{2(n+1)\delta}\right] \longrightarrow \text{choose } \delta = \frac{\pi}{n+1}$$ ☒

Remark: Jackson III proves that trig. polynomials are dense in $C_{2\pi}$.

- When does it hold that $S_n f \rightrightarrows f$?

→ Define S_n
→ Similar question for Cheb. pols

a sufficient condition → Dini-Lipschitz criterion.

We know that

$$\|S_n\|_\infty = \frac{1}{\pi}\int_0^\pi \left|\frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}\right| d\theta.$$

It can be shown [Powell, p. 192-193] that

$$(+) \quad \frac{4}{\pi^2}\ln(n+1) \le \|S_n\|_\infty \le 1 + \ln(2n+1).$$

[holds also for cont. fcions and Chebyshev expansion Rivlin 135]

Theorem (Dini-Lipschitz) Let $f \in C_{2\pi}$ and let

$$(++) \quad \lim_{\delta\to 0} |\omega(\delta)\ln(\delta)| = 0.$$ Then $S_n f \rightrightarrows f$ as $n \to \infty$.

Proof. S_n is linear and a projection $\Rightarrow$

$$\|f - S_n f\|_\infty \le (1 + \underbrace{\|S_n\|_\infty}_{(+)})\,\underbrace{E_n(f)}_{\text{Jackson III}}.$$

- $$\|f - S_n f\|_\infty \le (2 + \ln(2n+1))\,\frac{3}{2}\,\omega\left(\frac{\pi}{n+1}\right).$$

- use the same arguments and the assumption (++),

$$\ln(2n+1) \le \ln(2n+2) = \ln\left(\frac{2\pi}{\frac{\pi}{n+1}}\right) = \ln(2\pi) - \ln\left(\frac{\pi}{n+1}\right)$$

so that

$$\|f - S_n f\|_\infty \le \underbrace{\left(2 + \ln(2\pi) + \left|\ln\left(\frac{\pi}{n+1}\right)\right|\right)\frac{3}{2}\,\omega\left(\frac{\pi}{n+1}\right)}_{\to 0 \text{ as } n\to\infty}.$$ ☒

Theorem (Jackson IV). Let $f \in C_{2\pi}^{(k)}$, $n \ge 0$. Then

$$E_n(f) \le \left(\frac{\pi}{2(n+1)}\right)^k \|f^{(k)}\|_\infty.$$

Powell 194-195

Proof (idea). Show that $E_n(f) \le \frac{\pi}{2(n+1)} E_n(f')$ and use induction ☒

Extensions to algebraic polynomials (* Note)

Given $g \in C[-1,1]$, denote

maybe $e_n(g)$ → $d_n(g) \equiv \min_{q\in\mathcal{P}_n} \|g-q\|_\infty$. ↙ how to bound using previous results?

- Consider
$$f(x) \equiv g(\cos(x)) \quad \text{and} \quad q^* \in \mathcal{T}_n \text{ s.t.}$$
f is 2π periodic → satisfies the Haar cond, q^* is unique.
$$E_n(f) = \|f-q^*\|_\infty.$$

- $\cos(x)$ is even $\Rightarrow$ $f(x)$ is even, $f(x)=f(-x)$. We will show that also q^* is even:
$$\|f-q^*\|_\infty = \max_{-\pi\le\theta\le\pi} |f(\theta)-q^*(\theta)| = \max_{-\pi\le\theta\le\pi} |f(-\theta)-\underbrace{q^*(-\theta)}_{\tilde{q}^*(\theta)}| = \max_\theta |f(\theta)-\tilde{q}^*(\theta)|,$$
where $\tilde{q}^*(\theta)\equiv q^*(-\theta)$. $\Rightarrow$ $\tilde{q}^*$ is also best approx of f, use uniqueness $\Rightarrow$ $\tilde{q}^* = q^*$ $\Rightarrow$ $q^*(-\theta) = q^*(\theta)$.

