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6.1) $P(Y=k|X) = \frac{\lambda(x)^k e^{-\lambda(x)}}{k!}$ $\quad \lambda(x) = e^{\beta x}$, X nezávisí na β

i) $L(\beta) = \prod P(Y_i = y_i | X_i = x_i) \cdot P(X_i = x_i) \left(= \prod P(Y_i = y_i, X_i = x_i)\right)$

$= \prod_{i=1}^{m} \frac{(e^{\beta x_i})^{y_i} e^{-e^{\beta x_i}}}{y_i!} P(X_i = x_i)$

$\ell(\beta) = \sum \left(y_i x_i \beta - e^{\beta x_i} - \log(y_i!) + \log P(X_i = x_i)\right)$

$= \sum y_i x_i \cdot \beta - \sum e^{\beta x_i} + c$

$\ell'(\beta) = \sum x_i y_i - \sum e^{\beta x_i} \cdot x_i \overset{!}{=} 0$

$\ell''(\beta) = -\sum e^{\beta x_i} x_i^2 \qquad J_m(\beta) = m\, E\, e^{\beta X} X^2 \qquad \sqrt{m}(\hat\beta - \beta) \xrightarrow{\mathcal{D}} N\left(0, \frac{1}{E e^{\beta X} X^2}\right)$

6.2) $P(X_i = x) = \begin{cases} p & x \in \{-1, 1\} \\ 1-2p & x = 0 \end{cases}$ $\qquad$ def: $y := \#\{X_i = 0\}$

$L(p) = (1-2p)^y p^{m-y} \qquad \ell(p) = y \log(1-2p) + (m-y)\log p$

$\ell'(p) = \frac{-2y}{1-2p} + \frac{m-y}{p} = 0$

$-\cancel{2py} + m - 2mp - y + \cancel{2py} = 0$

$\hat p = (y-m)/(-2m) = (m-y)/2m = \sum I[X_i \neq 0]/2m$

$\ell''(p) = \frac{-2y \cdot 2}{(1-2p)^2} - \frac{m-y}{p^2} \qquad EY = E \sum I[X=0] = m P(X=0) = m(1-2p)$

$J_m(p) = \frac{4m\cancel{(1-2p)}}{(1-2p)^{\cancel{2}}} + \frac{m - m(1-2p)}{p^2} = \frac{4m}{1-2p} + \frac{2m}{p} =$

$= \frac{4mp + 2m - 4mp}{(1-2p)p} = m \cdot \frac{2}{p(1-2p)}$

$\sqrt{m}(\hat p - p) \xrightarrow{\mathcal{D}} N(0, p(1-2p)/2)$

6.3) $L(\theta) = \prod \frac{1}{2} \exp(-|x_i - \theta|) = \frac{1}{2^m} \exp\left\{-\sum |x_i - \theta|\right\}$

$\ell(\theta) = c - \sum_{i=1}^{m} |x_i - \theta| \implies$ maximalizovať $-\sum |x_i - \theta|$ vo θ

riešením je medián $\{x_i\}$. (riešenie online) Lemma 3.4 MSI

MLE: Vektorový parameter

6.4) $X \sim N(\mu, \sigma^2)$

i) $L(\mu, \sigma^2) = c \cdot (\sigma^m)^{-1} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$

$\ell(\mu, \sigma^2) = c - \frac{m}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$

$\nabla \ell = \left(\frac{1}{\sigma^2} \sum (x_i - \mu) \; ; \; -\frac{m}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (x_i - \mu)^2 \right)$

$\frac{\partial \ell}{\partial \mu} = 0 \implies \sum x_i = m\mu, \; \hat\mu = \overline{X} \qquad \frac{\partial \ell}{\partial \sigma^2} = 0 \implies m = \frac{\sum (x_i - \mu)^2}{\sigma^2}, \; \hat\sigma^2 = \frac{1}{m} \sum (x_i - \mu)^2 \Big|_{\mu = \hat\mu}$

ii) $H_\ell(\mu, \sigma^2) = \begin{pmatrix} -\frac{m}{\sigma^2} & -\frac{\sum(x_i - \mu)}{\sigma^4} \\ -\frac{\sum(x_i - \mu)}{\sigma^4} & \frac{m}{2\sigma^4} - \frac{\sum (x_i - \mu)^2}{(\sigma^2)^3} \end{pmatrix}$

