95) $X \sim N(\mu, \sigma^2) \quad H_0: \mu = \mu_0 \qquad \tau = \mu, \ \ \psi = \sigma^2$ Pr 64

$$L(\mu,\sigma^2) = \frac{1}{(\sqrt{2\pi\sigma^2})^m} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\} \qquad \ell(\mu,\sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$U(\mu,\sigma^2) = \Bigg(\underbrace{\frac{\sum(x_i-\mu)}{\sigma^2}}_{U_1}, -\frac{m}{2\sigma^2} + \frac{\sum(x_i-\mu)^2}{2\sigma^4}\Bigg) \qquad \hat{\mu} = \bar{X} \quad \hat{\sigma}^2 = \frac{1}{m}\sum(x_i-\bar{x})^2$$

$$\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$$

• pre $\tau = c_0$: $\quad \tilde{\sigma}^2 = \frac{1}{m}\sum(X_i-\mu_0)^2 \qquad \tilde{\theta} = (\mu_0, \tilde{\sigma}^2)$

$$-E\frac{\partial U}{\partial \theta^T}(\mu,\sigma^2) = \begin{pmatrix} +\frac{m}{\sigma^2} & 0 \\ 0 & \frac{m}{2\sigma^4}\end{pmatrix} = J_m(\theta) \qquad J(\theta) = \left(\begin{array}{c|c} \frac{1}{\sigma^2} & 0 \\ \hline 0 & \frac{1}{2\sigma^4}\end{array}\right)$$

$$J^{-1}(\theta) = \left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & 2\sigma^4\end{array}\right) \qquad J^{11}(\theta) = \sigma^2$$

• $LR := \cancel{2}.\left[\cancel{c} - \frac{m}{\cancel{2}}\log\hat{\sigma}^2 - \cancel{\frac{1}{2\hat{\sigma}^2}\sum(X_i-\bar{X})^2} - \cancel{c} + \frac{m}{\cancel{2}}\log\tilde{\sigma}^2 + \cancel{\frac{1}{2\tilde{\sigma}^2}\sum(X_i-\mu_0)^2}\right]$

$$= m\log\frac{\tilde{\sigma}^2}{\hat{\sigma}^2} = m\log\frac{\sum(X_i-\mu_0)^2}{\sum(X_i-\bar{X})^2}$$

• $W = m(\bar{X}-\mu_0)^2(\hat{\sigma}^2)^{-1}$

• $R = \frac{1}{m}\left(\frac{\sum(X_i-\mu_0)}{\tilde{\sigma}^2}\right)^2\tilde{\sigma}^2 = m\frac{(\bar{X}-\mu_0)^2}{\tilde{\sigma}^2}$

Każdy test odrzucamy gdy $T_m > \chi^2_1(1-\alpha)$

96) $X \sim \log N(\mu,\sigma^2) \quad H_0: \mu = \mu_0 \qquad \tau = \mu, \ \ \psi = \sigma^2$ Pr 65

$$\ell(\mu,\sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(\log x_i - \mu)^2$$

$$U = \left(\sum(\log x_i-\mu)/\sigma^2, \ -m/2\sigma^2 + \sum(\log x_i-\mu)^2/2\sigma^4\right) \qquad \hat{\mu} = \frac{1}{m}\sum\log X_i \quad \hat{\sigma}^2 = \frac{1}{2}\sum(\log X_i - \hat{\mu})^2$$

$$\tilde{\sigma}^2 = \frac{1}{m}\sum(\log X_i - \mu_0)^2$$

$$J(\theta) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{pmatrix} \Rightarrow J^{11}(\theta) = \sigma^2$$

• $LR = \ldots = m\log\left[\frac{\sum(\log X_i-\mu_0)^2}{\sum(\log X_i - \hat{\mu})^2}\right]$

