Problem 2 – solutions

(a)

We will specify the household's dynamic optimization problem. We want to maximize the sum of life-time utilities of all individuals in the economy. In other words, we want to maximize the value of the integral:

$$\max U = \int_0^{\infty} \frac{\left(c_t + g_t\right)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \tag{1}$$

The overall budget constraint of household at time t will be exactly the same as in the basic Ramsey model with the exception of the fact that household now has to pay taxes from its consumption:

$$\frac{d\left(Assets\right)}{dt} = L_t w_t + L_t r_t a_t - c_t L_t - \tau_C c_t L_t \tag{2}$$

We are interested in the change of holdings of assets per person i.e. $\dot{a}_t$. Since

$$a_t = \frac{Assets_t}{L_t} \tag{3}$$

then:

$$\dot{a}_t = \frac{d\left(Assets_t\right)}{dt}\frac{1}{L_t} - \frac{Assets_t}{L_t}\frac{\dot{L}_t}{L_t} = \frac{d\left(Assets_t\right)}{dt}\frac{1}{L_t} - na \tag{4}$$

If we now divide overall budget constraint by L_t, we can write the budget constraint in per capita terms:

$$\dot{a}_t = w_t + \left[r_t - n\right]a_t - \left(1+\tau_C\right)c_t \tag{5}$$

No Ponzi game restriction. Without this restriction, nothing in the model would prevent the household from borrowing an infinite amount of money and thus increasing the present value of its consumption to arbitrarily high levels. The household would never have to repay its debt because it can always borrow even more to cover the existing debt when it is due. The amount of debt would be ever-increasing. To prevent this, we restrict the present value of assets to be positive:

$$\lim_{t->\infty} a_t e^{-\int_0^t \left(r(v)-n\right)dv} \geq 0 \tag{6}$$

Where $-\int_0^t \left(r(v)-n\right)dv$ is continuous discount factor based on continuous discounting with changing interest rate r_t.

Therefore, household's dynamic optimization problem will be:

$$\max U = \int_0^{\infty} \frac{\left(c_t + g_t\right)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \tag{7}$$

s.t. $$\dot{a}(t) = w_t + \left[r_t - n\right] a_t - \left(1+\tau_C\right) c_t \tag{8}$$

$$\lim_{t->\infty} a_t e^{-\int_0^t \left(r(v)-n\right)dv} \geq 0 \tag{9}$$

(b)

For deriving the first order conditions, we will use the current value Hamiltonian:

$$H_{CV} = e^{(\rho-n)} H_{PV} = \frac{\left(c_t + g_t\right)^{1-\theta}}{1-\theta} + \mu_t \left(w_t + \left[r_t - n\right] a_t - \left(1+\tau_C\right) c_t\right) \tag{8}$$

1) $$\frac{\partial H_{CV}}{\partial c_t} = 0 \tag{9}$$

We compute the derivation:

$$\frac{\partial H_{CV}}{\partial c_t} = \frac{(1-\theta)\left(c_t + g_t\right)^{-\theta}}{1-\theta} - \mu_t \left(1+\tau_C\right) = 0 \tag{10}$$

which yields the equation:

$$\mu_t = \frac{\left(c_t + g_t\right)^{-\theta}}{1+\tau_C} \tag{11}$$

2) $$\frac{\partial H_{CV}}{\partial a_t} = (\rho - n)\mu_t - \dot{\mu}_t \tag{12}$$

$$\frac{\partial H_{CV}}{\partial a_t} = \left(r_t - n\right)\mu_t = (\rho - n)\mu_t - \dot{\mu}_t \tag{13}$$

After re-arrangement we obtain:

$$\frac{\dot{\mu}_t}{\mu_t} = \rho - n - r_t + n \tag{14}$$

which yields the equation:

$$\frac{\mu_t}{\dot{\mu}_t} = \rho - r_t \tag{15}$$

3) Transversality condition will have the form:

$$\lim_{t->\infty} a_t \mu_t e^{(\rho-n)t} = 0 \tag{16}$$

(c)

Since the whole tax revenue (which is equal to $\tau_C c_t L_t$) is always spent on government purchases, the flow budget constraint of the government will have the form:

$$\tau_C c_t L_t = G_t \tag{17}$$

We can also express this budget constraint in per capita terms:

$$\tau_C c_t = g_t \tag{18}$$

It is not necessary to specify No-Ponzi Game condition for the government behavior because we assume that government budget is always balanced. This means that government has no possibility to transfer funds to next period – it does not issue bonds. Everything that was collected in form of taxes will be spent on government purchases in the same period of time.

