NMSA332 Metoda maximální věrohodnosti

50) $X \sim alt(p)$

i) $L(p) = \prod p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$

$\ell(p) = \sum x_i \log p + (n - \sum x_i)\log(1-p)$

$\ell'(p) = \sum x_i / p + (n - \sum x_i)/(1-p) \stackrel{!}{=} 0$ $\qquad \ell''(p) = -\frac{\sum x_i}{p^2} - \frac{(n-\sum x_i)}{(1-p)^2}$

$(1-p)\sum x_i - (n - \sum x_i)p = 0$

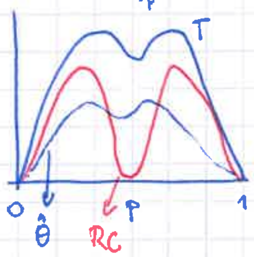
$p(-\sum x_i - n + \sum x_i) = -\sum x_i$ $\qquad J_n(p) = \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}$

$\hat{p} = \overline{X}$

$\sqrt{n}(\overline{X} - p) \xrightarrow{D} N(0, p(1-p))$

ii) invariance: $\theta = p(1-p)$ $\quad \hat{\theta} = \hat{p}(1-\hat{p}) = \overline{X}(1-\overline{X})$ $\quad g(t) = t(1-t)$ $\quad g'(t) = (1-t) - t = 1-2t$

$\sqrt{n}(\overline{X}(1-\overline{X}) - \theta) \xrightarrow{D} N(0, (1-2p)^2 p(1-p))$ $\quad p \neq 1/2$



iii) $\overline{X}$ je NNO, je eficientní; NNO θ je $T = \frac{n}{n-1}\overline{X}(1-\overline{X})$, není je eficientní; $\hat{\theta}$ není je nestranný

RC-MEZ PRO p: $\frac{p(1-p)}{n} = var(\overline{X}_n) = var(\overline{X})$

RC MEZ PRO $\theta = p(1-p)$: $\frac{(1-2p)^2 p(1-p)}{n}$

51) $X \sim Po(\lambda)$ $\quad L(\lambda) = e^{-n\lambda}\lambda^{\sum x_i}/\prod x_i!$ $\quad \ell(\lambda) = -n\lambda + \sum x_i \log\lambda + c$

i) $\ell'(\lambda) = -n + \sum x_i/\lambda = 0$ $\quad \hat{\lambda} = \overline{X}$

$\ell''(\lambda) = -\sum x_i/\lambda^2$ $\quad J_n(\lambda) = n/\lambda$ $\quad \sqrt{n}(\overline{X} - \lambda) \xrightarrow{D} N(0, \lambda)$

ii) $\theta = e^{-\lambda}$ $\quad g(t) = e^{-t}$ $\quad g'(t) = -e^{-t}$ $\quad \sqrt{n}(e^{-\overline{X}} - e^{-\lambda}) \xrightarrow{D} N(0, e^{-2\lambda}\cdot\lambda)$

iii) $\overline{X}$ je NNO, je eficientní λ; NNO θ je $T = \left(1 - \frac{1}{n}\right)^{\sum x_i}$, není je eficientní; RC MEZ PRO $\theta = e^{-\lambda}$: $\frac{\lambda e^{-2\lambda}}{n}$

RC MEZ PRO λ: $\frac{\lambda}{n}$

52) $X \sim Exp(\lambda)$ $\quad L(\lambda) = \lambda^n e^{-\lambda\sum x_i}$ $\quad \ell(\lambda) = n\log\lambda - \lambda\sum x_i$ $\quad \ell'(\lambda) = n/\lambda - \sum x_i = 0$

i) $\hat{\lambda} = 1/\overline{X}$ $\quad \ell''(\lambda) = -n/\lambda^2$ $\quad J_n(\lambda) = n/\lambda^2$

ii) $\sqrt{n}(1/\overline{X} - \lambda) \xrightarrow{D} N(0, \lambda^2)$

iii) $\frac{n-1}{\sum X_i}$ je NNO λ ale není je eficientní

53) $X \sim Ge(p)$ $\quad L(p) = p^n(1-p)^{\sum x_i}$ $\quad \ell(p) = n\log p + \sum x_i \log(1-p)$

i) $\ell'(p) = n/p - \sum x_i/(1-p) = 0$ $\quad \ell''(p) = -n/p^2 - \sum x_i/(1-p)^2$

