3. Approximation operators

$\mathcal{B}$ ... a normed linear space

$\mathcal{A}$ ... a set of approximating functions

An approximation operator ... any mapping from $\mathcal{B}$ to $\mathcal{A}$

$$\boxed{X : f \in \mathcal{B} \longmapsto X(f) \in \mathcal{A}}$$

- Nearly all numerical methods for calculating approximations are approx. operators.
  ↳ it is only necessary to select a unique element of $\mathcal{A}$

- Some definitions
  - X is a projection if $\boxed{X[X(f)] = X(f)}$, $f \in \mathcal{B}$

    a sufficient condition for X to be a projection is
    $$X(p) = p \quad \forall p \in \mathcal{A}.$$

  - X is linear if
    $$X(\lambda f) = \lambda X(f) \ \forall f \in \mathcal{B}, \lambda \in \mathbb{R}$$
    $$X(f+g) = X(f) + X(g), \quad \forall f \in \mathcal{B}, \forall g \in \mathcal{B}$$

    Usually,
    when X is linear and $\mathcal{A}$ is a finite-dimensional linear space
    → the calculation of $X(f)$ reduces to the solution of a system of linear eq.

  - norm of X. $\|X\|$ is the smallest real number s.t.
    $$\|X(f)\| \le \|X\| \, \|f\| \quad \forall f \in \mathcal{B}$$

    $\|X\|_p$ indicates that $\|X\|$ is derived from $\|f\|_p$

EXAMPLE: $\mathcal{B} = C[0,1]$, $\mathcal{A} = \mathcal{P}_1$ (lin. space ... real pol. of deg ≤ 1)

$X : f \in \mathcal{B} \mapsto p \in \mathcal{A}$ s.t. $p(0) = f(0)$, $p(1) = f(1)$.

X is a linear projection operator.

Choose a norm for X: $\|f\|_\infty = \max_{0 \le x \le 1} |f(x)|$. Then $\forall f$

$$\|X(f)\| = \underbrace{\|p\|}_{\text{lin.}} = \max(|p(0)|, |p(1)|) = \max(|f(0)|, |f(1)|) \le \|f\|.$$

i.e. $\boxed{\|X\| \le 1}$

For $f \equiv 1$ it holds that $\|X(f)\| = \|f\| = 1 \Rightarrow \|X\| = 1$ ∎

Note. When choosing $\|\cdot\|_2$ $\to$ X is unbounded in this norm

$$\|f\|_2 = \left(\int_0^1 [f(x)]^2\,dx\right)^{1/2}, \quad f \in C[0,1]$$

Consider functions f : $f(0) = f(1) = 1$ $\Rightarrow$ $X(f) \equiv 1$ $\Rightarrow$ $\|X(f)\|_2 = 1$.

But, we can choose f such that. $\|f\|_2$ is arbitrarily small

Hence, there is no number $\|X\|_2$ such that.

$$\|X(f)\|_2 \le \|X\|_2 \|f\|_2 \quad \forall f \in \mathcal{B}.$$

$\|X\|$.... sometimes called <u>the Lebesgue constant</u> of X.

(usually in connection with $\|\cdot\|_\infty$)

$\hookrightarrow$ can be used to <u>bound the error</u> of a best approximation.

<u>Theorem</u> Let $\mathcal{A}$ be a finite dimensional lin. subspace of $\mathcal{B}$, and let X be a <u>linear</u> operator from $\mathcal{B}$ to $\mathcal{A}$ that satisfies $X(p) = p \ \forall p \in \mathcal{A}$.

Then, $\forall f \in \mathcal{B}$ it holds that

$$\underbrace{\|f - X(f)\|}_{\text{error of the approx.}} \le (1 + \|X\|) \underbrace{\min_{p \in \mathcal{A}} \|f - p\|}_{\text{error of a best approx.}}.$$

<u>Proof.</u> Let p^* be a best approx. from $\mathcal{A}$ to f. (it exists, see previous thm).

