Ex. $X_1, \dots, X_n$ i.i.d $\sim f(x;\theta,\sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\theta|}{\sigma}}$

$$l_n(\theta,\sigma) = \log\left(\prod_{i=1}^{n} \frac{1}{2\sigma} e^{-\frac{|X_i-\theta|}{\sigma}}\right) =$$

$$= -n \log\sigma - \sum_{i=1}^{n} \frac{|X_i-\theta|}{\sigma} + c$$

$$\forall_{\sigma>0} \quad \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{n} \frac{|X_i-\theta|}{\sigma} = \tilde{m}_n$$

MEDIAN

$$\rightsquigarrow l_n(\tilde{m}_n,\sigma) = -n\log\sigma - \sum_{i=1}^{n} \frac{|X_i-\tilde{m}_n|}{\sigma}$$

$$\frac{\partial l_n(\tilde{m}_n,\sigma)}{\partial\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} |X_i-\tilde{m}_n|$$

$$\rightsquigarrow \tilde{\sigma}_n = \frac{1}{n} \sum_{i=1}^{n} |X_i-\tilde{m}_n|$$

Ex 40) $X_1, \dots, X_n$ i.i.d.

$$\hat{\theta}_n^{(H)} = \operatorname*{argmax}_{\theta \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \rho_H(X_i - \theta),$$

where $\rho_H(x) \begin{cases} x^2, & |x| \le k \\ k\left(|x| - \frac{k}{2}\right), & |x| > k \end{cases}$

$\rho_H(x)$ CONVEX $\rightsquigarrow$ $\underset{\rho'_H(x)}{\psi_H(x)} \begin{cases} x, & |x| \le k \\ k \operatorname{sign}(x), & |x| > k \end{cases}$

$\boxed{\text{TH } 10}$ $$\sqrt{n}\left(\hat{\theta}_n^{(H)} - \theta_H\right) \xrightarrow[n \to \infty]{d} N\left(0, \frac{\sigma_\psi^2}{\gamma^2}\right),$$

where $$\theta_H = \operatorname*{argmin}_{\theta \in \mathbb{R}} \mathbb{E}\, \rho_H(X_1 - \theta).$$

$\mathbb{E}\, \psi(X_1 - \theta_H) = 0$

Further:

$$\sigma_\psi^2 = \operatorname{var}\left(\psi(X_1 - \theta_H)\right) = \mathbb{E}\left[\psi(X_1 - \theta_H)\right]^2$$

$$= \int_{-\infty}^{\infty} \psi^2(x - \theta_H)\, dF(x)$$

$$= \int_{-\infty}^{\infty} \Big[(x - \theta_H)^2 \mathbb{1}\{|x - \theta_H| \le k\} + k^2 \underbrace{\operatorname{sign}^2(x - \theta_H)}_{=1} \mathbb{1}\{|x - \theta_H| > k\}\Big] dF(x)$$

$$= \int_{\theta_H - k}^{\theta_H + k} (x - \theta_H)^2\, dF(x) + \underbrace{k^2 P(|X_1 - \theta_H| > k)}_{k^2\left(F(\theta_H - k) + 1 - F(\theta_H + k)\right)}$$

$\boxed{\text{TH. 10}}$ $\quad \gamma = \left.\dfrac{\partial^2 M(\theta)}{\partial \theta^2}\right|_{\theta=\theta_M}$, $M(\theta) = \mathbb{E}\,\rho_H(X_1 - \theta)$

$$\frac{\partial M(\theta)}{\partial \theta} \underset{\substack{\uparrow \\ X_1 \text{ HAS A DENSITY WRT TO LEB. MEASURE}}}{=} - \mathbb{E}\,\psi_H(X_1 - \theta)$$

$$= - \mathbb{E}\Big[(X_1 - \theta)\,\mathbb{1}\{|X_1-\theta| \le k\} + k\,\mathrm{sign}(X_1-\theta)\,\mathbb{1}\{|X_1-\theta|>k\}\Big]$$

$$= - \int_{\theta-k}^{\theta+k} (x-\theta)\,dF(x) + k\,\underbrace{P(X_1 < \theta - k)}_{= F(\theta-k)} - k\,\underbrace{P(X_1 > \theta+k)}_{1 - F(\theta+k)}$$

$$\frac{\partial^2 M(\theta)}{\partial \theta^2} = -(\theta+k-\theta)\,f(\theta+k) + (\theta-k-\theta)\,f(\theta-k) + \int_{\theta-k}^{\theta+k} 1\,dF(x) + k\,f(\theta-k) + k\,f(\theta+k)$$

$$= F(\theta+k) - F(\theta-k)$$

$$\rightsquigarrow \gamma = F(\theta_M + k) - F(\theta_M - k)$$

LET X HAVE A DENSITY (WRT TO LEB. MEAS.)

