MINIMAX APPROXIMATION

Recall

Theorem (Weierstraß 1885). For any $f \in C[a,b]$ and for any $\varepsilon > 0$, there exists an algebraic polynomial of the form

$$p(x) = c_0 + c_1 x + \dots + c_n x^n, \qquad a \le x \le b$$

such that

$$\|f - p\|_\infty \le \varepsilon.$$

[Mergelyan's theorem 1951 -> generalization to the complex plane
S... compact, $\mathbb{C} \setminus S$ connected]

Idea of the proof.

Def. The operator $L : C[a,b] \to C[a,b]$ is <u>monotone</u>, if $\forall f$ and $\forall g \in C[a,b]$ s.t.

$$f(x) \ge g(x) \text{ it holds that } (Lf)(x) \ge (Lg)(x).$$

$\hookrightarrow$ useful when studying uniform convergence

Theorem (Bohman-Korovkin) Let L_i, $i = 0, 1, \dots$, be a sequence of linear monotone operators, $L_i : C[a,b] \to C[a,b]$. If the sequence $L_i f$ converges uniformly ($L_i f \rightrightarrows f$) to f for the functions

$$f(x) = 1, \quad f(x) = x, \quad \text{and } f(x) = x^2,$$

then $L_i f \rightrightarrows f \quad \forall f \in C[a,b]$.

[Proof Powell p. 62]

Bernstein operator

$f \in C[0,1]$ (use lin. transformation for general $[a,b]$)

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad 0 \le x \le 1$$

- it is linear
- it is not a projection

! interpolants
Cheb. polynomial are projecti.

-> consider $p(x) \begin{cases} 1 & \text{in } k/n \\ 0 & \text{in } \frac{j}{n},\ j \ne k \end{cases}$

Then

$$(B_n p)(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$\hookrightarrow$ is positive in $\frac{j}{n}$
$\Rightarrow \ne p$

$$B_n f \rightrightarrows f \quad \forall f \in C[0,1]$$

$\hookrightarrow$ use Bohman-Korovkin

Hence, $\forall f \in C[0,1]\ \exists \varepsilon > 0 : \|f - p\|_\infty < \varepsilon$ ☐

Introduction

Given $f \in C[a,b]$, $\mathcal{A}$ ... a linear space (finite dim)

Look for p^*:

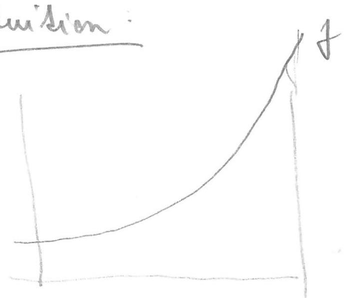
$$\|f - p^*\|_\infty = \min_{p \in \mathcal{A}} \|f - p\|_\infty$$

From previous, p^* exists.

$\mathcal{A}$ ... e.g. $\mathcal{P}_n$ ... algebraic polynomial, $\deg(p) \le n$. In general, span of functions satisfying „Haar condition".

Questions: Uniqueness, algorithm, Characterisation,

Intuition:



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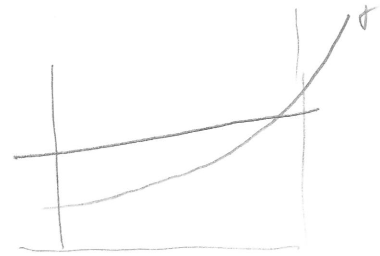
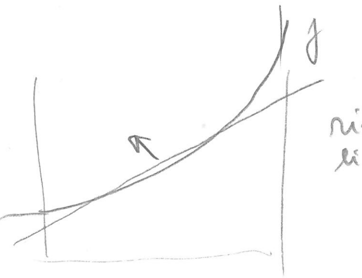
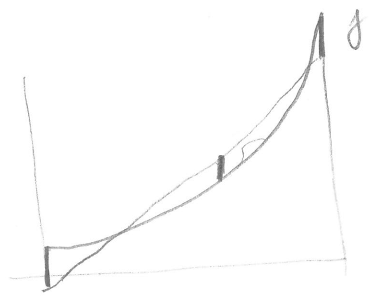


rotate the line a counter-clockwise



rise the line



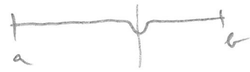
cannot be improved by a movement

Observation:
the error function
$e(x) \equiv f - p(x)$
changes the sign and maximum error is achieved at three points.

one only need consider the extreme values of $e(x)$.