Then,

- $$f - X(f) = f - p^* + p^* - X(f) \overset{X \text{ is a projection}}{=} f - p^* + X(p^*) - X(f) \underset{\text{linearity}}{=} (f - p^*) - X(f - p^*).$$

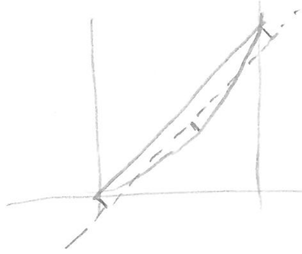
- $$\|f - X(f)\| \le \|f - p^*\| + \|X(f - p^*)\| \le (1 + \|X\|)\,\|f - p^*\|.$$

<u>Note.</u> In the previous example $\|X\| = 1$, i.e.

$$\|f - X(f)\|_\infty \le 2 \min_{p \in \mathcal{P}_1} \|f - p\|_\infty$$

$\Rightarrow$ X provides "almost" the best app.

<u>Note.</u> The inequality in the theorem is not sharp $\to$ we can find f such that $=$ holds. Consider X from the example, and $f = x^2$, $0 \le x \le 1$. Then $X(f) = x$. The best linear approx. in $\|\cdot\|_\infty$ of f is $p^*(x) = x - \frac{1}{8}$.



It holds that

$$\frac{1}{4} = \|x^2 - \underset{\substack{\downarrow \\ X(f)}}{x}\|_\infty = 2\,\|x^2 - \overset{p^*}{(x - \tfrac{1}{8})}\|_\infty.$$

Examples of approx operators and approx sets $\mathcal{A}$

$$X : C[a,b] \to \mathcal{A}$$

? ?

- Polynomial approximations $\mathcal{A} = \mathcal{P}_n$

  X determined using
  - interpolation conditions
  - best approximation in $\|\cdot\|_\infty$ or in $\|\cdot\|_2$
  - using Bernstein operator: on $[0,1]$

    $$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n}\right)$$

- Piecewise polynomial approximations

  $\mathcal{A} = \{ s : s$ is a continuous piecewise polynomial $\}$

  $$a = \xi_0 < \xi_1 < \dots < \xi_m = b$$

  s is a pol. of deg $\le k$ on each $[\xi_i, \xi_{i+1}]$.

  X determined using
  - interpolation cond, smoothness, boundary conditions.

- Approximation to periodic functions using trigonometric pols.

  $$\mathcal{A} = \left\{ q(x) : q(x) = \tfrac{1}{2}a_0 + \sum_{j=1}^{n} \big(a_j \cos(jx) + b_j \sin(jx) \big) \right\}$$

  $$X : C_{2\pi} \to \mathcal{A}$$

  - the best approximation in $\|\cdot\|_2$
  - "almost" best $\to$ approx. Fourier coef using a quadrature rule
    FFT

- Rational approximations

  $$X : C[a,b] \to \mathcal{A}$$

  $$\mathcal{A} = \left\{ r(x) : r(x) = \frac{p(x)}{q(x)},\ \deg(p) \le m,\ \deg(q) \le n \right\}$$

  - is not a linear space!
  - X is not a linear operator

  X determined using
  - interpolation conditions
  - the best approximation in $\|\cdot\|_\infty$
  - Padé approximation
    (a generalization of Taylor expansion)
    $\to$ the rational-function analogue of the Taylor polynomial approximation

  $f(x) \sim r_{m,n}$

  ↓ Taylor ↓

  agree to highest possible number of summands.

TESTS for uniform convergence [Powell, Chapter 17] (1)

Recall: A normed linear space is complete if every Cauchy sequence is convergent.