(d)

We take the first derivative of the equation (11) with respect to time. We obtain:

$$\dot{\mu}_t = \frac{-\theta\left(c_t + g_t\right)^{-\theta-1}\left(\dot{c}_t + \dot{g}_t\right)}{1+\tau_C} \tag{19}$$

Since it holds that:

$$g_t = \tau_C c_t \tag{20}$$

$$\dot{g}_t = \left(\tau_C c_t\right)' = \tau_C \dot{c}_t \tag{21}$$

we can rewrite the equation (19):

$$\dot{\mu}_t = \frac{-\theta\left(c_t + \tau_C c_t\right)^{-\theta-1}\left(\dot{c}_t + \tau_C \dot{c}_t\right)}{1+\tau_C} \tag{22}$$

which simplifies to:

$$\dot{\mu}_t = -\theta\left(\left(1+\tau_C\right)c_t\right)^{-\theta-1}\dot{c}_t \tag{23}$$

We also rewrite equation (11) in the same way:

$$\mu_t = \frac{\left(c_t + g_t\right)^{-\theta}}{1+\tau_C} = \frac{\left(c_t + \tau_C c_t\right)^{-\theta}}{1+\tau_C} = \frac{\left(\left(1+\tau_C\right)c_t\right)^{-\theta}}{1+\tau_C} \tag{24}$$

We divide equation (23) by equation (24):

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{-\theta\left(\left(1+\tau_C\right)c_t\right)^{-\theta-1}\dot{c}_t}{\dfrac{\left(\left(1+\tau_C\right)c_t\right)^{-\theta}}{1+\tau_C}} = \frac{-\theta\left(1+\tau_C\right)\dot{c}_t}{\left(1+\tau_C\right)c_t} = \frac{-\theta\dot{c}_t}{c_t} \tag{25}$$

From equations (15) and (25) we obtain:

$$\frac{-\theta \dot{c}_t}{c_t} = \rho - r_t \tag{26}$$

which yields the Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(r_t - \rho\right) \tag{27}$$

(e)

A profit-maximizing representative firm maximizes the profit function (see lecture notes for further explanation) in the form:

$$\max_{k_t} \Pi = Ak_t^{\alpha} - \left(r_t + \delta\right)k_t - w_t \tag{28}$$

We derive the first order condition with respect to capital:

$$\frac{\partial \Pi}{\partial c_t} = A\alpha k_t^{\alpha-1} - \left(r_t + \delta\right) = 0 => r_t = A\alpha k_t^{\alpha-1} - \delta \tag{29}$$

which tells us that in optimum marginal product of capital ($A\alpha k_t^{\alpha-1}$) minus depreciation must be equal to the interest rate r_t.

Similarly, marginal product of labor in optimum must be equal to marginal costs of labor (wage):

$$w_t = MPL = \frac{\partial F\left(K_t, L_t\right)}{\partial L_t} = \frac{\partial\left(L_t f\left(k_t\right)\right)}{\partial L_t} = f\left(k_t\right) + L_t f'\left(k_t\right)\frac{\partial k_t}{\partial L_t} =$$

$$= f\left(k_t\right) + L_t f'\left(k_t\right)\frac{\partial\left(\frac{K_t}{L_t}\right)}{\partial L_t} = f\left(k_t\right) - L_t f'\left(k_t\right)\frac{K_t}{L_t^2} = f\left(k_t\right) - f'\left(k_t\right)k_t \tag{30}$$

which yields the equation:

$$w_t = f\left(k_t\right) - f'\left(k_t\right)k_t = Ak_t^{\alpha} - A\alpha k_t^{\alpha-1}k_t = \left(1-\alpha\right)Ak_t^{\alpha} \tag{31}$$

We can now specify the competitive market equilibrium. In this equilibrium, it must hold that all of the capital stock must be owned by household members ($a_t = k_t$). Therefore, we can rewrite the budget constraint (8):

$$\dot{k}_t = w_t + \left[r_t - n\right]k_t - \left(1+\tau_C\right)c_t \tag{32}$$

We also plug the above derived expressions for r_t and w_t (equations (29) and (31) respectively) into this budget constraint and into the Euler equation (27):

$$\dot{k}_t = \left(1-\alpha\right)Ak_t^{\alpha} + \alpha Ak_t^{\alpha} - \left(\delta + n\right)k_t - \left(1+\tau_C\right)c_t \tag{33}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(\alpha Ak_t^{\alpha-1} - \delta - \rho\right) \tag{34}$$

