72) cont. $J_m(\theta,\eta) = \begin{pmatrix} m/\theta^2 & 0 \\ 0 & m/\eta^2 \end{pmatrix}$

$$\sqrt{m}\left(\begin{pmatrix}\hat\theta \\ \hat\eta\end{pmatrix} - \begin{pmatrix}\theta \\ \eta\end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}\theta^2 & 0 \\ 0 & \eta^2\end{pmatrix}\right)$$

b) $\sqrt{m}(\hat\theta - \theta) \xrightarrow{\mathcal{D}} N(0, \theta^2)$

c) $\theta \in [\hat\theta \mp u_{1-\alpha/2}\, \hat\theta/\sqrt{m}]$

d) ak $\theta_0 \in CI$ pre θ nezamietam H_0, inak zamietam.

nové d) exponenciálna rodina, úplná postač. pre $\frac{1}{\theta}$ je $\sum \frac{y_i}{x_i}$, $E\,\frac{y_i}{x_i} = \theta$ nestr. odhad, fce úplnej postač., najlepší nestranný.

8. asymptotické testy bez rušivých parametrov.

83) $X \sim alt(p)$ z Pr. 50 vieme

$U_m(p) = \dfrac{\sum X_i}{p} - \dfrac{m - \sum X_i}{1-p}$ $\quad \hat p = \bar X \quad J_m(p) = \dfrac{m}{p(1-p)}$

- $W_m = m(\bar X - p_0)^2 \cdot \dfrac{1}{\hat p(1-\hat p)} = \left[\sqrt{m}\, \dfrac{\bar X - p_0}{(\bar X(1-\bar X))^{1/2}}\right]^2 \iff$ klas. asympt. test

- $R_m = \dfrac{\left(\frac{\sum X_i}{p_0} - \frac{m - \sum X_i}{1-p_0}\right)^2}{m/(p_0(1-p_0))} = \left[\sqrt{m}\sqrt{p_0(1-p_0)}\left(\dfrac{\bar X}{p_0} - \dfrac{1-\bar X}{1-p_0}\right)\right]^2 = \left[\dfrac{\sqrt{m}}{\sqrt{p_0(1-p_0)}}(\bar X - p_0)\right]^2$

$\iff$ Wilsonova metóda

- $LR_m = 2\left[\sum X_i \log \bar X + (m - \sum X_i)\log(1-\bar X) - \sum X_i \log p_0 - (m - \sum X_i)\log(1-p_0)\right]$
$= 2m\left[\bar X \log \dfrac{\bar X}{p_0} + (1-\bar X)\log\dfrac{1-\bar X}{1-p_0}\right]$

každý test zamietam ak $T_m > \chi^2_1(1-\alpha)$

84) $X \sim Po(\lambda)$ z Pr 51 vieme

$U_m(\lambda) = -m + \sum X_i/\lambda \quad \hat\lambda = \bar X \quad J(\lambda) = 1/\lambda$

- $W_m = m(\bar X - \lambda_0)^2 \frac{1}{\bar X} = \left(\sqrt{m}\, \dfrac{\bar X - \lambda_0}{\sqrt{\bar X}}\right)^2$
- $R_m = \left(-m + \sum X_i/\lambda_0\right)^2 / (m/\lambda_0) = \left(\sqrt{m}\, \dfrac{\bar X - \lambda_0}{\sqrt{\lambda_0}}\right)^2$
- $LR_m = 2\left(-m\bar X + \sum X_i \log \bar X + m\lambda_0 - \sum X_i \log \lambda_0\right) = 2m\left[\log\left(\frac{\bar X}{\lambda_0}\right)\cdot \bar X - (\bar X - \lambda)\right]$

každý test zamietame ak $T_m > \chi^2_1(1-\alpha)$

85) $\tilde X \sim Po(\lambda)$ | ale $X = \tilde X \mid \tilde X > 0$

$P(X = k) = P(\tilde X = k \mid \tilde X > 0) = P(\tilde X = k, \tilde X > 0)/P(\tilde X > 0) = \dfrac{e^{-\lambda}\lambda^k/k!}{1 - e^{-\lambda}} \quad k \geq 1$