Def. The set $\{f_i : i = 0,1,2,\dots\}$ in a normed linear space $\mathcal{B}$ is called fundamental if $\forall f \in \mathcal{B}$ and $\forall \varepsilon > 0$ $\exists k$ and coefficients $\{a_i : i = 0,1,\dots,k\}$: $\| f - \sum_{i=0}^{k} a_i f_i \| < \varepsilon$.

countable, dense $\Rightarrow$ separable space

A normed space is separable $\Leftrightarrow$ it has a countable fundamental set

$\rightarrow$ Functional Anal. subjects

Example: $\mathcal{B} = C[a,b]$, $\|\cdot\| = \|\ \|_\infty$, $\{f_i(x) = x^i : a \le x \le b,\ i = 0,1,\dots\}$

Lemma Two bounded linear operators L_1 and L_2 ($\mathcal{B} \to \mathcal{B}$) are equal $\Leftrightarrow$ $L_1 f_i = L_2 f_i\ \forall i$.

Proof. ($\Rightarrow$) trivial

($\Leftarrow$) • Let $L_1 f_i = L_2 f_i\ \forall i$, but L_1 and L_2 different, i.e.,
$$\exists f \in \mathcal{B} : L_1 f \neq L_2 f.$$

• Define $\varepsilon = \dfrac{\|L_1 f - L_2 f\|}{\|L_1\| + \|L_2\|}$ and find $\phi = \sum_{i=0}^{k} a_i f_i$

such that $\|f - \phi\| < \varepsilon$.

• Then
$$\|L_1 f - L_2 f\| = \|L_1(f-\phi) - L_2(f-\phi)\| \le (\|L_1\| + \|L_2\|)\|f - \phi\| < (\|L_1\| + \|L_2\|)\varepsilon$$

contradiction with the def. of ε. ∎

Do not need

Given a sequence of linear operators X_m (from $\mathcal{B}$ to $\mathcal{B}$). When $X_m f \to f$ in the given norm $\forall f \in \mathcal{B}$?

$\exists M > 0 : \|X_m\| \le M\ \forall m$.

Lemma $\mathcal{B}$ ... a normed linear space, $\{f_i \in \mathcal{B}\}$ ... a fundamental set. Let $\{X_m : m = 0,1,\dots\}$ be a sequence of bounded linear operators $X_m : \mathcal{B} \to \mathcal{B}$.

Then $\lim_{m\to\infty} \|f - X_m f\| = 0$ $\forall f \in \mathcal{B}$ $\Leftrightarrow$ $\lim_{m\to\infty} \|f_i - X_m f_i\| = 0$, $i = 0,1,\dots$. $(*)$

s.t. $\{\|X_m\|\}$ is bounded.

Proof. ($\Rightarrow$) trivial

($\Leftarrow$) • Denote M a fixed upper bound on $\|X_m\|$

• f given, $\forall \varepsilon > 0$ $\exists k$ and $\{a_i\}$ s.t. $\|f - \phi\| < \frac{1}{2}\frac{\varepsilon}{M+1}$,
$$\phi = \sum_{i=0}^{k} a_i f_i.$$

• From $(*)$ $\exists N : \forall m \ge N$ $\|f_i - X_m f_i\| \le \frac{1}{2}\frac{\varepsilon}{\sum_{i=0}^{k}|a_i|}$ $i = 0,1,\dots,k$.

is smaller than a constant

- Then
$$\|f - X_n f\| \le \|(f-\phi) - X_n(f-\phi)\| + \|\phi - X_n\phi\|$$
$$\le (M+1)\|f-\phi\| + \left\|\sum_{i=0}^{k} a_i (f_i - X_n f_i)\right\|$$
$$\le (M+1)\|f-\phi\| + \sum_{i=0}^{k} |a_i|\,\|f_i - X_n f_i\|$$
$$\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$
$\boxtimes$

In summary,
- $\Rightarrow \forall \varepsilon\ \exists N : \forall n \ge N \quad \|f - X_n f\| < \varepsilon.$

Note: Many algs for calculating spline approx are bounded lin operators.

[Powell, 1981 book] uniform boundedness theorem 17.2.

<u>Theorem</u> $\mathcal{B}$ ... a complete normed linear space.
$\{X_n\}$ ... a sequence of linear operators, $X_n : \mathcal{B} \to \mathcal{B}$.