$f(x;\theta) = f(x-\theta)$, WHERE f IS SYMMETRIC

$\rightsquigarrow \theta_M = \theta$

$$\sigma_Y^2 = \int_{\theta_M - k}^{\theta_M + k} (x-\theta_M)^2 f(x-\theta_M)\,dx + k^2\left(F(\theta_M - k) + 1 - F(\theta_M + k)\right)$$

$$= \int_{-k}^{k} x^2 f(x)\,dx + \underbrace{k^2\left(F(-k) + 1 - F(k)\right)}_{2k^2(1-F(k))}$$

$$\gamma = F(\theta_M + k) - F(\theta_M - k) = F(k) - F(-k) =$$

$$= 2F(k) - 1$$

NUMERICAL ILLUSTRATIONS IN R

1. ROBUST ESTIMATION OF LOCATION

(A) CONTAMINATED NORMAL DISTRIB.:

$0.9 \cdot$ "$N(0,1)$" $+ \; 0.1 \cdot$ "$N(0,25)$"

huber(..., k = 1.5)

$$\sum_{i=1}^{n} \psi\left(\frac{X_i - \hat{\theta}_n^{(H)}}{S_n}\right) = 0,$$

$$\text{where } S_n = \frac{\underset{1 \le i \le n}{\text{med}} \{|X_i - \tilde{m}_n|\}}{\Phi^{-1}(0.75)}$$

$\psi(x) = x \, \mathbb{1}\{|x| \le k\} + k \, \text{sign}(x) \, \mathbb{1}\{|x| > k\}$

(B) $X_1, \dots, X_{51}$ i.i.d $N(0,1)$, $X_{52} = \dots = X_{100} = 10^{12}$

(C) $X_1, \dots, X_n$ i.i.d $0.9 \cdot$ "$N(0,1)$" $+ \; 0.1 \cdot$ "$N(0,25)$"

$n = 100$

(D) $X_1, \dots, X_n$ i.i.d $LN(\mu, \sigma^2)$,

SO THAT $\mathbb{E} X_1 = e^{\mu + \frac{\sigma^2}{2}} = 38\,525$

$\text{med}\, X_1 = e^{\mu} = 32\,870$

2. COMPARISON OF DIFF. ESTIM IN LINEAR MODEL

(A) $Y_i = \overset{1}{\beta_0} + \overset{1}{\beta_1} X_i + \varepsilon_i \,, \; X_i \sim U(0,1) \,, \; \varepsilon_i \sim Exp(1)$

$$(\hat{\beta}_0, \hat{\beta}_1)^T = \underset{b_0, b_1}{\operatorname{argmin}} \sum_{i=1}^{n} \rho(Y_i, X_i; b_0, b_1)$$

LS: $\rho(Y_i, X_i; b_0, b_1) = (Y_i - b_0 - b_1 X_i)^2$

LAD: $\rho(Y_i, X_i; b_0, b_1) = |Y_i - b_0 - b_1 X_i|$

HUBER: film ... LATER

(B) $Y_i \mid X_i \sim Exp\left(\frac{1}{2(1+X_i)}\right) \,, \; X_i \sim U(0,2)$ $\quad n = 10\,000$

$$\mathbb{E}[Y_i \mid X_i] = 2(1+X_i) = \overset{\beta_0^{LS}}{2} + \overset{\beta_1^{LS}}{2} X_i$$

$$\operatorname{med}(Y_i \mid X_i) = 2(1+X_i)\log(2) = \doteq \underbrace{1.39}_{\beta_0^{LAD}} + \underbrace{1.39}_{\beta_1^{LAD}} X_i$$

HUBER ?

$\hat{\beta}_0 \doteq 2 \,, \; \hat{\beta}_1 \doteq 1.35 \,, \; S_n \doteq 3.01$

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i \, \psi\left(\frac{Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i}{S_n}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\mathbf{X}_i = \begin{pmatrix} 1 \\ X_i \end{pmatrix}$

rlm(..., k = 1.345)

$$\hat{\underset{\sim}{\beta}}^{(0)} = \hat{\underset{\sim}{\beta}}^{(LS)}, \quad \hat{\varepsilon}_i^{(0)} = Y_i - \underset{\sim}{X}_i^T \hat{\underset{\sim}{\beta}}^{(0)}, \; i = 1, \dots, n$$

$$\hat{\sigma}_n^{(0)} = \frac{\underset{1 \le i \le n}{\mathrm{med}} \{ |\hat{\varepsilon}_i^{(0)}| \}}{\Phi^{-1}(0.75)}$$

ITER STEP:

$$\underset{(n,n)}{\mathbb{W}^{(k+1)}} = \mathrm{diag}\{ \hat{w}_1^{(k+1)}, \dots, \hat{w}_n^{(k+1)} \}$$

$$w_i^{(k+1)} = \frac{\psi\left(\frac{\hat{\varepsilon}_i^{(k)}}{\hat{\sigma}_n^{(k)}} \right)}{\frac{\hat{\varepsilon}_i^{(k)}}{\hat{\sigma}_n^{(k)}}}$$

$$\hat{\underset{\sim}{\beta}}^{(k+1)} = \left(\mathbb{X}^T \mathbb{W}^{(k)} \mathbb{X} \right)^{-1} \mathbb{X}^T \mathbb{W}^{(k)} \underset{\sim}{Y}$$

$$\hat{\varepsilon}_i^{(k+1)} = Y_i - \underset{\sim}{X}_i^T \hat{\underset{\sim}{\beta}}^{(k+1)}, \quad i = 1, \dots, n$$

$$\hat{\sigma}_n^{(k+1)} = \underset{1 \le i \le n}{\mathrm{med}} \{ |\hat{\varepsilon}_i^{(k+1)}| \}$$