In general, a best minimax operator from $C[a,b]$ to $\mathcal{A}$ is not linear

$X : f \mapsto p^*$. Consider $g \neq 0$



-> a small perturbation of 0.

Then p^* is also best app. to $f+g$

$$p^* = X(f+g) \neq X(f) + X(g).$$
$$\quad\quad\quad\quad\quad\quad p^* \quad\quad \neq 0$$

$\forall x \in Z_M$

$|f(x) - p^*(x)| < \|\tilde{e}\|_\infty = |\hat{e}(x)|$

$|\tilde{e}(x) - p(x)| < \hat{e}(x)$ $\forall x \in Z_M$

error function

- Let $\hat{p}^*$ be a trial approximation from $\mathcal{A}$ to f.

Denote $\hat{e}^*(x) = f(x) - \hat{p}^*(x)$, $x \in [a,b]$ ...

$$Z_M \equiv \{x \in [a,b] : |\hat{e}^*(x)| = \|\hat{e}^*(x)\|_\infty\}$$

a set where the error function takes its extreme values

Suppose that $\hat{p}^*$ is not optimal, and let $\underbrace{\hat{p}^* - p}_{p^*}$ be optimal for some $p \in \mathcal{A}$.

Then $\underbrace{|f(x) - p^*(x)|}_{< \|\hat{e}^*\|_\infty} = |\hat{e}^*(x) - p(x)| < |\hat{e}^*(x)|$ $\forall x \in Z_M$

$\|\hat{e}^*\|_\infty$

if $\hat{p}^*$ is not optimal => $\exists p$ s.t.

=> if $x \in Z_M$, then the sign of $\hat{e}^*(x)$ is the same as of $p(x)$.

=> $[f(x) - \hat{p}^*(x)]\, p(x) > 0$ $\forall x \in Z_M$

=> if $\nexists p \in \mathcal{A}$ s.t. $[f(x) - \hat{p}(x)]\, p(x) > 0$ $\forall x \in Z_M$, then $\hat{p}^*$ is a best minimax approximation

logic

$A \Rightarrow B \iff \neg B \Rightarrow \neg A$

A small generalization: Instead of $[a,b]$ consider Z ... any closed subset of $[a,b]$ (also a set of discrete points).

→ write as min max

<u>Theorem (Kolmogorov)</u>. Let A be a finite dim linear subspace of $C[a,b]$, $f \in C[a,b]$, let $Z \subseteq [a,b]$ be a closed subset, $p^* \in A$,

otherwise, a solution need not exist

and let Z_M be the set of point of Z at which the error $\{|f(x)-p^*(x)|;\, x \in Z\}$ takes its maximum value.

Then $p^* \in A$ minimizes

$$\max_{x\in Z} |f(x)-p(x)|, \quad p \in A$$

$\Leftrightarrow$

$$\nexists p \in A : \quad [f(x)-p^*(x)]p(x) > 0 \quad \forall\ x \in Z_M.$$

<u>Proof.</u> We have shown ($\Leftarrow$) (straightforward to extend to Z)

($\Rightarrow$) we shall show $\exists p \in A$:

$$(*) \quad [f(x)-p^*(x)]p(x) > 0 \quad \forall x \in Z_M$$

construct

$\Rightarrow$ p^* is not optimal (we can improve $f(x)-p^*(x)$)

