Ex. „WASTE"

$$\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \dots, \begin{pmatrix} Y_n \\ X_n \end{pmatrix} \quad \text{i.i.d}$$

BIO WASTE (Y_n), TOTAL (X_n)

$$\theta_x = \frac{\mathbb{E} Y_1}{\mathbb{E} X_1}, \quad \hat{\theta}_n = \frac{\overline{Y}_n}{\overline{X}_n}$$

PREVIOUSL:

$$\sqrt{n}\left(\hat{\theta}_n - \theta_x\right) \xrightarrow[n \to \infty]{d} N(0, \sigma^2),$$

WHERE $\sigma^2 = \frac{1}{(\mathbb{E} X_1)^2} \operatorname{var}(Y_1 - \theta_x X_1)$

ESTIM. OF σ^2:

$$\hat{\sigma}_n^2 = \frac{1}{(\overline{X}_n)^2} \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \hat{\theta} X_i)^2$$

STAND. ASYMPT. CONF. INT

$$\left(\hat{\theta}_n \mp u_{1-\alpha/2} \frac{\hat{\sigma}_n}{\sqrt{n}} \right)$$

NONPARAM. BOOTSTRAP

$$\begin{pmatrix} X^*_{1,b} \\ Y^*_{1,b} \end{pmatrix}, \dots, \begin{pmatrix} X^*_{n,b} \\ Y^*_{n,b} \end{pmatrix} \quad \text{i.i.d}$$

$$\rightsquigarrow R^*_{n,b} = \sqrt{n}\left(\hat{\theta}^*_{n,b} - \hat{\theta}_n\right), \quad b = 1, \dots, B$$

$$\hat{\theta}^*_{n,b} = \frac{\overline{Y}^*_{n,b}}{\overline{X}^*_{n,b}}$$

BASIC BOOT. CONF. INT

$$\left(\hat{\theta}_n - \frac{\mu^*_{n,B}(1-\alpha/2)}{\sqrt{n}}, \ \hat{\theta}_n - \frac{\mu^*_{n,B}(\alpha/2)}{\sqrt{n}} \right),$$

WHERE $\mu^*_{n,B}(\alpha)$ IS „SAMPLE" α - QUANTILE FROM THE „BOOT. SAMPLE" $R^*_{n,1}, \dots, R^*_{n,B}$

PERCENTILE CONF. INT.

$$\left(q^*_{n,B}\left(\tfrac{\alpha}{2}\right),\ q^*_{n,B}(1-\alpha/2) \right),$$

WHERE $q^*_{n,B}(\alpha)$ IS THE „SAMPLE" α-QUANTILE FROM THE „BOOT. SAMPLE": $\hat{\vartheta}^*_{n,1}, \ldots, \hat{\vartheta}^*_{n,B}$.

STUDENT. BOOT. CONF. INT.

$$\tilde{R}^*_{n,b} = \frac{\sqrt{n}\left(\hat{\vartheta}^*_{n,b} - \hat{\vartheta}_n\right)}{\hat{\sigma}^*_{n,b}},$$

WHERE $\hat{\sigma}^{*2}_{n,b} = \frac{1}{\left(\overline{X}^*_{n,b}\right)^2} \frac{1}{n-1} \sum_{i=1}^{n} \left(Y^*_{i,b} - \hat{\vartheta}^*_{n,b} X^*_{i,b} \right)^2$

$$\left(\hat{\vartheta}_n - \frac{\tilde{m}^*_{n,B}(1-\alpha/2)\,\hat{\sigma}_n}{\sqrt{n}},\ \hat{\vartheta}_n - \frac{\tilde{m}^*_{n,B}(\alpha/2)\,\hat{\sigma}_n}{\sqrt{n}} \right),$$

WHERE $\tilde{m}^*_{n,B}(\alpha)$ IS THE „SAMPLE" α-QUANTILE FROM THE „BOOT. SAMPLE" $\tilde{R}^*_{n,1}, \ldots, \tilde{R}^*_{n,B}$.

(Ex 62) $X_1, \dots, X_n$ i.i.d $U(0, \theta)$

$$\hat{\theta}_n = \max_{1 \le i \le n} \{X_i\}$$

$$n(\hat{\theta}_n - \theta_X) \xrightarrow[n \to \infty]{d} Z, \text{ WHERE } P(Z \le \cancel{x}) \begin{cases} e^{\frac{x}{\theta_X}}, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

LET $R_n^* = n(\hat{\theta}_n^* - \hat{\theta}_n)$,

WHERE $\hat{\theta}_n^* = \max_{1 \le i \le n} \{X_{i,b}^*\}$

NONPARAM. BOOT:

$X_{1,b}^*, \dots, X_{n,b}^*$ i.i.d from $F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \le x\}$

PARAM. BOOT:

$X_{1,b}^*, \dots, X_{n,b}^*$ i.i.d from $U(0, \hat{\theta}_n)$

Ex.