• $W = m(\hat{\mu}-\mu_0)^2(\hat{\sigma}^2)^{-1}$

• $R = \frac{1}{m}\left[\frac{\sum(\log X_i-\mu_0)}{\tilde{\sigma}^2}\right]^2 \cdot \tilde{\sigma}^2 = m\frac{(\hat{\mu}-\mu_0)^2}{\tilde{\sigma}^2}$

odrzucamy gdy $T_m > \chi^2_1(1-\alpha)$

97) $Y|N \sim Bi(\tilde m, p)$ pre $\tilde m = N$, $N \sim Po(\lambda)$ $\quad H_0: p = p_0 \quad \tau = p,\ \psi = \lambda$ Gn 70

$$L(p,\lambda) = \prod \binom{m_i}{y_i} p^{y_i}(1-p)^{m_i-y_i} \frac{\lambda^{m_i} e^{-\lambda}}{m_i!} = c\cdot p^{\sum y_i}(1-p)^{\sum m_i - \sum y_i}\lambda^{\sum m_i} e^{-m\lambda}$$

$$\ell(p,\lambda) = c + \sum y_i \log p + \left(\sum m_i - \sum y_i\right)\log(1-p) + \sum m_i \log\lambda - m\lambda$$

$$U(p,\lambda) = \left(\frac{\sum y_i}{p} - \frac{\sum m_i - \sum y_i}{1-p},\ \frac{\sum m_i}{\lambda} - m\right) \qquad \hat\lambda = \frac{\sum m_i}{m} \qquad \hat p = \frac{\sum y_i}{\sum m_i}$$

$\hat\lambda$ nezávisí na $p \Rightarrow \tilde\lambda = \hat\lambda$

$$J(p,\lambda) = \begin{pmatrix} \frac{\lambda}{p}+\frac{\lambda}{1-p} & 0 \\ 0 & \frac{1}{\lambda}\end{pmatrix} \qquad J^{11}(\theta) = \frac{1/1}{\frac{(1-p)\lambda + p\lambda}{p(1-p)}} = \frac{p(1-p)}{\lambda}$$

- $LR = 2\Big[\cancel{c} + \sum Y_i \log \hat p + \sum (N_i - Y_i)\log(1-\hat p) + \cancel{\sum N_i \log\hat\lambda} - \cancel{m\hat\lambda}$
$\quad - \cancel{c} - \sum Y_i \log p_0 - \sum(N_i - Y_i)\log(1-p_0) - \cancel{\sum N_i \log \hat\lambda} + \cancel{m\hat\lambda}\Big]$
$\quad = 2\left[\sum Y_i \log \hat p/p_0 + \sum(N_i - Y_i)\log[(1-\hat p)/(1-p_0)]\right]$

- $W = m(\hat p - p_0)^2 \left[\frac{\hat p(1-\hat p)}{\hat\lambda}\right]^{-1} = m\frac{(\hat p - p_0)^2}{\hat p(1-\hat p)}\cdot\hat\lambda$

- $R = \frac{1}{m}\left(\frac{\sum Y_i}{p_0} - \frac{\sum(N_i - Y_i)}{1-p_0}\right)^2 \frac{p_0(1-p_0)}{\hat\lambda} = \frac{1}{m}\left(\frac{\sum Y_i - \cancel{p_0\sum Y_i} - p_0\sum N_i + \cancel{p_0 \sum Y_i}}{p_0(1-p_0)}\right)^2\frac{p_0(1-p_0)}{\hat\lambda}$
$\quad = \frac{1}{m}\left(\sum N_i\left(\frac{\sum Y_i}{\sum N_i} - p_0\right)\right)^2 \frac{1}{\hat\lambda p_0(1-p_0)}$

zamietam ak $T_n > \chi^2_1(1-\alpha)$

98) $X \sim M(1, p_1, p_2, p_3, p_4)$ Pn 68

→ • štandardná parametrizácia $(p_1,p_2,p_3,p_4)^T = p \qquad y_j = \sum_{i=1}^m x_{ij}$ (počet j v súbore)

i) $L(p) = \prod p_j^{y_j} \qquad \ell(p) = \sum y_j \log p_j \qquad p_1 = \tau \quad (p_2,p_3,p_4) = \psi \quad H_0: p_1 = 1/4$