Therefore, we can summarize the equilibrium conditions as:

$$\dot{k}_t = Ak_t^{\alpha} - (\delta + n)k_t - (1+\tau_C)c_t \tag{35}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(\alpha A k_t^{\alpha-1} - \delta - \rho\right) \tag{36}$$

NPG condition:

$$\lim_{t \to \infty} k_t e^{-\int_0^t (r(v)-n)dv} \geq 0 \tag{37}$$

(f)

The conditions for the steady-state level of capital and consumption are:

$$\dot{k}_t = 0 \Rightarrow Ak_t^{\alpha} - (\delta + n)k_t - (1+\tau_C)c_t = 0 \tag{38}$$

$$\dot{c}_t = 0 \Rightarrow \alpha A k_t^{\alpha-1} = \delta + \rho \tag{39}$$

Under these conditions we can draw the phase diagram. Equations (38) and (39) tell us how to draw the curves (for further details about how to draw the curves see lecture notes) and "arrows":

Condition for $\dot{c}_t = 0$ is $f'(k_t^*) = \delta + \rho$, where $f'(k)$ is decreasing function in *k*. That implies:

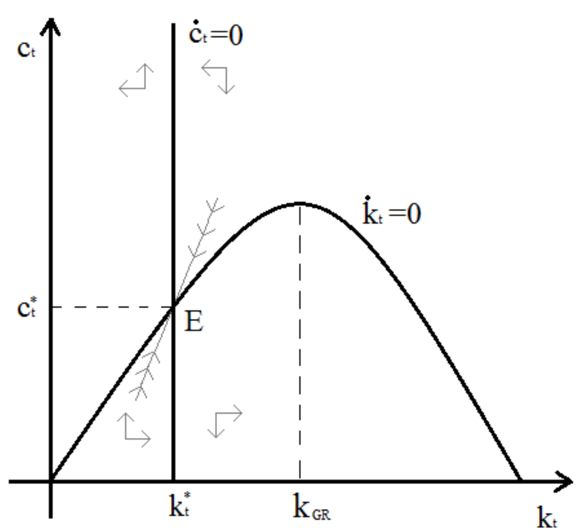
If $k_t > k_t^*$ (i.e. to the right of $\dot{c}_t = 0$ curve) => $f'(k) < \delta + \rho \Rightarrow \dot{c}_t < 0 \Rightarrow c \searrow$

If $k_t < k_t^*$ (i.e. to the left of $\dot{c}_t = 0$ curve) => $f'(k) > \delta + \rho \Rightarrow \dot{c}_t > 0 \Rightarrow c \nearrow$

Condition for $\dot{k}_t = 0$ is $f(k_t^*) - (\delta + n)k_t^* - (1+\tau_C)c_t^* = 0$. That implies:

If $c_t > c_t^*$ (i.e. above $\dot{k}_t = 0$ curve) => $f(k_t) - (\delta + n)k_t < (1+\tau_C)c_t \Rightarrow \dot{k}_t < 0 \Rightarrow k \searrow$

If $c_t < c_t^*$ (i.e. under $\dot{k}_t = 0$ curve) => $f(k_t) - (\delta + n)k_t > (1+\tau_C)c_t \Rightarrow \dot{k}_t > 0 \Rightarrow k \nearrow$



(g)

Because government has balanced budget, increased government purchases are financed through the increase of consumption tax ($\tau_C \nearrow$). This has following implication on the dynamics of our model:

1) $\dot{c}_t$ locus does not change – as it does not depend on the consumption tax
2) $\dot{k}_t$ locus changes – with higher tax, holding the amount of capital fixed we need lower consumption for $\dot{k}_t = 0$ to hold. Therefore, $\dot{k}_t$ locus shifts down.

Let's analyze two basic variations of this problem:

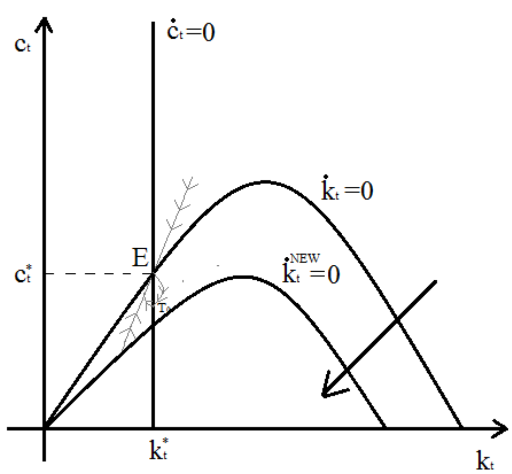
i) Change of taxation was not anticipated

Timing: at time t_0 government implements higher consumption tax and says it will be abolished at time t_1. So we will have two relevant locuses - $\dot{k}_t = 0$ valid to t_0 and from $t_1 \to \infty$, and $\dot{k}_t^{NEW} = 0$ valid from t_0 to t_1.