$X_1, \dots, X_n$ i.i.d

$$F_X(x;\lambda) \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$H_0: \mathcal{L}(X_i) \in \{ Exp(\lambda) : \lambda > 0 \}$ $\quad H_1:$ not H_0

$$KS_n = \sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F(x; \hat{\lambda}_n) \right| ,$$

WHERE $\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \leq x\}$

$$F(x; \hat{\lambda}_n) = 1 - e^{-\hat{\lambda}_n x} , \quad \hat{\lambda}_n = \frac{1}{\overline{X}_n}$$

$$KS^*_{n,b} = \sup_{x \in \mathbb{R}} \left| \hat{F}^*_{n,b}(x) - F(x, \hat{\lambda}^*_{n,b}) \right|$$

$\hat{F}^*_{n,b}$: ecdf $X^*_{1,b}, \dots, X^*_{n,b}$ $\qquad \hat{\lambda}^*_{n,b} = \frac{1}{\overline{X}^*_{n,b}}$

P-VALUE:

$$\frac{1 + \sum_{b=1}^{B} \mathbb{1}\{KS^*_{n,b} \geq KS_n\}}{B+1}$$

$$CM_n = \frac{1}{n} \sum_{i=1}^{n} \left(\hat{F}_n(X_i) - F(X_i; \hat{\lambda}_n) \right)^2$$

$$AD_n = \frac{1}{n} \sum_{i=1}^{n} \frac{\left(\hat{F}_n(X_i) - F(X_i; \hat{\lambda}_n) \right)^2}{F(X_i; \hat{\lambda}_n)\left(1 - F(X_i; \hat{\lambda}_n)\right)}$$

AVERAGE MARK 7TH GRADE

Ex „COMPARING EXPACTATIONS IN TWO-SAMPLE PROBLEMS"

BOYS $X_1, \dots, X_{n_1}$ i.i.d
GIRLS $Y_1, \dots, Y_{n_2}$ i.i.d } INDEPENDENT

$H_0: EX_1 = EY_1$

$H_1: EX_1 \neq EY_1$

1. $T_n = \dfrac{\overline{X}_{n_1} - \overline{Y}_{n_2}}{\sqrt{S^2_{n_1,n_2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$, $S^2_{n_1,n_2} = \dfrac{1}{n_1+n_2-2}\left[(n_1-1)S_X^2 + (n_2-1)S_Y^2\right]$

2. $\tilde{T}_n = \dfrac{\overline{X}_{n_1} - \overline{Y}_{n_2}}{\sqrt{\frac{S_X^2}{n_1} + \frac{S_Y^2}{n_2}}}$

3. NONPARAMETRIC BOOTSTRAP

$$\tilde{T}^*_{n,b} = \frac{\overline{X}^*_{n_1,b} - \overline{Y}^*_{n_2,b}}{\sqrt{\frac{S^{2*}_{X,b}}{n_1} + \frac{S^{2*}_{Y,b}}{n_2}}}$$

$X^*_{1,b}, \dots, X^*_{n_1,b}$ SAMPLE WITH REPLACEMENT FROM $X_1 - \overline{X}_{n_1}, \dots, X_{n_1} - \overline{X}_{n_1}$

$Y^*_{1,b}, \dots, Y^*_{n_2,b}$ — " — — " — $Y_1 - \overline{Y}_{n_2}, \dots, Y_{n_2} - \overline{Y}_{n_2}$

4. PERMUTATION TEST

$n = n_1 + n_2$

- DENOTE: $\mathcal{Z} = (Z_1, \dots, Z_n)^T = (X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2})^T$

$\rightsquigarrow (Z^*_{1,b}, \dots, Z^*_{n,b})^T$ BE A RANDOM PERMUTATION OF $\mathcal{Z}$