- without loss of generality $|p(x)| \le 1$ (otherwise scale s.t. (*) holds)
- given $e^*(x)$ and $p(x)$, denote

$$Z_0 \equiv \{x \in Z : p(x)\, e^*(x) \le 0\}$$ → complementary ineq. to (*)

Z_0 is closed and $Z_0 \cap Z_M = \{\emptyset\}$ no points in common

- Denote

$$d \equiv \max_{x\in Z_0} |e^*(x)| < \max_{x\in Z} |e^*(x)|$$

$\exists$; ↳ maximum is attained on Z_M, not on Z_0

if Z_0 is empty, define $d=0$.

- Denote

$$\theta \equiv \tfrac{1}{2}\left[\max_{x\in Z}|e^*(x)| - d\right] > 0. \quad \left(\stackrel{d<\max_{x\in Z}}{\Rightarrow}\ \theta < \max_{x\in Z}|e^*(x)|\right)$$

Since Z is closed $\Rightarrow$ $|e^*(x)-\theta p(x)|$ attaines its maximum on Z

$$\exists \xi \in Z : \ |e^*(\xi)-\theta p(\xi)| = \max_{x\in Z} |\ldots\ldots|.$$

$e^*(x)-\theta p(x) = f(x) - (p^*(x)+\theta p(x))$ → vylepšení

- Either $\xi \in Z_0$ or $\xi \notin Z_0$:

∘ $\xi \in Z_0$:
$$\max_{x\in Z}|e^*(x)-\theta p(x)| = |e^*(\xi)-\theta p(\xi)| \stackrel{\xi\in Z_0}{=} |e^*(\xi)| + \theta|p(\xi)|$$
$$\le d + \theta = \tfrac12 d + \tfrac12 \max_{x\in Z}|e^*(x)| < \max_{x\in Z}|e^*(x)|.$$

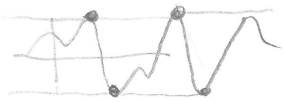
∘ $\xi \notin Z_0$ $\Rightarrow$ $e^*(\xi)p(\xi) > 0$
$$|e^*(\xi)-\theta p(\xi)| < \max(|e^*(\xi)|, \theta\overbrace{|p(\xi)|}^{\le 1}) = |e^*(\xi)| \le \max_{x\in Z}|e^*(x)|$$

∘ & ∘ $\Rightarrow$ $\max_{x\in Z}|e^*(x)-\theta p(x)| < \max_{x\in Z}|e^*(x)|$. ∎

To find out if a trial approx. is optimal, one only need consider the extreme values of the error function. (4)

The Haar condition

Recall the condition $(*)$ $[f(x)-p^*(x)]p(x) > 0 \ \forall x \in Z_M$



Motivation $\mathcal{A} = \mathcal{P}_m$

$p^* \in \mathcal{P}_m$ has at most m sign changes (m roots)

- If $f(x)-p^*(x)$ changes sign on Z_M more than m times $\Rightarrow \nexists p : (*) \Rightarrow p^*$ is best.
- If the number of sign changes on Z_M $\leq m \Rightarrow \exists p : (*) \Rightarrow p^*$ is not best.

$\to$ a more general class of functions, polynomials a special case

Def. An $\mathcal{A}$ ... $m+1$ dimensional subspace $\mathcal{A}$ of $C[a,b]$ is said to satisfy the Haar condition if the following condition is satisfied:

(1) $\phi \in \mathcal{A}, \phi \not\equiv 0 \Rightarrow$ number of roots of the equation $\{\phi(x) = 0 : a \leq x \leq b\}$ is less than $(m+1)$. ▨

$\boxed{(1) \Rightarrow (2)}$ (without proof)

(2) If $\{\xi_j : j = 1, \dots, k; \ 1 \leq k \leq m\}$ is any set of distinct points from (a,b) $\Rightarrow \exists$ an element of $\mathcal{A}$ that changes sign at these points and that has no other zeros. Moreover, $\exists p \in \mathcal{A}$ that has no zeros in $[a,b]$.

