Ex 2g) $X_i = \rho X_{i-1} + \varepsilon_i$, $\varepsilon_i \perp X_{i-1}$, $\varepsilon_i \sim N(0,\sigma^2)$

$i \in \mathbb{Z}$, $|\rho| < 1$, $\{\varepsilon_i, i \in \mathbb{Z}\}$ i.i.d.

→ WE OBSERVE $X_1, \dots, X_n$

$$f(x_1, \dots, x_n) = f(x_n | x_{n-1}, x_{n-2}, \dots, x_1) \cdot f(x_{n-1} | x_{n-2}, \dots) \cdot \dots \cdot f(x_2|x_1) f(x_1)$$

NOTE THAT:

* $\mathcal{L}(X_i | X_{i-1}, \dots, X_1) = \mathcal{L}(X_i | X_{i-1}) = N(\rho X_{i-1}, \sigma^2)$
* $X_1 \sim N\left(0, \frac{\sigma^2}{1-\rho^2}\right)$ ← AS $X_1 = \sum_{j=0}^{\infty} \varepsilon_{1-j} \rho^j$

  $var(X_1) = \sum_{j=0}^{\infty} \sigma^2 \rho^{2j} = \frac{\sigma^2}{1-\rho^2}$

$$f(x_1, \dots, x_n) = f(x_n | x_{n-1}) \cdot f(x_{n-1} | x_{n-2}) \cdot \dots \cdot f(x_1)$$

$$= \left(\prod_{i=2}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i - \rho x_{i-1})^2}{2\sigma^2}} \right) \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1-\rho^2}}} e^{-\frac{x_1^2}{2\frac{\sigma^2}{1-\rho^2}}}$$

$$\to \ell_n(\rho, \sigma^2) = \log f(X_1, \dots, X_n)$$

$$= -\frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=2}^{n} (X_i - \rho X_{i-1})^2 - \frac{X_1^2 (1-\rho^2)}{2\sigma^2} + \frac{1}{2} \log(1-\rho^2) + c$$

APPROX. SOLUTIONS: $\hat{\rho}_n = \dfrac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=2}^{n} X_{i-1}^2}$, $\hat{\sigma}_n^2 = \dfrac{1}{n}\sum_{i=2}^{n}\left(X_i - \hat{\rho}_n X_{i-1}\right)^2$

$$\frac{\partial \ell_n(\rho,\sigma^2)}{\partial \rho} = \frac{1}{\sigma^2}\sum_{i=2}^{n}\left(X_i - \rho X_{i-1}\right)X_{i-1} + \frac{X_1^2\rho}{\sigma^2} - \frac{\rho}{1-\rho^2} \overset{!}{=} 0$$

$$\frac{\partial \ell_n(\rho,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum_{i=2}^{n}\left(X_i-\rho X_{i-1}\right)^2}{2\sigma^4} + \frac{X_1^2(1-\rho^2)}{2\sigma^4} \overset{!}{=} 0$$

$$\frac{\partial^2 \ell_n(\rho,\sigma^2)}{\partial\rho\,\partial\rho} = -\frac{1}{\sigma^2}\sum_{i=2}^{n} X_{i-1}^2 - \frac{X_1^2}{\sigma^2} - \frac{\partial}{\partial\rho}\left(\frac{\rho}{1-\rho^2}\right)$$

$$\frac{\partial^2 \ell_n(\rho,\sigma^2)}{\partial\rho\,\partial\sigma^2} = -\frac{1}{\sigma^4}\sum_{i=2}^{n}\underbrace{\left(X_i-\rho X_{i-1}\right)}_{\varepsilon_i}X_{i-1} - \frac{X_1^2\rho}{\sigma^4}$$

$$\frac{\partial^2 \ell_n(\rho,\sigma^2)}{\partial\sigma^2\,\partial\sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=2}^{n}\left(X_i-\rho X_{i-1}\right)^2 - \frac{X_1^2(1-\rho^2)}{\sigma^6}$$

$$I_{11}(\rho,\sigma^2) = \lim_{n\to\infty} -\frac{1}{n}\,\mathbb{E}\left[-\frac{1}{\sigma^2}\sum_{i=2}^{n}X_{i-1}^2 - \frac{X_1^2}{n\sigma^2} - \dots\right]$$

$$= \frac{1}{\sigma^2}\mathbb{E}X_i^2 = \frac{\operatorname{var}X_i}{\sigma^2} = \frac{\cancel{\sigma^2}\,1}{1-\rho^2}$$

$$I_{12}(\rho,\sigma^2) = \mathbb{E}\left[\varepsilon_i X_{i-1}\right] = \mathbb{E}\varepsilon_i\,\mathbb{E}X_{i-1} = 0$$