$P(\tilde X > 0) = 1 - e^{-\lambda}\lambda^0/0! = 1 - e^{-\lambda}$

$$L(\lambda) = \prod \frac{e^{-\lambda}\lambda^{x_i}/x_i!}{(1-e^{-\lambda})} = \frac{e^{-m\lambda}\lambda^{\Sigma x_i}/\prod x_i!}{(1-e^{-\lambda})^m}$$

$$\ell(\lambda) = -m\lambda + \Sigma x_i \log\lambda - \Sigma \log x_i! - m\log(1-e^{-\lambda})$$

$$U(\lambda) = -m + \Sigma x_i/\lambda + - \frac{me^{-\lambda}}{1-e^{-\lambda}} \overset{!}{=} 0$$

$$\hookrightarrow \lambda \frac{e^{\lambda}}{e^{\lambda}-1} = \bar{X}$$

$$H_\ell(\lambda) = -\Sigma x_i/\lambda^2 - \frac{me^{\lambda}}{(e^{\lambda}-1)^2}$$

$$J_m(\lambda) = \frac{m}{\lambda(1-e^{-\lambda})} + \frac{me^{\lambda}}{(1-e^{\lambda})^2}$$

$$EX = \frac{1}{1-e^{-\lambda}}\sum_{x=1}^{\infty} x\, e^{-\lambda}\frac{\lambda^x}{x!} = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = \frac{\lambda}{1-e^{-\lambda}}$$

a) $\sqrt{m}(\hat\lambda - \lambda) \xrightarrow{\mathcal{D}} N\left(0, \left[\frac{1}{\lambda(1-e^{-\lambda})} + \frac{e^{\lambda}}{(1-e^{\lambda})^2}\right]^{-1}\right)$

b) numericky $\hat\lambda \approx 0{,}31$ $\quad \sqrt{m}(\hat\lambda-\lambda) \overset{as}{\approx} N(0, 0{,}033)$ Mathematica

c) $\theta = P(X=0) = e^{-\lambda}$ $\quad \hat\theta = e^{-\hat\lambda} \doteq 0{,}73$

interval Δ-metodou $g(t) = e^{-t}$ $\quad g'(\lambda) = -e^{-\lambda}$

$$\sqrt{m}(\hat\theta - \theta) \xrightarrow{d} N\left(0, e^{-2\lambda}\left[\frac{1}{\lambda(1-e^{-\lambda})} + \frac{e^{\lambda}}{(1-e^{-\lambda})^2}\right]^{-1}\right)$$

CI: $\theta \in [0{,}725;\ 0{,}735]$ $\quad$ pozorované $\theta = 0{,}75$.

d) $\hat\theta = 0{,}73$. , resp. θ v intervale hore v c)

86) $X \sim Exp(\lambda)$ $\quad$ z príkladu Cv 52

$U_m(\lambda) = m/\lambda - \Sigma X_i$ $\quad \hat\lambda = 1/\bar{X}$ $\quad J(\lambda) = 1/\lambda^2$

- $W_m = m\left(\frac{1}{\bar{X}} - \lambda_0\right)^2 \bar{X}^2$
- $R_m = (m/\lambda_0 - \Sigma X_i)^2/(m/\lambda_0^2)$
- $LR_m = 2m\left(-\log\bar{X} - (\bar{X})^{-1}\bar{X} - \log\lambda_0 + \bar{X}\lambda_0\right)$

87) $X \sim Ge(p)$ $\quad$ Cv 53

- $W_m = m\left(\frac{1}{1+\bar{X}} - p_0\right)^2\left(\frac{1}{\hat{p}^2} + \frac{1}{\hat{p}(1-\hat{p})}\right)$
- $R_m = \left(\frac{m}{p_0} - \frac{\Sigma X_i}{1-p_0}\right)^2 \Big/ \left(m\left(\frac{1}{p_0^2} + \frac{1}{p_0(1-p_0)}\right)\right)$
- $LR_m = 2m\left(\log\hat{p} + \bar{X}\log(1-\hat{p}) - \log p_0 - \bar{X}\log(1-p_0)\right)$