$J_m(\mu, \sigma^2) = \begin{pmatrix} m/\sigma^2 & 0 \\ 0 & \frac{m}{2\sigma^4} \end{pmatrix} \qquad \sqrt{m}\left(\begin{pmatrix} \overline{X} \\ \frac{m-1}{m} S^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right)$

iii) IS pre μ: $\left[\ \bar X \mp \dfrac{u_{1-\alpha/2}\sqrt{\frac{n-1}{n}S^2}}{\sqrt n}\ \right]$ porovnaj $\left[\ \bar X \mp t_{n-1}(1-\alpha/2)\dfrac{S}{\sqrt n}\ \right]$

iv) $\hat\theta = \hat\mu + u_\alpha \hat\sigma$

$g(s,t) = s + u_\alpha\sqrt t \qquad \nabla g = \left(1, \dfrac{u_\alpha}{2\sqrt{\sigma^2}}\right) \qquad \nabla g\, J_n^{-1}\, \nabla g' = \dfrac1n\left(\sigma^2 + u_\alpha^2 2\sigma^4\right)$

$\sqrt n\left((\hat\theta) - \theta\right) \xrightarrow{D} N\left(0, \sigma^2\left(1 + \dfrac{u_\alpha^2}{2}\right)\right)$ $\qquad \dfrac1n\begin{pmatrix}1 & \dfrac{u_\alpha}{2\sigma}\end{pmatrix}\begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}\begin{pmatrix}1\\ \dfrac{u_\alpha}{2\sigma}\end{pmatrix}$

AC medián $\qquad = \dfrac1n\left(\sigma^2 + \dfrac{u_\alpha^2\sigma^2}{2}\right)$

65) $L(\mu,\sigma^2) = \sigma^{-n}(\prod x_i)^{-1} c \cdot \exp\left\{-\dfrac{1}{2\sigma^2}\sum(\log x_i - \mu)^2\right\}$

i) $\ell(\mu,\sigma^2) = -\dfrac n2 \log\sigma^2 + c - \dfrac{1}{2\sigma^2}\sum(\log x_i - \mu)^2$

$\nabla\ell = \left(\dfrac{\sum(\log x_i - \mu)}{\sigma^2}, -\dfrac{n}{2\sigma^2} + \dfrac{1}{2\sigma^4}\sum(\log x_i-\mu)^2\right)$

$\nabla\ell = 0 \Rightarrow \hat\mu = \dfrac1n\sum\log x_i \qquad \hat\sigma^2 = \dfrac1n\sum(\log x_i - \hat\mu)^2$

ii) $H_\ell = \begin{pmatrix} -n/\sigma^2 & -\dfrac{\sum(\log x_i-\mu)}{\sigma^4} \\ -\dfrac{\sum(\log x_i - \mu)}{\sigma^4} & \dfrac{n}{2\sigma^4} - \dfrac{1}{\sigma^6}\sum(\log x_i-\mu)^2\end{pmatrix}$ $\qquad \log X \sim N(\mu,\sigma^2)$

$J_n = \begin{pmatrix} +n/\sigma^2 & 0 \\ 0 & +\dfrac{n}{2\sigma^4}\end{pmatrix}$ $\qquad \sqrt n\left(\begin{pmatrix}\hat\mu\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}\right)$

iii) $\left\{\begin{pmatrix}\mu\\ \sigma^2\end{pmatrix} : n\left(\begin{pmatrix}\hat\mu\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right)^T \begin{pmatrix}1/\hat\sigma^2 & 0\\ 0 & 1/2\hat\sigma^4\end{pmatrix}\left(\begin{pmatrix}\hat\mu\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \le \chi^2_2(1-\alpha)\right\}$

tj. $n\left[\dfrac{(\hat\mu-\mu)^2}{\hat\sigma^2} + \dfrac{(\hat\sigma^2 - \sigma^2)^2}{2\hat\sigma^4}\right] \le \chi^2_2(1-\alpha)$

iv) $\left[\hat\mu - \dfrac{u_{1-\alpha}\hat\sigma}{\sqrt n}, \infty\right)$ dolný odhad

66) $X \sim \lambda e^{-\lambda(x-\delta)}$; $x > \delta$

i) $L(\lambda,\delta) = \lambda^n \exp\left\{-\lambda\sum(x_i - \delta)\right\} I[\min x_i > \delta]$