$(X^*_{1,b}, \dots, X^*_{n_1,b}) = (Z^*_{1,b}, \dots, Z^*_{n_1,b})^T$

$(Y^*_{1,b}, \dots, Y^*_{n_2,b}) = (Z^*_{n_1+1,b}, \dots, Z^*_{n_1+n_2,b})^T$

Ex 70

TWO-SAMPLE KOLM.-SMIRNOV TEST

$$K_{n_1,n_2} = \sup_x \Big| \underbrace{F_{n_1}(x)}_{\text{ecdf from } X_1,\dots,X_{n_1}} - \underbrace{G_{n_2}(x)}_{\text{ecdf } Y_1,\dots,Y_{n_2}} \Big|$$

$$K^*_{n_1,n_2,b} = \sup_x \Big| \underbrace{F^*_{n_1,b}(x)}_{\text{ecdf } Z^*_{1,b},\dots,Z^*_{n_1,b}} - \underbrace{G^*_{n_2,b}(x)}_{\text{ecdf } Z^*_{n_1+1,b},\dots,Z^*_{n_1+n_2,b}} \Big|$$

P VALUE:
$$\frac{1 + \sum_{b=1}^{B} \mathbb{1}\{K^*_{n_1,n_2,b} \geq K_{n_1,n_2}\}}{B+1}$$

Ex 71

$\begin{pmatrix} Z_1 \\ I_1 \end{pmatrix}, \ldots, \begin{pmatrix} Z_N \\ I_N \end{pmatrix}$

$Z_N \in \{1, 2, \ldots, 6\}$, $I_N \in \{A, B, C\}$

$$n_{jk} = \sum_{i=1}^{N} \mathbb{1}\{Z_i = j, I_i = k\}$$

PERM. RESAMPLE:

$\begin{pmatrix} Z^*_{1,b} \\ I_1 \end{pmatrix}, \ldots, \begin{pmatrix} Z^*_{N,b} \\ I_N \end{pmatrix}$, WHERE

$(Z^*_{1,b}, \ldots, Z^*_{N,b})$ IS PERMUT. OF $(Z_1, \ldots, Z_N)$

$$X^2 = \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{\left(n_{jk} - \frac{n_{j+} n_{+k}}{N}\right)^2}{\frac{n_{j+} n_{+k}}{N}}$$

$$\sqrt{n}\left(\hat{\underset{\sim}{\beta}}_n(\tau)-\underset{\sim}{\beta}(\tau)\right) \xrightarrow[n\to\infty]{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \mathbb{V}\right), \quad \mathbb{V}=\left(\mathbb{E}\left[\underset{\sim}{X}\underset{\sim}{X}^T f_{Y|\underset{\sim}{X}}\left(F^{-1}_{Y|\underset{\sim}{X}}(\tau)\right)\right]\right)^{-1} \mathbb{E}\underset{\sim}{X}\underset{\sim}{X}^T \left(\mathbb{E}\left[\underset{\sim}{X}\underset{\sim}{X}^T f_{Y|\underset{\sim}{X}}\left(F^{-1}_{Y|\underset{\sim}{X}}(\tau)\right)\right]\right)^{-1}$$

QUANTILE REGRESSION AND BOOTSTRAP

$$F^{-1}_{Y|X}(\tau) = \beta_0(\tau) + \beta_1(\tau)\, X$$

FOOD EXPEND. INCOME

FOR SIMPLICITY ONLY $\tau = 0.25$ AND $\tau = 0.75$ CONSIDERED

NONPAR. BOOTSTRAP : $\begin{pmatrix} X^*_{1,b} \\ Y^*_{1,b}\end{pmatrix}, \dots, \begin{pmatrix} X^*_{n,b} \\ Y^*_{n,b}\end{pmatrix}$ FROM $\begin{pmatrix} X_1 \\ Y_1\end{pmatrix}, \dots, \begin{pmatrix} X_n \\ Y_n\end{pmatrix}$

SUMMARY(..., SE = "BOOT") :

$$\hat{\mathbb{V}}_{n,B} = \frac{1}{B-1}\sum_{b=1}^{B}\left(\hat{\underset{\sim}{\beta}}^*_{n,b} - \overline{\underset{\sim}{\beta}}^*_{n,B}\right)\left(\hat{\underset{\sim}{\beta}}^*_{n,b} - \overline{\underset{\sim}{\beta}}^*_{n,B}\right)^T,$$

WHERE $\overline{\underset{\sim}{\beta}}^*_{n,B} = \frac{1}{B}\sum_{b=1}^{B}\hat{\underset{\sim}{\beta}}^*_{n,b}$

AS. NORM. + BOOTSTRAP ESTIM. OF VARIANCE

$$\left(\hat{\beta}_1 \mp u_{1-\alpha/2}\sqrt{\hat{v}_{22,B}}\right),$$

WHERE $\hat{v}_{22,B}$ IS THE SECOND DIAGONAL ELEMENT OF $\hat{\mathbb{V}}_{n,B}$