88) $\binom{X}{Y} \overset{iid}{\sim} f(x,y) = \beta x\, e^{-\beta x y}$ $\quad x \in (0,1),\ y > 0$

a) $L(\beta) = \beta^m \prod x_i \cdot e^{-\beta\Sigma x_i y_i}$

$$\ell(\beta) = m\log\beta + c - \beta\Sigma x_i y_i$$

$$U(\beta) = \frac{m}{\beta} - \Sigma x_i y_i = 0$$

$$\hat\beta = \underline{\underline{\frac{m}{\Sigma x_i y_i}}}$$

cont. es, $\frac{\partial U}{\partial \beta}(\beta) = -m/\beta^2$ $J(\beta) = 1/\beta^2$

- $W_m = (\hat\beta - \beta)^2 m / \hat\beta^2$
- $R_m = \left(\frac{m}{\beta_0} - \sum x_i y_i\right)^2 / (m/\beta_0^2)$
- $LR_m = m2\left(\log\hat\beta - \hat\beta \frac{1}{m}\sum x_i y_i - \log\beta_0 + \beta_0 \frac{1}{m}\sum x_i y_i\right)$

90) $X \sim \text{Logist}(\theta)$ R Pn 59.

esplicita ilm

- $R_m = \left(m - \sum_{i=1}^{m} \frac{2e^{-x_i}}{e^{-\theta_0} - e^{-x_i}}\right)^2 / \left(\frac{m}{3}\right)$

91) $X \sim N(\mu, \sigma^2)$ R Gn 64

- $W_m = \left(\left(\bar X, \frac{1}{m}\sum(x_i - \bar X)^2\right) - (\mu, \sigma^2)\right)\begin{pmatrix} m/\hat\sigma^2 & 0 \\ 0 & m/2\hat\sigma^4 \end{pmatrix}\begin{pmatrix} \bar X - \mu \\ \frac{1}{m}\sum(x_i - \bar X)^2 - \sigma^2 \end{pmatrix}$

$$= \frac{(\bar X - \mu)^2 m}{\hat\sigma^2} + \frac{(\hat\sigma^2 - \sigma^2) m}{2\hat\sigma^4}$$

- $R_m = \left(\frac{1}{\sigma_0^2}\sum(x_i - \mu_0), -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(x_i - \mu_0)^2\right)\begin{pmatrix} \sigma_0^2/m & 0 \\ 0 & 2\sigma_0^4/m \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_0^2}\sum(x_i - \mu_0) \\ -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(x_i - \mu_0)^2 \end{pmatrix}$

$$= \frac{\sigma_0^2}{m}\left(\sum(x_i - \mu_0)\right)^2 + \frac{2}{m}\left(-\frac{m}{2} + \frac{1}{2\sigma_0^2}\sum(x_i - \mu_0)^2\right)^2$$

- $LR_m = 2m\left(-\frac{\log\hat\sigma^2}{2} - \frac{1}{2\hat\sigma^2}\frac{1}{m}\sum(x_i - \bar X)^2 + \frac{\log\sigma_0^2}{2} + \frac{1}{2\sigma_0^2}\frac{1}{m}\sum(x_i - \mu_0)^2\right)$

paragoniamo a $\chi^2_2(1-\alpha)$

92) $X \sim \log N(\mu, \sigma^2)$ Gn 65

a) $\hat{\mu} = \frac{1}{m}\sum \log X_i \qquad \hat{\sigma}^2 = \frac{1}{m}\sum(\log X_i - \hat{\mu})^2$

b) $\sqrt{m}\left(\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}\right)$

c) $\left\{\mu, \sigma^2 : m(\hat{\mu}-\mu, \hat{\sigma}^2-\sigma^2)\begin{pmatrix}1/\hat{\sigma}^2 & 0\\ 0 & 1/2\hat{\sigma}^4\end{pmatrix}\begin{pmatrix}\hat{\mu}-\mu\\ \hat{\sigma}^2-\sigma^2\end{pmatrix} \leq \chi^2_2(1-\alpha)\right\}$