$\ell(\lambda,\delta) = n\log\lambda - \lambda\sum(x_i-\delta) + \log I[\min x_i > \delta]$

$= n\log\lambda - \lambda\sum x_i + n\lambda\delta$ ak $\min x_i > \delta$

pre pevné $\lambda > 0$ maximalizuje $\hat\delta = \min X_i$

pre dané δ $\dfrac{\partial\ell(\lambda,\delta)}{\partial\lambda} = \dfrac n\lambda - \sum x_i + n\delta = 0 \Rightarrow \hat\lambda = \dfrac{1}{\bar X - \hat\delta}$

ii) $\hat\lambda \xrightarrow{P} \lambda$ z pr 51 (52) (pr posunutí a Slutsky vete)

$\hat\delta$ sa správa ako $Exp(n\lambda)$ posunuté o $\delta \Rightarrow \hat\delta \xrightarrow{P} \delta$ $\Big\}\ \begin{pmatrix}\hat\lambda\\ \hat\delta\end{pmatrix} \xrightarrow{P} \begin{pmatrix}\lambda\\ \delta\end{pmatrix}$

iii) $n \cdot Exp(n\lambda) \sim Exp(\lambda)$, tj. $n\hat\delta \sim Exp(\lambda) + \delta n$

$P(n(\hat\delta - \delta) \le x) \longrightarrow (=) 1 - e^{-\lambda x}$

tj. $(\sqrt n)^2(\hat\delta - \delta) \xrightarrow{D} Exp(\lambda)$

67) $X \sim R(a,b) \qquad L(a,b) = \left[\dfrac{1}{b-a}\right]^n I[a < \min x_i \le \max x_i < b]$

i) L je maximalizované ak $b-a$ je minimalizované tak, že $a < \min X_i < \max X_i < b$

$\begin{pmatrix}\hat a\\ \hat b\end{pmatrix} = \begin{pmatrix}\min X_i\\ \max X_i\end{pmatrix}$. $\qquad$ ii) z MSI: $\left.\begin{matrix}\hat a \xrightarrow{P} a\\ \hat b \xrightarrow{P} b\end{matrix}\right\} \Rightarrow \begin{pmatrix}\hat a\\ \hat b\end{pmatrix} \xrightarrow{P} \begin{pmatrix}a\\ b\end{pmatrix}$

iii) $P(\hat b \le x) = \left(\dfrac{x-a}{b-a}\right)^n \quad x \in [a,b]$

1 ak $x \ge 0$

$P(n(\hat b - b) \le x) = P(\hat b \le b + x/n) = \left(\dfrac{b + x/n - a}{b-a}\right)^n = \left(1 + \dfrac{x/(b-a)}{n}\right)^n \xrightarrow[n\to\infty]{} e^{x/(b-a)}$ pre $x < 0$

$n(b - \hat b) \xrightarrow{D} Exp\left(\dfrac{1}{b-a}\right)$

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68) $X \sim M(1, p_1 \dots p_k)$ $\quad L(p) = \prod p_j^{y_j} I[\sum p_j = 1]$ $\quad y_j = \#\{X_i : X_i = (0..0,\overset{j\text{-tý el.}}{1},0..0)\}$

i) Lagrangeove multiplikátory:

maximalizujeme $\ell(p) = \sum y_j \log p_j$ za podmienky $\sum p_i = 1$

tj. $\sum y_j \log p_j + \lambda(1-\sum p_j) = f(p,\lambda)$

$$\frac{\partial}{\partial p_i} f(p,\lambda) = \frac{y_i}{p_i} - \lambda = 0 \Rightarrow p_i = y_i/\lambda$$

$$\frac{\partial}{\partial \lambda} f(p,\lambda) = 1 - \sum p_j = 0 \Rightarrow \sum \frac{y_i}{\lambda} = 1 \Rightarrow \hat\lambda = \sum y_i \Rightarrow \hat p_i = \frac{y_i}{\sum y_i} = \frac{\sum_{i=1}^m [X_{ij}=1]}{m}$$

ii) definujeme $\hat p = \frac{1}{m}\sum_{i=1}^m \left(I[X_{i1}=1], I[X_{i2}=1], \dots, I[X_{ik}=1]\right)' = \frac{1}{m}\sum X_i$

ide o priemer iid náhodných vektorov $\quad E\, I[X_{ij}=1] = P(X_i = (0..0,\overset{j\text{-tý el.}}{1},0..0)) = p_j$