$n - np - p\sum x_i = 0$ $\quad J_n(p) = n/p^2 + \frac{\cancel{(1-p)}}{p}\frac{n}{(1-p)^{\cancel{2}}} = n\left(\frac{1}{p^2} + \frac{1}{p(1-p)}\right)$

$p(n + \sum x_i) = n$

$\hat{p} = \frac{n}{n + \sum x_i} = \frac{1}{1+\overline{X}}$ $\quad \sqrt{n}\left(\frac{1}{1+\overline{X}} - p\right) \xrightarrow{D} N(0, p^2(1-p))$

ii) $\theta = p(1-p)$ $\quad \hat{\theta} = \frac{1}{1+\overline{X}}\left(1 - \frac{1}{1+\overline{X}}\right) = \frac{1\cdot\overline{X}}{(1+\overline{X})^2}$

$g(t) = t(1-t)$ $\quad g'(t) = 1-2t$ $\quad \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, (1-2p)^2 p^2(1-p))$

54) $X \sim R(\theta - 1/2, \theta + 1/2)$ $\quad L(\theta) = \prod I[\theta - 1/2 < x_i < \theta + 1/2] = I[\theta - 1/2 < \min x_i]\, I[\max x_i < \theta + 1/2]$

$= \begin{cases} 1 & \text{až } \theta < \min X_i + 1/2 \text{ a } \theta > \max X_i - 1/2 \\ 0 & \text{jinak} \end{cases}$ $\Rightarrow$ $\hat{\theta}$ je kterákoli z hodnot $\left(\max X_i - \frac{1}{2}, \min X_i + \frac{1}{2}\right)$

i)

ii) R MSI: $\min X_i \xrightarrow{P} \theta - 1/2$ $\Rightarrow$ $\max X_i - 1/2 \xrightarrow{P} \theta$ $\Rightarrow$ každý odhad je taky konzistentní

$\max X_i \xrightarrow{P} \theta + 1/2$ $\quad \min X_i + 1/2 \xrightarrow{P} \theta$

55) $X \sim R\{1..M\}$ $\quad L(M) = \prod \frac{1}{M} I[x_i \in \{1..M\}] = \frac{1}{M^n} I[1 \le \min X_i \le \max X_i \le M] =$

i) $= \begin{cases} 1/M^n & \text{až } M \ge \max X_i \\ 0 & \text{jinak} \end{cases}$ $\Rightarrow$ $\hat{M} = \max X_i$

ii) $P(\hat{M} \le k) = P(X_i \le k)^n = \left(\frac{k}{M}\right)^n$ $\quad \cancel{f_{\hat{M}}(k) = n k^{n-1}/M^n}$ $= P(\hat{M} \le M - \lceil\varepsilon\rceil)$

$P(|\hat{M} - M| > \varepsilon) = P(\hat{M} < M - \varepsilon) \le P(\hat{M} \le M - \varepsilon) \stackrel{\le}{=} \left(\frac{M-\varepsilon}{M}\right)^n = \left(1 - \frac{\varepsilon}{M}\right)^n \xrightarrow[n\to\infty]{} 0$

až $\varepsilon \in (0, M)$

56) $Y_i \sim N(\theta x_i, 1)$ $\quad L(\theta) = c\exp\{-\frac{1}{2}\sum(y_i - \theta x_i)^2\}$ $\quad \ell(\theta) = c - \frac{1}{2}\sum(y_i - \theta x_i)^2$ $\quad$ iii)

i) $\ell'(\theta) = \sum(y_i - \theta x_i)x_i = 0$ $\quad \ell'' = -\sum x_i^2$

$\sum y_i x_i = \theta\sum x_i^2$ $\quad$ ii) $E\hat{\theta} = \frac{1}{\sum x_i^2}\sum x_i EY_i = \theta$ nestranný $\quad RC_n = 1/\sum x_i^2$