d) $(\mu, \sigma) = (0,1) : H_0$

- $W_m = m(\hat{\mu}-\mu_0, \hat{\sigma}^2-\sigma^2)\begin{pmatrix}1/\hat{\sigma}^2 & 0\\ 0 & 1/2\hat{\sigma}^4\end{pmatrix}\begin{pmatrix}\hat{\mu}-\mu_0\\ \hat{\sigma}^2-\sigma^2\end{pmatrix}$

- $R_m = \left(\sum \frac{\log X_i - \mu_0}{\sigma_0^2}, -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(\log X_i - \mu_0)^2\right)\begin{pmatrix}\sigma_0^2/m & 0\\ 0 & 2\sigma_0^4/m\end{pmatrix}\begin{pmatrix}\sum \log X_i\\ -\frac{m}{2} + \frac{1}{2}\sum(\log X_i)^2\end{pmatrix}$

$= \frac{(\sum \log X_i)^2}{m} + \frac{2}{m}\left(-\frac{m}{2} + \frac{1}{2}\sum(\log X_i)^2\right)^2$

- $LR_m = 2m\left(-\frac{\log \hat{\sigma}^2}{2} - \frac{1}{2\hat{\sigma}^2}\frac{1}{m}\sum(\log X_i - \hat{\mu})^2 + 0 + \frac{1}{2m}\sum(\log X_i)^2\right)$

porovnávam s $\chi^2_2(1-\alpha)$

e) $\sqrt{m}(\hat{\mu}-\mu) \xrightarrow{d} N(0, 2\sigma^4)$

zamietam $H_0: \mu = \mu_0$ ak $\mu_0 \notin \left[\hat{\mu} \mp u_{1-\alpha/2}\frac{\sqrt{2\hat{\sigma}^4}}{\sqrt{m}}\right]$

93) $Y|X \sim N(\beta_1 X + \beta_2 X^2, 1)$

$L(\beta) = \prod c \cdot \exp\left\{-\frac{1}{2}(y_i - \beta_1 x_i - \beta_2 x_i^2)^2\right\} = c \cdot \exp\left\{-\frac{1}{2}\sum(y_i - \beta_1 x_i - \beta_2 x_i^2)^2\right\}$

$\ell(\beta) = -\frac{1}{2}\sum(y_i - \beta_1 x_i - \beta_2 x_i^2)^2$

$U(\beta) = \left(\sum \frac{2}{2}(y_i - \beta_1 x_i - \beta_2 x_i^2)x_i, \sum x_i^2(y_i - \beta_1 x_i - \beta_2 x_i^2)\right)$

$U'(\beta) = \begin{pmatrix}-\sum x_i^2 & -\sum x_i^3\\ -\sum x_i^3 & -\sum x_i^4\end{pmatrix} \qquad J_m(\beta) = m\begin{pmatrix}EX^2 & EX^3\\ EX^3 & EX^4\end{pmatrix}$

$H_0: \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$R_m = \left(\cancel{\sum(Y_i - \beta_1 X_i - \beta_2 X_i^2)\cdot X_i}, \sum X_i^2 Y_i\right)\frac{1}{m}\cancel{\begin{pmatrix}EX^2 & EX^3\\ EX^3 & EX^4\end{pmatrix}}^{-1}\begin{pmatrix}\sum Y_i X_i\\ \sum Y_i X_i^2\end{pmatrix}$

$\sum_{i=1}^{m} Y_i X_i$

porovnávam s $\chi^2_2(1-\alpha)$

$\begin{pmatrix}\frac{1}{m}\sum_{i=1}^{m} X_i^2, & \frac{1}{m}\sum_{i=1}^{m} X_i^3\\ \frac{1}{m}\sum_{i=1}^{m} X_i^3, & \frac{1}{m}\sum_{i=1}^{m} X_i^4\end{pmatrix}^{-1}$