$E\hat p = p$ $\quad \operatorname{var} I[X_{ij}=1] = p_j - p_j^2$

$\operatorname{cov} I[X_{ij}=1]\, I[X_{ij'}=1] = 0 - p_j p_{j'}$

$$\operatorname{var} \hat p = \frac{1}{m}\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_k \\ -p_1p_2 & p_2(1-p_2) & \dots & -p_2p_k \\ & & \ddots & \\ -p_1p_k & -p_2p_k & \dots & p_k(1-p_k)\end{pmatrix} =: \frac{1}{m}\Sigma(p)$$

CLV: $\sqrt{m}(\hat p - p) \xrightarrow{D} N_k(0, \Sigma(p))$

69) ako v pr 68: $X \sim M(1, p_1 \dots p_4)$ $\quad X \sim (CHCH, DD, CHD, DCH)^T$

$= M(1, p, q, \frac{1-p-q}{2}, \frac{1-p-q}{2})$

i) $\hat p = \frac{\#\{CHCH\}}{m} = 604/1987$ $\quad \hat q = \frac{\#\{D,D\}}{m} = 609/1987$

$\theta(p,q) = \frac{2p}{1+p-q}$ $\quad \hat\theta = \frac{2\hat p}{1+\hat p - \hat q}$ $\quad g(s,t) = \frac{2s}{1+s-t}$ $\quad \nabla g = \left(\frac{2(1-t)}{(1+s-t)^2}, \frac{2s}{(1+s-t)^2}\right)'$

$\nabla g((p,q)) = \left(\frac{2(1-q)}{1+p-q}, \frac{2p}{(1+p-q)^2}\right)'$ $\quad \sqrt{m}\left(\begin{pmatrix}\hat p\\ \hat q\end{pmatrix} - \begin{pmatrix}p\\ q\end{pmatrix}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix} p(1-p) & -pq \\ -pq & q(1-q)\end{pmatrix}\right)$

Δ-veta: $\sqrt{m}(\hat\theta - \theta) \xrightarrow[\text{metóda}]{D} N\left(0, \frac{4p(1-q)(1-p-q)}{(1+p-q)^4}\right)$

ii) IS: $\left[\frac{2\hat p}{1+\hat p - \hat q} \mp \frac{u_{1-\alpha/2}}{\sqrt{m}} \sqrt{\frac{4\hat p(1-\hat q)(1-\hat p - \hat q)}{(1+\hat p + \hat q)^4}}\right] = [0{,}58;\ 0{,}63]$ $\quad \hat\theta = 0{,}609$

70) $P(Y=i, N=j) = P(Y=i \mid N=j)P(N=j) \Rightarrow Y|N \sim Bi(N,p)$ $\quad N \sim Po(\lambda)$

i) $L(p,\lambda) = \prod P(Y_i = y_i, N_i = n_i) = \prod P(Y_i = y_i \mid N_i = n_i) \cdot \prod P(N_i = n_i)$

$\ell(p,\lambda) = \sum \log P_p(Y_i = y_i \mid N_i = n_i) + \sum \log P_\lambda(N_i = n_i)$

$\frac{\partial \ell}{\partial p}$ = "vierohodnosť z $Bi(N_i,p)$", tj. $\frac{\sum y_i}{p} - \frac{\sum(n_i - y_i)}{1-p} = 0 \Rightarrow \hat p = \left(\frac{\sum n_i}{\sum y_i}\right)^{-1}$

$\frac{\partial \ell}{\partial \lambda}$ = "vierohodnosť z $Po(\lambda)$", tj. $\hat\lambda = \frac{1}{m}\sum n_i$

ii) H = ako vždy, derivujeme zvlášť p a λ

$$H = \begin{pmatrix} -\frac{\sum y_i}{p^2} - \frac{\sum(n_i - y_i)}{(1-p)^2} & 0 \\ 0 & -\frac{\sum n_i}{\lambda^2}\end{pmatrix} \qquad J_m = \begin{pmatrix} \frac{m\lambda}{p} + \frac{m\lambda}{1-p} & 0 \\ 0 & m/\lambda\end{pmatrix}$$