$\frac{\sum y_i x_i}{\sum x_i^2} = \hat{\theta}$ $\quad var\,\hat{\theta} = \frac{\sum x_i^2 \cdot 1}{(\sum x_i^2)^2} = \frac{1}{\sum x_i^2} = RC$

57) $P(X=k) = \frac{1}{1-(1-p)^m}\binom{m}{k} p^k (1-p)^{m-k}$ $k=1,2..m$

i) $L(p) = \frac{1}{(1-(1-p)^m)^m}\prod_{i=1}^{m}\binom{m}{x_i} \cdot p^{\sum x_i}(1-p)^{\sum(m-x_i)}$

$\ell(p) = -m\log(1-(1-p)^m) + c + \sum x_i \log p + \sum(m-x_i)\log(1-p)$

$\ell'(p) = \frac{-m}{1-(1-p)^m}\cdot m(1-p)^{m-1} + \frac{\sum x_i}{p} - \frac{\sum(m-x_i)}{1-p} = 0$

$\ell''(p) =$ mathematica skript

58) $X\sim \theta x^{\theta-1}e^{-x^\theta} I[X>0]$ $L(\theta)=\theta^m\left(\prod x_i\right)^{\theta-1}e^{-\sum x_i^\theta} I[\min X_i>0]$

$x^\theta = e^{\theta\log x}$

i) $\ell(\theta)=m\log\theta+(\theta-1)\log\left(\prod x_i\right)-\sum x_i^\theta$

$\ell'(\theta)=\frac{m}{\theta}+\sum\log x_i - \sum x_i^\theta \log x_i = 0$

$\ell''(\theta) = -\frac{m}{\theta^2} - \sum x_i^\theta(\log x_i)^2$

} spojitá klesajúca fce, $\ell'(0)=\infty$, $\ell'(\infty)=-\infty$ jediné riešenie

ii) $EX = \int_0^\infty \theta x^\theta e^{-x^\theta}dx = \int_0^\infty t^{1/\theta}e^{-t}dt = \Gamma\left(\frac1\theta+1\right)$

$x^\theta=t$
$x=t^{1/\theta}$

$E X^\theta(\log X)^2 =$ mathematica $= c'$

$\sqrt{m}(\hat\theta-\theta)\to N\left(0,\left(\frac{1}{\theta^2}+c'\right)^{-1}\right)$

59) $f(x)=e^{-(x-\theta)}/(1+e^{-(x-\theta)})^2$ $L(\theta)=e^{-\sum x_i+m\theta}/\prod(1+e^{-(x_i-\theta)})^2$

$\ell(\theta) = -\sum x_i + m\theta - \sum\log(1+e^{-(x_i-\theta)})^2$

i) $\ell'(\theta)=m-2\sum\frac{1\cdot e^{-(x_i-\theta)}}{1+e^{-(x_i-\theta)}} = m-2\sum\frac{e^{-x_i}}{e^{-\theta}+e^{-x_i}}=0$

$\ell''(\theta) = +2\sum\frac{e^{-x_i}(-e^{-\theta})}{(e^{-\theta}+e^{-x_i})^2}$

} spojitá kles. fce $\ell'(-\infty)=m$, $\ell(\infty)=-m$ jediné riešenie

ii) $J_m(\theta)=\frac{m}{3}$

$E\frac{2e^{-X}e^{-\theta}}{(e^{-\theta}+e^{-X})^2} = \int_R \frac{2e^{-x}e^{-\theta}}{(e^{-\theta}+e^{-x})^2}\frac{e^{-x}e^{\theta}}{(1+e^{-(x-\theta)})^2}dx = \frac{2e^{-2\theta}}{e^{-2\theta}}\int_1^\infty\frac{(t-1)}{t^4} = 2\left[\frac{-1}{2t^2}+\frac{1}{3t^3}\right]_1^\infty = \frac13$