$\hat p_j = y_j/m$ pre $\tilde p$ maximalizuj $\sum y_j \log p_j$ na podm. $p_1 = 1/4$, $\sum_{j=2}^4 p_j = \frac{3}{4}$

tj $f(p_2,p_3,p_4,\lambda) = y_1 \log 1/4 + y_2\log p_2 + y_3 \log p_3 + y_4 \log p_4 + \lambda\left(\frac34 - \sum p_j\right)$

$\partial/\partial p_j\, f = y_j/p_j - \lambda = 0 \qquad \partial/\partial\lambda\, f = 3/4 - \sum p_j = 0$

$\Rightarrow p_j = y_j/\lambda \quad \to \quad 3/4 - 1/\lambda \sum y_j = 0 \Rightarrow \lambda = 4/3\sum y_j$

$\Rightarrow \tilde p_j = \frac{y_j}{\sum_{j=2}^4 y_j}\cdot\frac{3}{4} \qquad j = 2,3,4$

- $LR = 2\left(\sum_{j=1}^4 Y_j \log(Y_j/m) - Y_1 \log 1/4 - \sum_{j=2}^4 Y_j \log\left(\frac{Y_j \cdot \frac34}{\sum_{i=2}^4 Y_i}\right)\right)$

- Fisher. inf. nemôžeme počítať lebo nemáme ako zobrať do úvahy podm. $\sum_{j=1}^4 p_j = 1$

→ • parametrizácia $(p_1, p_2, p_3, 1-p_1-p_2-p_3) \qquad p = (p_1,p_2,p_3)^T$

$L(p) = \prod_{j=1}^3 p_j^{y_j}\cdot(1-p_1-p_2-p_3)^{m-y_1-y_2-y_3}$ Mathematica $\quad \hat p_j = y_j/m \qquad 1 - \sum_{j=1}^3 \hat p_j = y_4/m$

$\tilde p = \left(1/4,\ 3/4\,\frac{y_2}{m-y_1},\ 3/4\,\frac{y_3}{m-y_1}\right)^T \xRightarrow{\text{suma}} 1 - \sum_{j=1}^3 \tilde p_j = \frac{3/4\, Y_4}{\sum_{i=2}^4 Y_i}$

- $LR = 2\left(\sum_{j=1}^3 Y_j \log(Y_j/m) + Y_4\log(Y_4/m) - Y_1 \log 1/4 - \sum_{j=2}^3 Y_j\log\left(\frac{Y_j\, 3/4}{m-Y_1}\right) - Y_4 \log\left(\frac{3/4\, Y_4}{m-Y_1}\right)\right)$

99) z Pr 98 a R skriptu dostaneme $\hat{p} \doteq (0{,}240;\ 0{,}258;\ 0{,}267;\ 0{,}235)$

a) $LR = 60{,}08$ $\quad H_0: p_1 = 1/4 \quad \chi^2_1(1-\alpha) = 3{,}84 \quad$ p-val $= 9\cdot 10^{-15}$ $\Rightarrow$ zamietam

b) $H: p_1 = p_2 \quad LR = 76{,}41 \quad$ p-val $\doteq 2\cdot 10^{-18}$ $\Rightarrow$ zamietam

c) $H: p_3 = 1{,}1\, p_1 \quad LR = 1{,}39 \quad$ p-val $= 0{,}239$ $\Rightarrow$ nezamietam

100) $(X_i, Y_i)^T \sim$ náh. výber kde $Y_i \mid X_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ $\quad X_i$ nezáv. na β, σ^2
pre $\sigma = 1$ sme riešili v Pr 23

$H_0: \beta_1 = 0 \quad H_1: \beta_1 \neq 0$

$$L(\beta,\sigma^2) = c \cdot \prod \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right\} = c(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum (y_i - \beta_0 - \beta_1 x_i)^2\right\}$$

$$\ell(\beta,\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum (y_i - \beta_0 - \beta_1 x_i)^2 \overset{!}{=} 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \qquad \hat{\beta}_1 = \frac{\sum (Y_i - \bar{Y})(x_i - \bar{X})}{\sum (x_i - \bar{x})^2}$$

pre $\hat{\beta}_0$ a $\hat{\beta}_1$ v Pr 22 (Dupač, Hušková) Veta 7.1

na H_0: $\quad \ell(\beta_0, \sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i - \beta_0)^2$