- q^* is even $\Rightarrow$ is is a lin combination of $\cos(jx)$, $j=0,\dots,n$
$\Rightarrow$ — " — of $\cos^j(x)$ $j=0,\dots,n$
$$\Rightarrow \quad q^*(x) = \sum_{j=0}^n \gamma_j (\cos x)^j \equiv \tilde{p}(\cos x), \qquad \tilde{p}(s) = \sum_{j=0}^n \gamma_j s^j.$$

- Then
$$d_n(g) = \min_{q\in\mathcal{P}_n}\|g-q\|_\infty \le \|g-\underset{\in\mathcal{P}_n}{\tilde{p}}\|_\infty = \|f-q^*\|_\infty = E_n(f).$$
- On the other hand, let p^* be the best approx. to g, Then denote $\tilde{q}\in\mathcal{T}_n$, $\tilde{q}(x) = p^*(\cos(x))$.
$$E_n(f) \le \|f-\tilde{q}\|_\infty = \|g-p^*\|_\infty = \min_{q\in\mathcal{P}_n}\|g-q\|_\infty = d_n(g).$$

$$\boxed{E_n(f) = d_n(g)}$$ → Jackson I, II, III holds also for polynomials, Jackson IV only in a weaker form.

Theorem (Jackson V). Let $g\in C[-1,1]$, $n\ge 0$. Then
$$d_n(g) \le \frac{3}{2}\,\omega\!\left(\frac{\pi}{n+1}\right),$$ if g is Lipschitz (L), then
$$d_n(g) \le \frac{\pi L}{n+1},$$ if $g\in C^{(1)}_{[-1,1]}$, then $d_n(g) \le \frac{\pi}{n+1}\|g'\|_\infty$.

If $g\in C^{(k)}[-1,1]$, then
$$d_n(g) \le \frac{(n-k)!}{n!}\left(\frac{\pi}{2}\right)^k \|g^{(k)}\|_\infty \quad \text{for } n\ge k.$$
↳ want to use result → composed funct. ↳ without proof. ☐
$$\frac{\partial}{\partial x} g(x) = \frac{\partial}{\partial\theta} f(\cos(\theta)) = -f'(\cos(\theta))\cdot\sin(x)$$

Note Cheb. expansion of $f(x)$ in $[-1,1]$ coincides with the Fourier cosine series for $g(\theta) \equiv f(\cos\theta)$, $\theta \in [0,\pi]$.

$$\overset{\theta}{[0,\pi]} \underset{\cos^{-1}}{\overset{\cos}{\longleftrightarrow}} \overset{x}{[-1,1]} \qquad x = \cos\theta$$

$$T_k(x) = \cos(k\theta) \quad k \geq 1$$

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \qquad a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k}{\sqrt{1-x^2}}dx = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos(k\theta)\,d\theta$$

$$= \sum_{k=0}^{\infty} a_k \cos(k\theta), \qquad = \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos(k\theta)\,d\theta$$

$$= \cancel{\sum_{k=0}^{\infty}} g(\theta)$$

$\hookrightarrow$ even

Note even × odd

$$\sum_{j=0}^{m} a_j \cos(jx) + \sum_{j=1}^{n} b_j \sin(jx) \longrightarrow \text{even}$$

(a_0: 1/2)

$$\Rightarrow \sum_{j=1}^{m} b_j \sin(jx) = \sum_{j=1}^{n} b_j\,(\sin(jx)) = -\sum_{j=1}^{n} b_j \sin(jx)$$

$$\Rightarrow 2\sum_{j=1}^{m} b_j \sin(jx) \equiv 0 \Rightarrow b_j = 0$$

Note The theory of best L_1 approx [Powell, 164-172]

$$\min_{p\in\mathcal{A}} \|f-p\|_1 = \min_{p\in\mathcal{A}} \int_a^b |f(x)-p(x)|\,dx$$

... standard setting $f\in C[a,b]$, $\mathcal{A}$ lin space

$\hookrightarrow$ ~~instead of error fction~~ define the sign function which corresponds to p:

$$s(x) = \begin{cases} -1 & \text{if } f(x) < p(x) \\ 0 & \text{if } f(x) = p(x) \\ 1 & \text{if } f(x) > p(x) \end{cases}$$