$1+e^{-(x-\theta)}=t$ $t-1=e^{-x}e^\theta$
$(-1)\cdot e^{-(x-\theta)}dx=dt$ $(t-1)e^{-\theta}=e^{-x}$
$-e^\theta e^{-x}dx=dt$

$\sqrt m(\hat\theta-\theta)\xrightarrow{D} N(0,3)$

60) $X\sim N(\theta,\theta^2)$ $L(\theta)=c\cdot\theta^{-m}\exp\left\{-\frac{1}{2\theta^2}\sum(x_i-\theta)^2\right\}$ $\ell(\theta)=-m\log\theta+c-\frac{1}{2\theta^2}\sum(x_i-\theta)^2$

i) $\ell'(\theta)=-\frac m\theta+\frac{1\cdot\cancel{2}}{\cancel{2}\theta^2}\sum(x_i-\theta)+\frac{1\cdot\cancel{2}}{\cancel{2}\theta^3}\sum(x_i-\theta)^2=0$

$-m+\frac{\sum x_i-m\theta}{\theta}+\frac{\sum x_i^2-2\theta\sum x_i+m\theta^2}{\theta^2}=0$

$-m+\frac{\sum x_i}{\theta}-\cancel{m}+\frac{\sum x_i^2}{\theta^2}-\frac{2\sum x_i}{\theta}+\cancel{m}=0$

$-m\theta^2+\theta\sum x_i-2\theta\sum x_i+\sum x_i^2=0$

$\theta^2 m+\theta\sum x_i-\sum x_i^2=0$

$\frac{-\sum x_i\pm\sqrt{(\sum x_i)^2+4\sum x_i^2\, m}}{2m}=\hat\theta$

ii) $\hat\theta = \left[-\frac12\bar X+\frac12\sqrt{(\bar X)^2+4\frac1m\sum X_i^2}\right] \xrightarrow[m\to\infty]{s.j.} -\frac\theta2+\frac12\sqrt{\theta^2+4\cdot2\theta^2} = -\frac\theta2+\frac12\sqrt{9\theta^2}=\theta$

$EX^2=2\theta^2$

iii) $\ell''(\theta)=\frac{m}{\theta^2}+\left(-\frac{2}{\theta^3}\sum(x_i-\theta)+\frac{1(-m)}{\theta^2}\right)+\left(\frac{-3}{\theta^4}\sum(x_i-\theta)^2-\frac{2\sum(x_i-\theta)}{\theta^3}\right)$

$J_m(\theta)=-\frac{m}{\theta^2}+\frac{m}{\theta^2}+\frac{3m}{\theta^2}=\frac{3m}{\theta^2}$ $\sqrt m(\hat\theta-\theta)\xrightarrow{D}N\left(0,\frac{\theta^2}{3}\right)$

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61) $P(Y=k|X) = \dfrac{\lambda(x)^k e^{-\lambda(x)}}{k!}$ $\quad \lambda(x) = e^{\beta x}$, X nezávisí na β

i) $L(\beta) = \prod P(Y_i = y_i | X_i = x_i) \cdot P(X_i = x_i) \left(= \prod P(Y_i = y_i, X_i = x_i)\right)$
$= \prod_{i=1}^{n} \dfrac{(e^{\beta x_i})^{y_i} e^{-e^{\beta x_i}}}{y_i!} P(X_i = x_i)$

$\ell(\beta) = \sum \left(y_i x_i \beta - e^{\beta x_i} - \log(y_i!) + \log P(X_i = x_i)\right)$