$\boxed{(1) \Leftrightarrow (3)}$

(3) If $\{\phi_i : i = 0, 1, \dots, m\}$ is any basis of $\mathcal{A}$, and if $\{\xi_j : j = 0, \dots, m\}$ is any set of $(m+1)$ distinct points in $[a,b]$, then the matrix $A = \{a_{ij}\}_{i,j=0}^{m}$, $a_{ij} = \phi_i(\xi_j)$ is nonsingular.

Proof: $(1) \Rightarrow (3)$ Let (1) hold but (3) fails.

A singular $\Rightarrow \exists \alpha_i, \ \sum \alpha_i^2 > 0 : \ \sum_{i=0}^{m} \alpha_i \begin{bmatrix} \phi_i(\xi_0) \\ \vdots \\ \phi_i(\xi_m) \end{bmatrix} = 0 \leftarrow \exists \ \xi_0, \dots, \xi_m$ distinct

$\Rightarrow \phi(x) \equiv \sum_{i=0}^{m} \alpha_i \phi_i(x)$ has $m+1$ distinct zeros $\to$ contradiction.

$(3) \Rightarrow (1)$

If (1) fails $\Rightarrow \exists \phi \in \mathcal{A} : \phi(\xi_i) = 0, \ i = 0, \dots, m, \ \xi_i$ distinct

$\phi \in \mathcal{A} \Rightarrow \phi = \sum_{i=0}^{m} \alpha_i \phi_i \Rightarrow$ the vectors $\begin{bmatrix} \phi_i(\xi_0) \\ \vdots \\ \phi_i(\xi_m) \end{bmatrix}$ are lin. dependent.

▨

A satisfies the Haar condition ... Haar space.

Any basis for a Haar space is called a Chebyshev system.

Examples on $\mathbb{R}$: [Cheney, Light, p. 6 Chapter 1]

- $1, x, x^2, \dots, x^m$
- $e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_m x}$ $\quad (\lambda_1 < \lambda_2 < \dots < \lambda_m)$
- $1, \cosh x, \sinh x, \dots, \cosh mx, \sinh mx$

On $(0, \infty)$

- $x^{\lambda_1}, \dots, x^{\lambda_m}$ $\quad (\lambda_1 < \lambda_2 < \dots < \lambda_m)$
- $(x+\lambda_1)^{-1}, \dots, (x+\lambda_m)^{-1}$, $\quad (0 \le \lambda_1 < \dots < \lambda_m)$

on the circle $\mathbb{R}/2\pi$

- $1, \cos\theta, \sin\theta, \dots, \cos m\theta, \sin m\theta$ ← prove it

Theorem (Chebyshev) Let A be an $(n+1)$-dimensional linear subspace of $C[a,b]$ that satisfies the Haar condition, let $f \in C[a,b]$. Then p^* is the best minimax approximation from A to f

$\iff$ $\exists$ $n+2$ points $\{\xi_i : i = 0, \dots, n+1\}$, ← Chebyshev alternation points

$$a \le \xi_0 < \xi_1 < \dots < \xi_{n+1} \le b$$

s.t.

(+) $$|f(\xi_i) - p^*(\xi_i)| = \|f - p^*\|_\infty, \quad i = 0, \dots, n+1$$

and

$$f(\xi_{i+1}) - p^*(\xi_{i+1}) = -[f(\xi_i) - p^*(\xi_i)], \quad i = 0, \dots, n.$$

Proof. Idea → use Kolmogorov for $Z = [a,b]$ and the Haar condition.