At time t_0 the new locus $\dot{k}_t^{NEW} = 0$ starts to hold. Therefore the current value of $c = c^*$ is too high. As we cannot adjust k immediately, consumption jumps down to the point T_0. (Note, that it will not jump under the $\dot{k}_t^{NEW} = 0$ locus, as in that case it would diverge from temporal steady-state and it will also not jump directly into new steady-state, as people know that the change will be temporary and staying in the temporal steady-state would prohibit them to return back to the old balanced growth path).

In the period $t_0 \to t_1$: as consumption is still above equilibrium level, we rule ourselves by up-left arrow, i.e. decreasing capital and increasing consumption.

At time t_1 the old locus $\dot{k}_t = 0$ holds, we just arrived at balanced growth path and converge to the old steady-state.



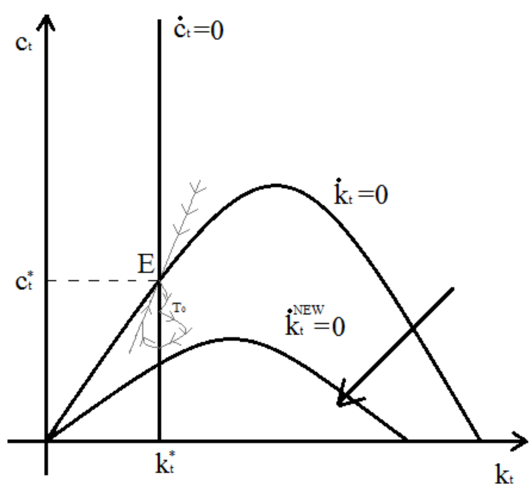
(ii) Change of taxation was anticipated.

Timing: at time t_0 government announces that it plans to introduce higher consumption tax at time t_1 and says it will be abolished at time t_2. Again we will have two relevant locuses but with different timing - $\dot{k}_t = 0$ valid from t_0 to t_1 and from $t_2 \rightarrow \infty$ and $\dot{k}_t^{NEW} = 0$ is valid from t_1 to t_2.

At time t_0 old locus $\dot{k}_t = 0$ still holds. However, consumers anticipate future increase in tax and as they prefer to smooth consumption, they will adjust current *c*. It will therefore jump down to the point T_0.
In the period from t_0 to t_1: the old locus still holds, so we follow down-right arrows from point T_0 – consumption decreases and capital increases.

At time t_1 the new locus $\dot{k}_t^{NEW} = 0$ starts to hold, we are above it and follow corresponding arrows – first we further decrease consumption together with decreasing capital, then after crossing $\dot{c}_t$ locus we increase consumption and further decrease capital.

At time t_2 the old locus $\dot{k}_t = 0$ holds again, we just arrived at balanced growth path and converge to the old steady-state.



(h)

Social's planner problem (maximize utility of average consumer s.t. aggregate feasibility constraint):

$$\max_{c_t} \int_0^\infty \frac{(c_t + g_t)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \tag{40}$$

s.t. $\quad Y_t = C_t + I_t + G_t \qquad (41)$

The feasibility constraint can be (after dividing all terms by L_t and substituting for $i_t = \dot{k}_t + (\delta + n) k_t$ written as:

$$\dot{k}_t = Ak_t^{\alpha} - c_t - g_t - (n+\delta) k_t \tag{42}$$

The present value Hamiltonian will be:

$$H_{PV} = \frac{(c_t + g_t)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} + \lambda_t \left(Ak_t^{\alpha} - c_t - g_t - (n+\delta) k_t \right) \tag{43}$$