$EY = E\,E[Y|N] = E\,Bi(N,p) = EN\,p = \lambda p$

$E(N-Y) = E\,E[N-Y|N] = E[N - Bi(N,p)] = \lambda(1-p)$

$EN = \lambda$

$$\sqrt{m}\left(\begin{pmatrix}\hat p\\ \hat\lambda\end{pmatrix} - \begin{pmatrix}p\\ \lambda\end{pmatrix}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix}\frac{p(1-p)}{\lambda} & 0 \\ 0 & \lambda\end{pmatrix}\right)$$

72) $P(Y=1 \mid x) = \dfrac{e^{\beta' x}}{1+e^{\beta' x}}$ $\quad P(Y=0 \mid x) = 1 - P(Y=1 \mid x)$

$S(x,\beta) := e^{\beta' x} / (1+e^{\beta' x})$

$L(\beta) = \prod S(x_i,\beta)^{y_i} (1-S(x_i,\beta))^{1-y_i} = e^{\sum y_i \beta' x_i} / \prod (1+e^{\beta' x_i})$

$\ell(\beta) = \sum y_i \beta' x_i - \sum \log(1+e^{\beta' x_i})$

$\nabla \ell(\beta) = \sum y_i x_i - \sum \dfrac{x_i e^{\beta' x_i}}{1+e^{\beta' x_i}} = \sum x_i (y_i - S(x_i,\beta))$

$H_\ell(\beta) = -\sum x_i \left(\dfrac{x_i' e^{\beta' x_i}(1+e^{\beta' x_i}) - e^{\beta' x_i} e^{\beta' x_i} x_i'}{(1+e^{\beta' x_i})^2} \right) = -\sum x_i x_i' \dfrac{e^{\beta' x_i}}{1+e^{\beta' x_i}} \dfrac{1}{1+e^{\beta' x_i}} =$

$= -\sum x_i x_i' S(x_i,\beta)(1-S(x_i,\beta)) = -X'WX$ pre $X = \begin{pmatrix} x_1' \\ \vdots \\ x_m' \end{pmatrix}$

$W := \mathrm{diag}\left(S(x_1,\beta)(1-S(x_1,\beta)), \dots, S(x_m,\beta)(1-S(x_m,\beta))\right)$

a) $\sqrt{m}(\hat{\beta} - \beta) \xrightarrow{d} N_p(0, [E(X'WX)]^{-1} m)$

b) $\beta_1 \in [\hat{\beta}_1 \mp u_{1-\alpha/2} (E X'WX)^{-1}_{11}]$.

73) $X \sim Exp(1/\eta)$ $\quad Y \mid X \sim Exp(1/x\theta)$

a) $L(\theta,\eta) = (\prod x_i)^{-1} \theta^{-m} \eta^{-m} \exp\left\{ -\sum \dfrac{y_i}{x_i \theta} - \sum \dfrac{x_i}{\eta} \right\}$

$\ell(\theta,\eta) = c - m \log \theta - m \log \eta - \sum \dfrac{y_i}{x_i \theta} - \sum \dfrac{x_i}{\eta}$

$\nabla \ell(\theta,\eta) = \left(-\dfrac{m}{\theta} + \sum \dfrac{y_i}{x_i \theta^2} \, ; \, -\dfrac{m}{\eta} + \sum \dfrac{x_i}{\eta^2} \right) \stackrel{!}{=} 0 \Rightarrow \hat{\eta} = \bar{X} \quad \hat{\theta} = \dfrac{1}{m} \sum \dfrac{Y_i}{X_i}$

$H_\ell(\theta,\eta) = \begin{pmatrix} \dfrac{m}{\theta^2} - \sum \dfrac{2 y_i}{x_i \theta^3} & , \; 0 \\ 0 & , \; \dfrac{m}{\eta^2} - \sum \dfrac{2 x_i}{\eta^3} \end{pmatrix}$

$EX = \eta \quad E\left[\dfrac{Y}{X}\right] = E\left[E\left[\dfrac{Y}{X} \,\middle|\, X\right]\right] = E\left[\dfrac{1}{X} E[Y \mid X]\right] = E\left[\dfrac{X\theta}{X}\right] = \theta$