($\Leftarrow$) $e^*(x) = f(x) - p^*(x)$ changes sign $(n+1)$ times on Z_M $\overset{(1)}{\Rightarrow}$ $\nexists p \in A$:

$$[f(x) - p^*(x)]\, p(x) > 0 \quad \forall x \in Z_M$$

↳ such a pol. would have $n+1$ roots

($\Rightarrow$) p^* is the best $\Rightarrow$ $\nexists p \in A : [f(x) - p^*(x)]\, p(x) > 0 \quad \forall x \in Z_M$

How many times changes $e^*(x)$ sign on Z_M? (2)

- Less than $n+1$ times $\Rightarrow$ we can find at most $n+1$ subintervals s.t.

→ $e^*(x)$ does not change the sign on $x \in Z_M$ lying in given subinterval

$\Rightarrow \exists p : [f(x) - p^*(x)]\, p(x) > 0 \quad x \in Z_M$

p^* is not best.

Hence $e^*(x)$ changes sign on Z_n at least $n+1$ times $\Rightarrow$ $(+)$. ☒

Note. This can be formulated for any compact Z containing at least $n+2$ points.

Def. Any set of $n+2$ distinct points in $[a,b]$ is called a reference.

$$a \le \xi_0 < \xi_1 \dots < \xi_{n+1} \le b$$

The case Z contains just $n+2$ distinct point $\rightarrow$ important

Theorem. Let $\mathcal{A}$ be an $(n+1)$-dimensional subspace of a Haar subspace of $C[a,b]$ that satisfies the Haar condition. Let $\{\xi_i : i = 0,\dots,n+1\}$ be a reference, and let $f \in C[a,b]$. Then $p^* \in \mathcal{A}$ minimizes

$$\max_{i=0,\dots,n+1} |f(\xi_i) - p(\xi_i)| \quad \text{over } p \in \mathcal{A}$$

$\Leftrightarrow$

$$(++) \quad f(\xi_{i+1}) - p^*(\xi_{i+1}) = -[f(\xi_i) - p^*(\xi_i)], \quad i = 0,\dots,n.$$

Proof. Use Cheb theorem & $\mathcal{A} = \{\xi_i : i = 0\dots,n+1\}$ ☒.

p^* satisfying $(++)$ can be computed:

- Denote $h \equiv f(\xi_0) - p^*(\xi_0)$ (unknown) $\Rightarrow$ $f(\xi_i) - p^*(\xi_i) = (-1)^i h$, $i = 0,\dots,n+1$
- Choose a basis of $\mathcal{A}$ ... $\{\phi_j : j = 0,\dots,n\}$
- Consider $p^*(x) = \sum_{j=0}^{n} \alpha_j \phi_j(x)$
- Substitute to $(++)$

$$f(\xi_{i+1}) - \sum_{j=0}^{n} \alpha_j \phi_j(\xi_{i+1}) = -[(-1)^i h], \quad i = 0,\dots,n$$

$$\& \quad f(\xi_0) - \sum_{j=0}^{n} \alpha_j \phi_j(\xi_0) = h$$

$\rightarrow$ $n+2$ equations, $n+2$ unknowns α_j and h

$$\begin{array}{c} \xi_0 \rightarrow \\ \\ \\ \xi_{n+1} \rightarrow \end{array} \left[\begin{array}{c|c} & 1 \\ & -1 \\ A & 1 \\ & -1 \\ & \vdots \end{array} \right] \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \\ h \end{bmatrix} = \begin{bmatrix} f(\xi_0) \\ \vdots \\ \vdots \\ f(\xi_{n+1}) \end{bmatrix}$$

$\uparrow \phi_0 \qquad \uparrow \phi_n$

$\rightarrow$ from previous theorem $\exists$ solution $\forall f \in C[a,b]$ (for any right hand side) $\Rightarrow R(A) = \mathbb{R}^{n+2} \Rightarrow A$ is nonsingular

A does not depend on f $\Rightarrow \exists!$ solution ☒

Remark. Choose ϕ_j such that (→ Cheb. pol) $\phi_j(\xi_i)$ can be efficiently evaluated, to compute their values of $p^*(y)$ by Clenshaw alg.