Similar characterisation theorems

Theorem 14.1 $f\in C[a,b]$. Let $p^*\in\mathcal{A}$ be any element of $\mathcal{A}$ s.t.

$$Z = \{x : f(x) = p^*(x),\ a\leq x\leq b\}$$

is either empty or is composed of a finite number of intervals and discrete points.

p^* is a best L_1 approx from $\mathcal{A}$ to f $\Leftrightarrow$

$$\left|\int_a^b s^*(x)p(x)\,dx\right| \leq \int_Z |p(x)|\,dx \quad \forall p\in\mathcal{A}.$$

Theorem 14.2 $\dim\mathcal{A} = n+1$ ... the Haar ~~system~~ space. Let p^* be a best L_1 approx to f. If the number of zeros of $e^*(x) = f(x)-p^*(x)$ is finite, then e^* changes sign at least $n+1$ times.

Theorem 14.3 If $\mathcal{A}$ is the Haar space, $f\in C[a,b]$ $\longrightarrow$ there is just one best approx.

14.4 $\mathcal{A}$ ... the Haar space, $\dim \mathcal{A} = n+1$, $\mathcal{A} \subset C[a,b]$

Let $f \in C[a,b]$ s.t. $f(x) - p^*(x)$ has exactly $n+1$ zeros, p^* is the best L_1 approx. Then the positions of zeros of the error function do not depend on f.

→ important for calculating p^* → e.g. $\mathcal{A} =$ polynomials

14.5 $\mathcal{A} = \mathcal{P}_n$, f satisfies assumptions of 14.4. $[a,b] = [-1,1]$.

Then the zeros of $e(x) = f(x) - p^*(x)$ have the values

$$\xi_i = \cos\left(\frac{(n+1-i)\pi}{n+1}\right), \quad i = 0, \dots, n.$$

↙ abscissae of the extrema of T_{n+2}

⇓

The best L_1 approx. satisfies the interpolation cond.

$$\boxed{f(\xi_i) = p^*(\xi_i), \quad i = 0, \dots, n.}$$

(provided that the error function changes the sign only at these points)

Discrete L_1 approx. ... a linear programming problem.

Jackson (IV). **Proof** $f \in C_{2\pi}^{(k)}$ Powell 194–195

- First show that $E_n(f) \leq \frac{\pi}{2(n+1)} \|f' - r\|_\infty \quad \forall r \in \mathcal{T}_n$
- Use (modif.) proof of Jackson I (Lemma 1)

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta \underbrace{f'(\theta+x+\pi)}_{\pm r}\,d\theta$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}_{\text{const}} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta (f'-r)(\theta+x+\pi)\,d\theta + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\, r(x+\theta+\pi)\,d\theta}_{\phi(x) \in \tilde{\mathcal{T}}_n}$$

$$\Rightarrow \min_{q \in \mathcal{T}_n} \|f - q\|_\infty = \min_{s \in \mathcal{T}_n} \left\| \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\left(f'(\theta+x+\pi) - r(\theta+x+\pi)\right)d\theta - q \right\|_\infty$$

(Lemma 2)

$$\leq \min \max \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} (\theta - q)(f'(\theta+x+\pi) - r(\theta+x+\pi))\,d\theta \right|$$

$$\leq \underbrace{\min_q \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta - q(\theta)|\,d\theta}_{\leq \frac{\pi}{2(n+1)}} \cdot \|f' - r\|_\infty$$

- Then

$$E_n(f) \leq \frac{\pi}{2(n+1)} \overbrace{\min_{r \in \mathcal{T}_n} \|f' - r\|_\infty}^{E_n(f')}$$

$$\leq \left(\frac{\pi}{2(n+1)}\right)^k \|f^{(k)}\|_\infty.$$

Idea

Show it just once, then apply to Jackson I with $r = 0$.

Periodic differentiable:

$$E_n(f) \leq \frac{\pi}{2(n+1)} E_n(f')$$

non periodic

$$E_n(g) \leq \frac{\pi}{2(n+1)} E_{n-1}(g')$$