If the sequence of norms $\{\|X_n\| : n = 0, 1, \dots\}$ is unbounded, then $\exists f^* \in \mathcal{B}$ s.t. $X_n f^*$ diverges ($\|X_n f^*\| \to \infty$).

<u>Proof</u>. Want to show that $X_n f^*$ diverges $\to$ it is sufficient to work with a subsequence X_{n_j} satisfying $\|X_{n_j}\| \ge (20_j) 4^j$.

- Without loss of generality (for simplicity) assume $\|X_n\| \ge 20n\,4^n$.
- We will construct a Cauchy sequence whose limit f^* is s.t. $\|X_n f^*\| \to \infty$.
- First, find elements $\phi_n \in \mathcal{B}$: $\|\phi_n\| = 1$ & $\|X_n \phi_n\| \ge 0.8\|X_n\|$
  ↳ they exist from the definition of $\|X_n\|$
- Define the sequence $\{f_i\}$ by $f_0 = \phi_0$
$$f_k = f_{k-1} \quad \text{if} \quad \|X_k f_{k-1}\| \ge k + \left(\tfrac{1}{4}\right)^{k+1}\|X_k\|$$
$$f_k = f_{k-1} + \tfrac{3}{4}\left(\tfrac{1}{4}\right)^k \phi_k \quad \text{otherwise}$$
- The sequence $\{f_i\}$ is Cauchy: For $j > k$ it holds that
$$\|f_j - f_k\| = \left\|\sum_{i=k+1}^{j} (f_i - f_{i-1})\right\| \le \sum_{i=k+1}^{j} \tfrac{3}{4}\left(\tfrac{1}{4}\right)^i \underbrace{\|\phi_i\|}_{1} = \tfrac{3}{4}\left(\tfrac{1}{4}\right)^{k+1} \underbrace{\sum_{i=0}^{j-k-1} \left(\tfrac{1}{4}\right)^i}_{\frac{1-\left(\frac{1}{4}\right)^{j-k}}{1-\frac{1}{4}}} < \left(\tfrac{1}{4}\right)^{k+1}.$$
$\Rightarrow \exists$ limit $f^* \in \mathcal{B}$ satisfying $\|f^* - f_k\| \le \left(\tfrac{1}{4}\right)^{k+1}$.
- For the sequence $\{f_k\}$ it holds that
$$\|X_k f_k\| \ge k + \left(\tfrac{1}{4}\right)^{k+1}\|X_k\|. \qquad (**)$$

③

$(**)$ holds for $k=0$. For $k \geq 1$, either $f_k = f_{k-1}$ and $(**)$ or

$$f_k = f_{k-1} + \frac{3}{4}\left(\frac{1}{4}\right)^k \phi_k, \text{ if } \|X_k f_{k-1}\| < k + \left(\frac{1}{4}\right)^{k+1}\|X_k\|.$$

Then

$$\|X_k f_k\| \geq \Big\| \frac{3}{4}\left(\frac{1}{4}\right)^k \overbrace{X_k \phi_k}^{0.8 = \frac{4}{5}} \Big\| - \|X_k f_{k-1}\|$$

$$> \frac{3}{5}\left(\frac{1}{4}\right)^k \|X_k\| - \left(k + \left(\frac{1}{4}\right)^{k+1}\|X_k\|\right)$$

$$\left(\tfrac{1}{4}\right)^k \|X_k\| \underbrace{\left(\frac{3}{5} - \frac{1}{4}\right)}_{\frac{7}{20} = \frac{1}{4} + \frac{1}{10}}$$

$$= \left[k + \left(\frac{1}{4}\right)^{k+1}\|X_k\|\right] + \underbrace{\left[\frac{1}{10}\left(\frac{1}{4}\right)^k \|X_k\| - 2k\right]}_{\geq 0 \text{ see } \|X_k\| \geq 20 k 4^k}$$

$$\geq k + \left(\frac{1}{4}\right)^{k+1}\|X_k\|.$$

i.e. $(**)$ holds.