$$I_{22}(\rho,\sigma^2) = \frac{1}{2\sigma^4} - \frac{1}{\sigma^6}\mathbb{E}\varepsilon_i^2 = \frac{1}{2\sigma^4} - \frac{\sigma^2}{\sigma^6} = \frac{1}{2\sigma^4}$$

$$\rightsquigarrow I(\rho,\sigma^2) = \begin{pmatrix} \frac{\cancel{\sigma^2}\,1}{1-\rho^2}, & 0 \\ 0, & \frac{1}{2\sigma^4} \end{pmatrix}$$

$$\rightsquigarrow \sqrt{n}\left(\begin{pmatrix}\hat{\rho}_n \\ \hat{\sigma}_n^2\end{pmatrix} - \begin{pmatrix}\rho \\ \sigma^2\end{pmatrix}\right) \xrightarrow[n\to\infty]{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{1-\rho^2}{\cancel{\sigma^2}}, & 0 \\ 0, & 2\sigma^4\end{pmatrix}\right)$$

$$\sqrt{n}\left(\hat{\underline{\theta}}_n - \underline{\theta}_X\right) \xrightarrow[n\to\infty]{d} N_p\left(\underline{0},\, I^{-1}(\theta_X)\right)$$

PROB. OF SUCCESS

PLACEBO, DRUG

Ex 34) $Y_{ij},\ i \in \{1, \dots I\},\ j \in \{0, 1\}$ INDEPENDENT, $Y_{ij} \sim Bin(n_{ij}, p_{ij})$

MEDIC. CENTERS (AGE GROUPS)

NUMBER OF PATIENTS IN THE GROUP i WITH THE TREATMENT j

ASSUMED MODEL:

$$\log\left(\frac{p_{ij}}{1-p_{ij}}\right) = \psi_i + \underset{\uparrow}{\tau}\, \mathbb{1}\{j=1\}$$

TREATMENT EFFECT

$\boxed{\text{GOAL: INFERENCE ON } \tau}$

JOINT DENSITY, $\underset{\sim}{\psi} = (\psi_1, \dots, \psi_I)^T$

$$f(\underset{\sim}{y}; \tau, \underset{\sim}{\psi}) = \prod_{i=1}^{I} \prod_{j=0}^{1} \underbrace{\binom{n_{ij}}{y_{ij}}}_{\text{CONST}} p_{ij}^{y_{ij}} (1-p_{ij})^{n_{ij}-y_{ij}}$$

$$= \prod_{i=1}^{I} \prod_{j=0}^{1} c_{ij} \exp\left\{ y_{ij} \log p_{ij} + (n_{ij} - y_{ij}) \log(1-p_{ij}) \right\}$$

$$= \exp\left\{ \sum_{i=1}^{I} \sum_{j=0}^{1} \left[y_{ij} \log\left(\frac{p_{ij}}{1-p_{ij}}\right) + n_{ij} \log(1-p_{ij}) \right] \right\} \cdot c$$

$$= \exp\left\{ \sum_{i=1}^{I} \sum_{j=0}^{1} \left[y_{ij} (\psi_i + \tau\, \mathbb{1}\{j=1\}) + n_{ij} \log(1-p_{ij}) \right] \right\} \cdot c$$

$$= \exp\left\{ \sum_{i=1}^{I} \underbrace{(y_{i1} + y_{i2})}_{y_{i+}} \psi_i + \tau \underbrace{\sum_{i=1}^{I} y_{i1}}_{y_{+1}} + \sum_{i=1}^{I}\sum_{j=0}^{1} n_{ij} \log(1-p_{ij}) \right\} \cdot c$$

$\rightsquigarrow \underset{\sim}{S}(\underset{\sim}{y}) = (y_{1+}, \dots, y_{I+})^T$

$\rightsquigarrow l_n^{(c)}(\tau) = \log f(\underset{\sim}{y}; \tau, \underset{\sim}{\psi}) - l_{n, \underset{\sim}{S}(\underset{\sim}{y})}(\tau, \underset{\sim}{\psi})$

WE NEED THE DISTRIBUTION OF $\underset{\sim}{S}(Y) = (Y_{1+}, \dots, Y_{I+})^T$.