Uniqueness

- $\|\cdot\|_\infty$ is not strictly convex

we will explain that uniqueness follows from the Haar condition

Idea: Suppose p^* and q^* are two best approx. satisfying $(+)$.
Consider the Cheb. alternation points that correspond, e.g., to p^* $\{\xi_i\}$

Define

$$r^*(x) = \underbrace{q^*(x) - p^*(x)}_{\in \mathcal{A} \dots \text{ satisfies the Haar condition}} = [f(x) - p^*(x)] - [f(x) - q^*(x)]$$

at ξ_i, $i = 0, \dots, n+1$ it holds that

$$r^*(\xi_i) = \underbrace{[f(\xi_i) - p^*(\xi_i)]}_{\substack{\pm \|f-p^*\|_\infty \\ \text{changes sign}}} - \underbrace{[f(\xi_i) - q^*(\xi_i)]}_{\leq \|f-q^*\|_\infty = \|f-p^*\|}$$

$\downarrow$

this term determines the sign at ξ_i

$(-1)^i r^*(\xi_i)$

$\Rightarrow$ either $(-1)^i r^*(\xi_i) \geq 0 \quad \forall i = 0, \dots, n+1$
or $(-1)^i r^*(\xi_i) \leq 0 \quad \forall i = 0, \dots, n+1$
does not change a sign

Lemma: Let $\mathcal{A}$ be an $(n+1)$-dim linear subspace of $C[a,b]$ that satisfies the Haar condition. Let $\{\xi_i : i = 0, \dots, n+1\}$ be a reference and let $r(x) \in \mathcal{A}$ satisfies the condition

$$(-1)^i r(\xi_i) \geq 0, \quad i = 0, \dots, n+1.$$

Then $r(x) \equiv 0$.

without proof

$\rightarrow$ trivial if Z is interval $\Downarrow$ r has $n+2$ roots

For general Z must be shown.

Hence, under the above assumptions, there is just one minimax approximation from $\mathcal{A}$ to f.

$\mathcal{A} \subset C[a,b]$

Theorem (de la Vallée Poussin) Let $\mathcal{A}$ be an $(n+1)$ dim lin sub of $C[a,b]$ that satisfies the Haar condition. Let $q \in \mathcal{A}$ be any element of $\mathcal{A}$, and let $\{\xi_i : i = 0, \dots, n+1\}$ be a reference s.t.

$$\operatorname{sign}[f(\xi_{i+1}) - q(\xi_{i+1})] = -\operatorname{sign}[f(\xi_i) - q(\xi_i)], \quad i = 0, 1, \dots, n.$$

Then it holds that

Draw a picture

$$\min_i |f(\xi_i) - q(\xi_i)| \leq \min_{p \in \mathcal{A}} \max_i |f(\xi_i) - p(\xi_i)|$$

$$\overset{\text{triv}}{\leq} \min_{p \in \mathcal{A}} \|f - p\|_\infty \leftarrow \text{this is optimal, can be bounded from below and above}$$

$$\leq \|f - q\|_\infty.$$

Moreover, the first inequality is strict unless all the numbers $\{|f(\xi_i) - q(\xi_i)| : i = 0, \dots, n+1\}$ are equal. *nejsou-li stejné, ostře nerovnost*

Proof. We just prove the first inequality.

Consideration, what implies?