We derive the first-order conditions:

$$\frac{\partial H_{PV}}{\partial c_t} = 0: \ (c_t + g_t)^{-\theta} e^{-(\rho-n)t} - \lambda_t = 0 \tag{44}$$

$$\frac{\partial H_{PV}}{\partial g_t} = 0: (c_t + g_t)^{-\theta} e^{-(\rho-n)t} - \lambda_t = 0 \tag{45}$$

$$\lim_{t \to \infty} \lambda_t a_t = 0 \tag{46}$$

We have indeterminacy of choice of c_t and g_t. Let's denote the path of optimal consumption when government spending is zero ($g_t = 0$ for all t) by $c_t^0 {}_{t=0}^{\infty}$. As private consumption and public expenditures are perfect substitutes, any choice of g_t s.t. $g_t \leq c_t^0$ for all t brings the same utility as by zero government spending and therefore is socially optimal. Also, when we consider decentralized equilibrium from previous parts, because all proceedings of consumption tax are redistributed in form of government expenditures, decentralized equilibrium is also socially optimal. The role of the perfect substitutability is crucial, as it assures that agent gets the same utility from c_t and g_t and therefore he is indifferent between them.

(i)

If household preferences have this utility function, the optimization problem will be:

$$\max U = \int_0^{\infty} \frac{\left(c_t^{\gamma} g_t^{1-\gamma}\right)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \tag{47}$$

$$\text{s.t.} \quad \dot{a}(t) = w_t + [r_t - n] a_t - (1+\tau_C) c_t \tag{48}$$

$$\lim_{t -> \infty} a_t e^{-\int_0^t (r(v)-n) dv} \geq 0 \tag{49}$$

The utility function can be rewritten as:

$$\frac{\left(c_t^{\gamma} g_t^{1-\gamma}\right)^{1-\theta}}{1-\theta} = \frac{\left(c_t^{\gamma} (\tau_C c_t)^{1-\gamma}\right)^{1-\theta}}{1-\theta} = \frac{\left(\tau_C^{1-\gamma} c_t\right)^{1-\theta}}{1-\theta} \tag{50}$$

Let us derive the first-order conditions. We will use the current value Hamiltonian:

$$H_{CV} = \frac{\left(\tau_C^{1-\gamma} c_t\right)^{1-\theta}}{1-\theta} + \mu_t \left[w_t + [r_t - n] a_t - (1+\tau_C) c_t \right] \tag{51}$$

1)

$$\frac{\partial H_{CV}}{\partial c_t} = 0: \ \frac{(1-\theta)\left(\tau_C^{1-\gamma} c_t\right)^{-\theta} \tau_C^{1-\gamma}}{1-\theta} - (1+\tau_C) \mu_t = 0 \tag{52}$$

This can be rewritten as:

$$\mu_t = \frac{\left(\tau_C^{1-\gamma} c_t\right)^{-\theta} \tau_C^{1-\gamma}}{\left(1+\tau_C\right)} \tag{53}$$

2)

$$\frac{\partial H_{CV}}{\partial a_t} = (\rho - n)\mu_t - \dot{\mu}_t : \ (r_t - n)\mu_t = (\rho - n)\mu_t - \dot{\mu}_t \tag{54}$$

which yields the equation:

$$\frac{\dot{\mu}_t}{\mu_t} = (\rho - r_t) \tag{55}$$

3) Transversality condition:

$$\lim_{t\to\infty} \mu_t e^{(\rho-n)t} a_t = 0 \tag{56}$$

We compute the first derivative of equation (53) with respect to time:

$$\dot{\mu}_t = \frac{-\theta\left(\tau_C^{1-\gamma} c_t\right)^{-\theta-1} \tau_C^{1-\gamma} \dot{c}_t \tau_C^{1-\gamma}}{\left(1+\tau_C\right)} \tag{57}$$

We divide equation (57) by equation (53):

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{\dfrac{-\theta\left(\tau_C^{1-\gamma} c_t\right)^{-\theta-1} \tau_C^{1-\gamma} \dot{c}_t \tau_C^{1-\gamma}}{\left(1+\tau_C\right)}}{\dfrac{\left(\tau_C^{1-\gamma} c_t\right)^{-\theta} \tau_C^{1-\gamma}}{\left(1+\tau_C\right)}} = \frac{-\theta\left(\tau_C^{1-\gamma} c_t\right)^{-\theta-1} \tau_C^{1-\gamma} \dot{c}_t \tau_C^{1-\gamma}}{\left(\tau_C^{1-\gamma} c_t\right)^{-\theta} \tau_C^{1-\gamma}} = \frac{-\theta \dot{c}_t}{c_t} \tag{58}$$

Equations (55) and (58) imply:

$$\frac{-\theta \dot{c}_t}{c_t} = \rho - r_t \tag{59}$$

which yields the Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(r_t - \rho\right) \tag{60}$$

So basically, the Euler equation does not change.