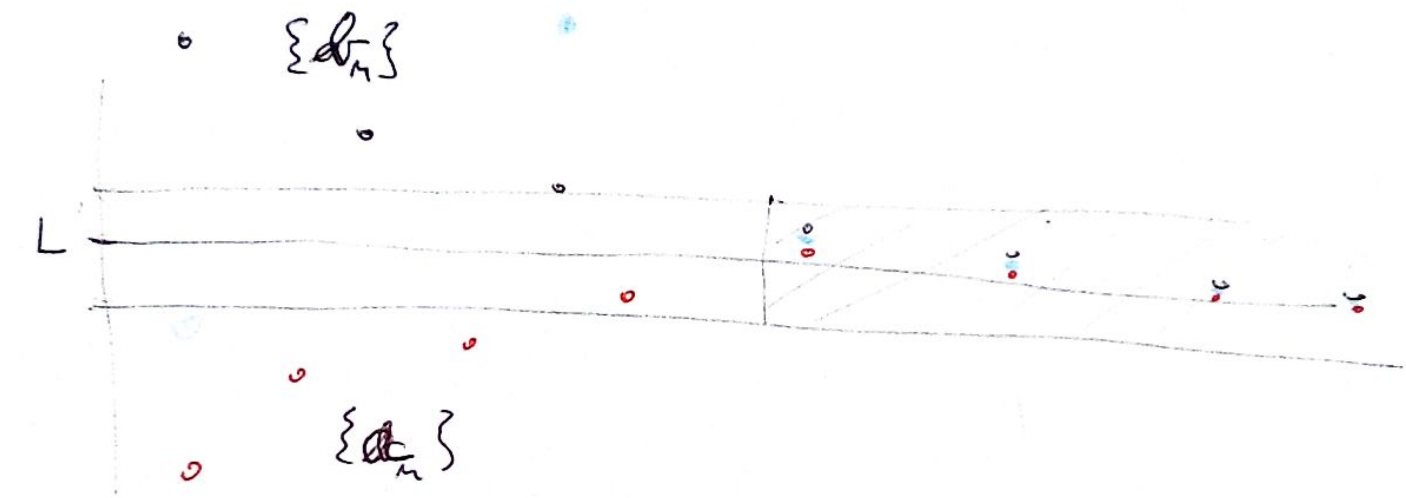
6. cvičení

$$\lim_{n\to+\infty} \frac{1000^n}{n!}$$



$$c_n \qquad \lim_{n\to+\infty} c_n = L$$

$$a = 1000$$

$$\frac{a^n}{n!} = \frac{\overbrace{a\cdot a\cdot \ldots \cdot a}^{n\times}}{1\cdot 2\cdot \ldots \cdot n} \overset{n>1000}{=} \underbrace{\frac{\overbrace{a\cdot a \ldots a}^{1000\times}}{1\cdot 2\cdot \ldots \cdot 1000}}_{\substack{\in \mathbb{R},\ >0 \\ =:C}} \overbrace{\frac{a}{1001}\cdot\frac{a}{1002}\cdot \ldots \cdot \frac{a}{n}}^{<1}$$

$$0 \le \frac{a^n}{n!} \le \frac{1000C}{n} \Rightarrow \left(\begin{array}{ll} a_n := 0 & a_n \le c_n \le b_n \\ b_n := \dfrac{1000C}{n} & \lim\limits_{n\to+\infty} a_n = 0 \\ c_n := \dfrac{a^n}{n!} & \lim\limits_{n\to+\infty} b_n = 0 \end{array}\right)$$

+ Věta o strážnících 2.6

$$\Rightarrow \lim \frac{a^n}{n!} = 0$$

2) $\lim\limits_{n\to+\infty} \frac{n!}{n^n} = 0$, protože

$$0 \leq \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot n}{\underbrace{n\cdot n\cdot n\cdot\ldots\cdot n}_{\leq 1}} \leq \frac{1}{n}$$

$$\begin{array}{ccccc} a_n & & c_n & & b_n \\ \downarrow{\scriptstyle n\to+\infty} & & \downarrow & & \downarrow{\scriptstyle n\to+\infty} \\ 0 & = & 0 & = & 0 \end{array}.$$

Růstová škála

a^n, $n \in \mathbb{R}$ je "$\underbrace{\text{hodně menší}}_{<<<}$" než $n! <<< n^n$

pro velká n

$\lim\limits_{n\to+\infty} \frac{n}{2^n} = 0$, protože

$$0 \leq \frac{n}{2^n} = \frac{n}{(1+1)^n} = \frac{n}{\sum\limits_{k=0}^{n}\binom{n}{k}} \leq \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

$$\left(\cancel{1+n} + \frac{n(n-1)}{2} + \cancel{\ldots + n + 1}\right)$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ a_n & \leq & c_n & \leq & b_n \\ \downarrow{\scriptstyle n\to+\infty} & & \downarrow & & \downarrow{\scriptstyle n\to+\infty} \\ 0 & = & 0 & = & 0 \end{array}.$$