Suppose $\exists s \in A$:

$$\min_i |f(\xi_i) - q(\xi_i)| \geq \max_i |f(\xi_i) - s(\xi_i)|$$

$$\Rightarrow r(x) = s(x) - q(x) = \underbrace{[f(x) - q(x)]}_{\text{dominance at } \xi_i} - [f(x) - s(x)]$$

satisfies

$(-1)^i r(\xi_i) \geq 0 \quad \forall i$ or $(-1)^i r(\xi_i) \leq 0 \quad \forall i$ $\Rightarrow r(x) \equiv 0$ and $q(x) \equiv s(x)$

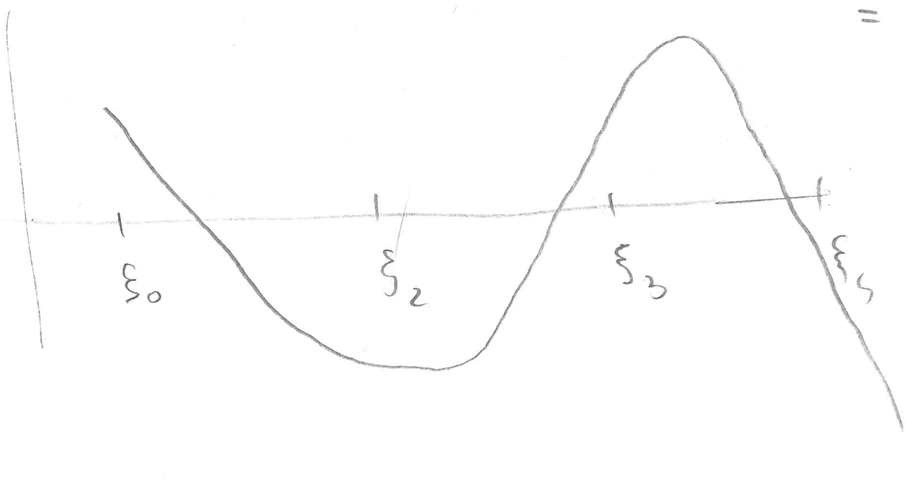
$\Rightarrow \{|f(\xi_i) - q(\xi_i)| : i = 0, \dots, n+1\}$ are equal and the equality holds.

Hence the first inequality holds.

- If the equality holds, then, from the previous $\{|f(\xi_i) - q(\xi_i)|\}$ are equal.
- If $\{|f(\xi_i) - q(\xi_i)|\}$ are equal, then $\min_i |f(\xi_i) - q(\xi_i)| = \max_i |f(\xi_i) - q(\xi_i)|$ $\Rightarrow q$ is best discrete so that the equality holds.
- Hence, the first inequality is equality $\Leftrightarrow \{|f(\xi_i) - q(\xi_i)|\}$ are equal.
- If they are not all equal $\to$ strict inequality holds. $\boxtimes$

$>$ $\to$ contradiction

$=$ $\to$ are the same



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<u>Proof</u>

- Suppose $\exists s \in \mathcal{A}$: $\min_i |f(\xi_i) - q(\xi_i)| > \max_i |f(\xi_i) - s(\xi_i)|$

$$\Rightarrow r(x) = \underbrace{s(x) - q(x)}_{\in \mathcal{A}} = \underbrace{(f(x) - q(x))}_{\text{dominant at } \xi_i} - (f(x) - s(x))$$

lemma $r \equiv 0$

$\Rightarrow (-1)^i r(\xi_i) > 0$ $\quad i = 0, 1, \dots, n+1$ $\quad \Rightarrow$ $\boxed{r \text{ ~~has~~ } n+1 \text{ roots}}$

or $(-1)^i r(\xi_i) < 0$

contradiction

$\Rightarrow$ $\leq$ holds.

- Suppose $=$ holds, then

$\exists s \in \mathcal{A}$: $\min |f(\xi_i) - q(\xi_i)| = \max |f(\xi_i) - s(\xi_i)|$

$\Rightarrow r(x) = s(x) - q(x)$ has the property

$(-1)^i r(\xi_i) \geq 0$ $\quad \Rightarrow r(x) \equiv 0$
≤ 0

$\forall i$ $\quad \Rightarrow q = s$

$\Rightarrow$

$\min |f(\xi_i) - q(\xi_i)| = \max |f(\xi_i) - q(\xi_i)|$

$\Rightarrow$ all the numbers are $=$ ∎