$\Rightarrow$ štandardný normálny model $\Rightarrow$ $\tilde{\beta}_0 = \bar{Y} \quad \tilde{\sigma}^2 = \frac{1}{n}\sum (Y_i - \tilde{\beta}_0)^2 \quad \tilde{\beta}_1 = 0$

$$LR = 2\left[\cancel{c} - \frac{n}{2}\log\hat{\sigma}^2 - \underbrace{\frac{1}{2\hat{\sigma}^2}\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}_{n/2} - \cancel{c} + \frac{n}{2}\log\tilde{\sigma}^2 + \underbrace{\frac{1}{2\tilde{\sigma}^2}\sum (Y_i - \tilde{\beta}_0)^2}_{n/2}\right]$$

$$= \frac{2n}{2}\left[\log\left[\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right]\right]$$

$$= n \log \frac{\sum (Y_i - \tilde{\beta}_0)^2}{\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}$$

zamietam ak $LR > \chi^2_1(1-\alpha)$

101) náhodný výber (X_i, Y_i) ako v Pr 42 kde

$$P(Y=1 \mid X=x) = \frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}} \qquad P(Y=0 \mid X=x) = 1 - P(Y=1 \mid X=x)$$

$\hat{\alpha}, \hat{\beta}$ numericky v R skripte

$$\ell(\alpha,\beta) = \sum y_i(\alpha + \beta x_i) - \sum \log(1 + e^{\alpha+\beta x_i})$$

$$U(\alpha,\beta) = \begin{pmatrix} \sum y_i - \sum e^{\alpha+\beta x_i}/(1+e^{\alpha+\beta x_i}) \\ \sum x_i y_i - \sum x_i e^{\alpha+\beta x_i}/(1+e^{\alpha+\beta x_i}) \end{pmatrix} \Big\} U_2$$

• na $H_0: \beta = 0$ $\quad \ell(\alpha) = \sum y_i \alpha - \sum \log(1+e^{\alpha})$

$$U(\alpha) = \sum y_i - \sum \frac{e^{\alpha}}{1+e^{\alpha}} \overset{!}{=} 0 \Rightarrow \bar{Y} = e^{\alpha}/(1+e^{\alpha}) \qquad \tilde{\alpha} = \log\frac{\bar{Y}}{1-\bar{Y}} \quad \tilde{\beta} = 0$$

$$LR = 2\left[\sum Y_i(\hat{\alpha} + \hat{\beta} x_i) - \sum \log(1 + e^{\hat{\alpha}+\hat{\beta}x_i}) - \sum Y_i \tilde{\alpha} + \sum \log(1 + e^{\tilde{\alpha}})\right]$$

Informačná matica: z Pr 42 máme pre $X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$ a $W = \operatorname{diag}\Big(\underbrace{\frac{e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2}}_{w_i}\Big)$

že $\mathcal{J}_n(\alpha,\beta) = X'WX = \begin{pmatrix} \sum w_i & \sum w_i x_i \\ \sum w_i x_i & \sum w_i x_i^2 \end{pmatrix}$

$\tau = \beta$, $H_0: \beta = 0$ – pre $\mathcal{J}^{11}$ prvok $(2,2)$ matice $[\mathcal{J}_n(\alpha,\beta)/n]^{-1}$

$$R = \frac{1}{n}[U_2(\tilde{\alpha},0)]^2 \mathcal{J}^{11}(\tilde{\alpha},0) \qquad W = n\left((\hat{\alpha},\hat{\beta}) - (\hat{\alpha},0)\right)^2 / \mathcal{J}^{11}(\hat{\alpha},\hat{\beta})$$

šikmé porovnanie $\sim \chi^2_1(1-\alpha)$