1. MISSPECIFIED POISSON REGRESSION

ASSUMED MODEL:

$$\mathcal{L}(\underset{\text{COUNT}}{Y_i} \mid \underset{\text{CONCENTRATION}}{C_i}) \sim \mathcal{P}o\Big(\overbrace{\exp\{\beta_0 + \beta_1 C_i\}}^{\lambda(\beta,\, X_i)}\Big)$$

DENOTE: $X_i = \begin{pmatrix} 1 \\ C_i \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$

$$f(y, x; \beta) = \underbrace{f_{Y|X}(y \mid x; \beta)}_{P_\beta(Y = y \mid X = x) = \frac{[\lambda(\beta; x)]^y e^{-\lambda(\beta; x)}}{y!}} \cdot f_X(x)$$

$$\rightarrow \rho(y, x; b) = -\log f(y, x; b) =$$

$$= -y \log \lambda(b; x) - \lambda(b; x)$$

$$= -y\, x^T b + \exp\{x^T b\} + c$$

$$\hat{\beta}_n = \underset{b \in \mathbb{R}^2}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} \left(-Y_i X_i^T b + \exp\{X_i^T b\}\right)$$

OK, if $\mathbb{E}[Y_i \mid X_i] = \exp\{X_i^T \beta_x\}$

$$\rightarrow \psi(y, x; b) = \frac{\partial \rho(y, x; b)}{\partial b} = -y x + \exp\{x^T b\} x$$

$$= -x\left(y - \exp\{x^T b\}\right)$$

$$\Sigma(\beta_x) = \operatorname{var}\left(\psi(Y_i, X_i, \beta_x)\right) =$$

$$= \mathbb{E}\left[X_i X_i^T \left(Y_i - \exp\{X_i^T \beta_x\}\right)^2\right]$$

$$\hat{\Sigma}_n = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T (Y_i - X_i^T \hat{\beta}_n)^2$$

$$D\psi(y, x; b) = \frac{\partial \Psi(y, x; b)}{\partial b} = x x^T \exp\{x^T b\}$$

$$\rightarrow \Gamma(\beta_*) = \mathbb{E}\, X_1 X_1^T \exp\{X_1^T \beta_*\}$$

$$\rightarrow \hat{\Gamma}_n = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T \exp\{X_i^T \hat{\beta}_n\}$$

WHEN WE BELIEVE THAT $\mathcal{L}(Y_i | X_i) \sim Po(\lambda(\beta, X_i))$, THEN $\hat{\Gamma}_n$ IS USUALLY USED AS THE ESTIMATE OF FIM $I(\beta_*)$ → „VCOV(FIT.GLM)"

$$\widehat{avar}(\hat{\beta}_n) = \frac{1}{n} \hat{\Gamma}_n^{-1} \hat{\Sigma}_n \hat{\Gamma}_n^{-1}$$ ⇐ „SANDWICH"

SIMULATION EXPERIMENT

$X_i = \begin{pmatrix} 1 \\ C_i \end{pmatrix}$, $C_i \sim$ DISCR. UNIF $(\{0; \frac{1}{2}; \dots; 4\})$

$Y_i | X_i \sim \exp\{\beta^T X_i\} \cdot \varepsilon_i$, $\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

where $\varepsilon_i \sim \frac{Bin(20, \frac{1}{2})}{20}$, $\mathbb{E}\varepsilon_i = 1$

$\varepsilon_i \perp C_i$

NOTE THAT: $\mathbb{E}[Y_i | X_i] = \exp\{\beta^T X_i\} \mathbb{E}\varepsilon_i = \exp\{\beta^T X_i\}$

2. WHITE ESTIMATOR OF ASYMPT. VARIANCE IN A LINEAR MODEL

CASE STUDY: $Y_i = \beta_0 + \beta_1 X_i + \sigma(X_i)\,\varepsilon_i$

$$X_i \sim U(0,2), \quad \varepsilon_i \sim \mathrm{Exp}(1) - 1 \qquad \mathbb{E}\varepsilon_i = 0$$

$$X_i \perp \varepsilon_i, \quad \sigma(X_i) = e^{X_i}$$

$$\underset{\sim}{x} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

$$\rho(y, \underset{\sim}{x}; \underset{\sim}{b}) = \frac{1}{2}\left(y - \underset{\sim}{x}^T \underset{\sim}{b}\right)^2$$

$$\underset{\sim}{\psi}(y, \underset{\sim}{x}; \underset{\sim}{b}) = -\left(y - \underset{\sim}{x}^T \underset{\sim}{b}\right)\underset{\sim}{x}$$

$$\mathbb{D}\underset{\sim}{\psi}(y; \underset{\sim}{x}; \underset{\sim}{b}) = \underset{\sim}{x}\,\underset{\sim}{x}^T$$

$$\underset{\sim}{X_i} = \begin{pmatrix} 1 \\ X_i \end{pmatrix}$$

$$\hat{\Pi}_n = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X_i}\,\underset{\sim}{X_i}^T,$$

$$\hat{\Sigma}_n = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X_i}\,\underset{\sim}{X_i}^T\left(Y_i - \underset{\sim}{X_i}^T \hat{\underset{\sim}{\beta}}_n\right)^2$$

STANDARD HOMOSC. ASYMPT. VARIANCE

$$\mathrm{avar}(\hat{\underset{\sim}{\beta}}) = \frac{1}{n}\left(\mathbb{E}\,\underset{\sim}{X_i}\,\underset{\sim}{X_i}^T\right)^{-1} \cdot \mathrm{var}(\varepsilon_i)$$

$$\rightsquigarrow \widehat{\mathrm{avar}}(\hat{\underset{\sim}{\beta}}) = \frac{1}{n}\,\hat{\Pi}_n^{-1}\,\frac{1}{n-2}\sum_{i=1}^{n}\left(Y_i - \underset{\sim}{X_i}^T \hat{\underset{\sim}{\beta}}_n\right)^2$$

HETEROSCEDASTICITY

$$\mathrm{avar}(\hat{\underset{\sim}{\beta}}) = \frac{1}{n}\left(\mathbb{E}\,\underset{\sim}{X_i}\,\underset{\sim}{X_i}^T\right)^{-1} \cdot \left(\mathbb{E}\,\underset{\sim}{X_i}\,\underset{\sim}{X_i}^T\left(Y_i - \underset{\sim}{X_i}^T \underset{\sim}{\beta}_x\right)^2\right) \cdot \left(\mathbb{E}\,\underset{\sim}{X_i}\,\underset{\sim}{X_i}^T\right)^{-1}$$

$$\widehat{\operatorname{avar}}(\hat{\beta}_n) = \frac{1}{n} \hat{\Pi}_n^{-1} \hat{\Sigma}_n \hat{\Pi}_n^{-1}$$

„HC3 - version" – $\hat{\Sigma}_n^{(HC3)} = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T \left(\frac{Y_i - X_i^T \hat{\beta}_n}{1 - h_{ii}} \right)^2$