- $\lim\limits_{n\to+\infty} \frac{n}{a^n} = 0$, protože

$a>1 \Rightarrow a = 1+\varepsilon$, kde $\varepsilon = a-1>0$

$$0 \le \frac{n}{a^n} = \frac{n}{(1+\varepsilon)^n} = \frac{n}{\sum\limits_{k=0}^{n} \binom{n}{k}\varepsilon^k} \le \frac{\cancel{n}}{\frac{\cancel{n}(n-1)}{2}\cdot\varepsilon^2} = \frac{2}{\varepsilon^2}\cdot\frac{1}{n-1}$$

- $\lim\limits_{n\to+\infty} \frac{n^2}{a^n} = 0$, protože

$a>1$

$$0 \le \frac{n^2}{a^n} \le \frac{n^{\cancel{2}}}{\frac{\cancel{n}(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\varepsilon^3} = \frac{6}{\varepsilon^3}\,\frac{n^{\cancel{2}}}{(n-1)(n-2)} = \frac{6}{\varepsilon^3}\underbrace{\frac{n}{n^2}}_{\to 0}\underbrace{\frac{1}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}}_{\to 1} \xrightarrow{n\to+\infty} 0$$

- $\lim\limits_{n\to+\infty} \frac{n^k}{a^n} =$

$a>1$, $k\in\mathbb{N}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot\ldots\cdot k}$$

$$\lim_{n\to+\infty}\left(\frac{n}{a^{n/k}}\right)^k = \lim_{n\to+\infty}\left(\frac{n}{(a^{1/k})^n}\right)^k =$$

$$= \lim_{n\to+\infty}\left(\frac{n}{b^n}\right)^k = \cancel{\lim_{n\to+\infty}}\ 0, \text{ indukcí a AL.}$$

$b := a^{1/k} > 1$

$$\lim_{n\to+\infty}\frac{3^n+n^5}{n^6+n!}=\lim_{n\to+\infty}\frac{3^n\left(1+\frac{n^5}{3^n}\right)}{n!\left(\frac{n^6}{n!}+1\right)}=0.$$

$$\frac{3^n}{n!}\xrightarrow[n\to+\infty]{}0,\quad \frac{n^5}{3^n}\to 0,\quad 1+\frac{n^5}{3^n}\to 1,\quad \frac{n^6}{n!}\to 0,\quad \frac{n^6}{n!}+1\to 1,\quad \to\frac{1}{1}=1$$

ad 4: $$\left|\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\right|\le\frac{\sqrt[3]{n^2}}{n+1}\xrightarrow[n\to+\infty]{}0$$

$$-\left(\frac{\sqrt[3]{n^2}}{n+1}\right)\le\frac{\sqrt[3]{n^2}\sin(n!)}{n+1}\le\frac{\sqrt[3]{n^2}}{n+1}$$

$$\to 0 \qquad \to 0 \qquad \to 0$$

$\lim_{n\to+\infty}\sqrt[n]{a}=1$, Dk: označme $\sqrt[n]{a}=1+\varepsilon_n$, $\varepsilon_n>0$

$a>1$

$$a=(1+\varepsilon_n)^n=\sum_{k=0}^{n}\binom{n}{k}\varepsilon_n^k$$
$$\ge\binom{n}{1}\varepsilon_n^{1}=n\,\varepsilon_n$$

$$\Rightarrow 0<\varepsilon_n\le\frac{a}{n}$$
$$\downarrow \quad \downarrow \quad \downarrow\ n\to+\infty$$
$$0=0=0$$

Doma: $k\in\mathbb{N}$

$$\lim_{n\to+\infty}\sqrt[n]{n^k}=1$$

$\lim_{n\to+\infty}\sqrt[n]{n}=1$; $\sqrt[n]{n}=1+\varepsilon_n$

$$n=(1+\varepsilon_n)^n=\sum_{k=0}^{n}\binom{n}{k}\varepsilon_n^k\ge\binom{n}{2}\varepsilon_n^2$$
$$\ge\frac{n(n-1)}{2}\varepsilon_n^2$$

$$\Rightarrow \sqrt{\frac{2}{n-1}}\ge\varepsilon_n>0 \Rightarrow \varepsilon_n\xrightarrow[n\to+\infty]{}0$$