- Finally

$$\|X_m f^*\| \geq \|X_m f_m\| - \|X_m(f^* - f_m)\|$$

$$\geq m + \left(\frac{1}{4}\right)^{m+1}\|X_m\| - \|X_m\| \underbrace{\|f^* - f_m\|}_{\leq \left(\frac{1}{4}\right)^{m+1}} \geq m. \quad \blacksquare$$

① Discussion

$\mathcal{B}$ complete, X_m $\boxed{\text{projection linear}}$

②

$$\|f - X_m f\| \leq \underbrace{(1 + \|X_m\|)}_{\to \infty \text{ how fast?}} \underbrace{\min_{p \in \mathcal{A}_m} \|f - p\|}_{\to 0 \text{ how fast?}}$$

$\mathcal{A}_m = \text{span}\{f_0, \ldots, f_m\}$ fundamental

③ Interpolation $\mathcal{B} = C[a,b]$, $\mathcal{A}_m = P_m$, $\|\cdot\|_\infty$

Previous theorem
$\|X_m\| \to \infty \Rightarrow \exists f^* : \|f^* - X_m f^*\| \to \infty$

However, if f is "more" continuous (Lipschitz), it can happen

$$\|X_m\|_\infty \sim \ln m, \quad \min_{p \in P_m} \|f - p\|_\infty \sim \frac{1}{m}$$

and $X_m f \rightrightarrows f$ (i.e. $\|f - X_m f\|_\infty \to 0$).

Note Weierstraß

Spaces $C[a,b]$

Def. The operator $X : C[a,b] \to C[a,b]$ is monotone if $\forall f$ and $\forall g \in C[a,b]$ s.t. $f(x) \geq g(x)$ it holds that

$$(Xf)(x) \geq (Xg)(x).$$

→ moreover, if linear, then monotone $\Leftrightarrow$ $X(f) \geq 0 \ \forall f \geq 0$

Theorem (Bohman - Korovkin). Let X_n, $n = 0, 1, \dots$ be a sequence of linear monotone operators, $X_n : C[a,b] \to C[a,b]$.

$\| \|_\infty \to 0$ If $X_n f \rightrightarrows f$ for the functions $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, then $X_n f \rightrightarrows f$ $\forall f \in C[a,b]$. [Powell, Theorem 6.2]

Example: Bernstein operator, $f \in C[0,1]$

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

- it is linear
- it is not a projection

consider polynomial $p(x) = \begin{cases} 1 & \text{in } \frac{k}{n} \\ 0 & \text{in } \frac{j}{n}, j \neq k \end{cases}$

Then $(B_n p)(x) = \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{\text{it is positive in } \frac{j}{n}}$

$\Rightarrow (B_n p)(x) \neq p(x)$.

- it is monotone $B_n f \geq 0$ $\forall f \geq 0$
- it can be shown $B_n x^j \rightrightarrows x^j$, $j = 0, 1, 2$

E.g. $j = 0$, $f \equiv 1$: $(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1$

binomial theorem

similarly $B_n x = x$ (case $j = 1$)

$$B_n x^2 = x^2 \underbrace{- \frac{1}{n} x^2 + \frac{1}{n} x}_{\to 0}$$

$\Rightarrow$ $\underbrace{B_n f}_{\text{a polynomial of degree } n} \rightrightarrows f$ $\Rightarrow$ $\forall f \in C[0,1]$ $\forall \varepsilon$ $\exists n$:

$\| f - p \|_\infty < \varepsilon$,

where $p \in \mathbb{P}_n$.

$\Downarrow$ 70 years

Weierstraß 1885

Runge 1886

generalisation to $\mathbb{C}$ plane

MERGELYAN'S theorem 1951

S compact, $\mathbb{C} \setminus S$ connected

f continuous on S, analytic inside S (holomorphic)