$= \sum y_i x_i \cdot \beta - \sum e^{\beta x_i} + c$

$\ell'(\beta) = \sum x_i y_i - \sum e^{\beta x_i} \cdot x_i \overset{!}{=} 0$

$\ell''(\beta) = -\sum e^{\beta x_i} x_i^2 \qquad J_n(\beta) = n \, E e^{\beta X} X^2 \qquad \sqrt{n}(\hat\beta - \beta) \xrightarrow{\mathcal{D}} N\left(0, \dfrac{1}{E e^{\beta X} X^2}\right)$

62) $P(X_i = x) = \begin{cases} p & x \in \{-1, 1\} \\ 1-2p & x = 0 \end{cases}$ $\quad$ dg: $y := \#\{X_i = 0\}$

$L(p) = (1-2p)^y p^{n-y} \qquad \ell(p) = y \log(1-2p) + (n-y)\log p$

$\ell'(p) = \dfrac{-2y}{1-2p} + \dfrac{n-y}{p} = 0$

$-\cancel{2py} + n - 2np - y + \cancel{2py} = 0$

$\hat p = (y-n)/(-2n) = (n-y)/2n = \sum I[X_i \neq 0]/2n$

$\ell''(p) = \dfrac{-2y \cdot 2}{(1-2p)^2} - \dfrac{n-y}{p^2}$ $\qquad EY = E\sum I[X=0] = n P(X=0) = n(1-2p)$

$J_n(p) = \dfrac{4n\cancel{(1-2p)}}{(1-2p)^{\cancel{2}}} + \dfrac{n - n(1-2p)}{p^2} = \dfrac{4n}{1-2p} + \dfrac{2n}{p} =$

$= \dfrac{4np + 2n - 4np}{(1-2p)p} = n \cdot \dfrac{2}{p(1-2p)}$

$\sqrt{n}(\hat p - p) \xrightarrow{\mathcal{D}} N(0, p(1-2p)/2)$

63) $L(\theta) = \prod \frac{1}{2} \exp(-|x_i - \theta|) = \dfrac{1}{2^n} \exp\left\{-\sum |x_i - \theta|\right\}$

$\ell(\theta) = c - \sum_{i=1}^{n} |x_i - \theta| \Rightarrow$ maximalizovať $-\sum|x_i - \theta|$ vo θ

riešením je medián $\{x_i\}$. (riešenie online) Lemma 3.4 MSI

MLE: Vektorový parameter

64) $X \sim N(\mu, \sigma^2)$

i) $L(\mu, \sigma^2) = c \cdot (\sigma^n)^{-1} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$

$\ell(\mu, \sigma^2) = c - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$

$\nabla \ell = \left(\frac{1}{\sigma^2} \sum (x_i - \mu) \; ; \; -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (x_i - \mu)^2 \right)$

$\dfrac{\partial \ell}{\partial \mu} = 0 \Rightarrow \sum x_i = n\mu, \; \hat\mu = \bar X \qquad \dfrac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow n = \dfrac{\sum (x_i - \mu)^2}{\sigma^2}, \; \hat\sigma^2 = \frac{1}{n} \sum (x_i - \mu)^2 \Big|_{\mu = \hat\mu}$

ii) $H_\ell(\mu, \sigma^2) = \begin{pmatrix} -\frac{n}{\sigma^2} & -\frac{\sum(x_i - \mu)}{\sigma^4} \\ -\frac{\sum(x_i - \mu)}{\sigma^4} & \frac{n}{2\sigma^4} - \frac{\sum (x_i - \mu)^2}{(\sigma^2)^3} \end{pmatrix}$

$J_n(\mu, \sigma^2) = \begin{pmatrix} n/\sigma^2 & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix} \qquad \sqrt{n}\left(\begin{pmatrix} \bar X \\ \frac{n-1}{n} S^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right)$