$$\rightsquigarrow P(Y_{i+} = y) = P(Y_{i0} + Y_{i1} = y) = \sum_{k \in K_i} P(Y_{i1} = k) P(Y_{i0} = y-k),$$

where $K_i = \{\max(0, y - n_{i0}), \dots, \min(y, n_{i1})\}$

$$\underbrace{P(Y_{i+} = y)}_{f_i(y;\, \tau, \psi_i)} = \sum_{k \in K_i} P(Y_{i1} = k)\, P(Y_{i0} = y-k)$$

$$= \sum_{k \in K_i} \binom{n_{i1}}{k} \left(\frac{p_{i1}}{1-p_{i1}}\right)^{k} (1-p_{i1})^{n_{i1} \cancel{-k}}$$

$$\binom{n_{i0}}{y-k} \left(\frac{p_{i0}}{1-p_{i0}}\right)^{y-k} (1-p_{i0})^{n_{i0} - \cancel{(y-k)}}$$

$$= \sum_{k \in K_i} \binom{n_{i1}}{k} \binom{n_{i0}}{y-k} e^{(\psi_i + \tau)k}\, e^{\psi_i (y-k)} \cdot (1-p_{i1})^{n_{i1}} (1-p_{i0})^{n_{i0}}$$

$$= \underbrace{\sum_{k \in K_i} \binom{n_{i1}}{k} \binom{n_{i0}}{y-k} e^{\tau k}}_{A_i(\tau, y)} e^{\psi_i y} (1-p_{i1})^{n_{i1}} (1-p_{i0})^{n_{i0}}$$

$$\rightarrow\; l_{n, S(Y)}(\tau, \psi) = \log\left(\prod_{i=1}^{I} f_i(Y_{i+};\, \tau, \psi_i) \right)$$

$$= \sum_{i=1}^{I} \Big[\log\big(A_i(\tau, Y_{i+})\big) + Y_{i+}\psi_i + n_{i1} \log(1-p_{i1}) + n_{i0} \log(1-p_{i0}) \Big]$$

$$l_n(\tau, \psi) = \log f(Y;\, \tau, \psi) =$$

$$= \sum_{i=1}^{I} \Big[Y_{i+}\psi_i + \tau Y_{i1} + n_{i0} \log(1-p_{i0}) + n_{i1} \log(1-p_{i1}) \Big]$$

$$l_n^{(c)}(\tau) = l_n(\tau, \psi) - l_{n, S(Y)}(\tau, \psi)$$

$$= \sum_{i=1}^{I} \Big[\tau Y_{i1} - \log\big(A_i(\tau, Y_{i+})\big) \Big]$$

$$= \tau \underbrace{\sum_{i=1}^{I} Y_{i1}}_{Y_{+1}} - \sum_{i=1}^{I} \log\big(A_i(\tau, Y_{i+})\big)$$

$$\ell_n^{(c)}(\tau) = \tau Y_{+1} - \sum_{i=1}^{I} \log\left(A_i(\tau, Y_{i+})\right)$$

→ MAXIMUM CONDIT. LIKEL. ESTIM.:

$$\hat{\tau}_n = \underset{\tau > 0}{\arg\max}\ \ell_n^{(c)}(\tau)$$

<u>RAO SCORE TEST</u> (based on COND. LIK. $\ell_n^{(c)}(\tau)$)

$H_0: \tau = \tau_0 \qquad H_1: \tau \neq \tau_0$

$$R_n^{(c)} = \frac{1}{n} \frac{\left[U_n^{(c)}(\tau_0)\right]^2}{I_n^{(c)}(\tau_0)},$$

$A_i(\tau, Y_i) = \sum_{k \in K_i} \binom{m_{i1}}{k}\binom{m_{i0}}{Y_{i+}-k} e^{\tau k}$

where

$$U_n^{(c)}(\tau) = \frac{\partial \ell_n^{(c)}(\tau)}{\partial \tau} = Y_{+1} - \sum_{i=1}^{I} \frac{A_i'(\tau, Y_{i+})}{A_i(\tau, Y_{i+})}$$

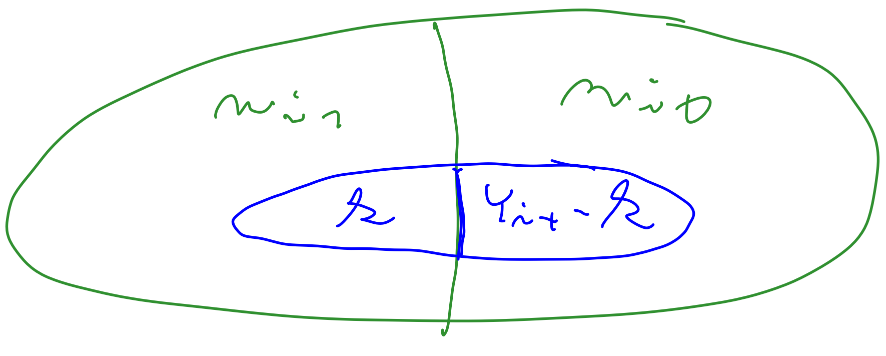
$$= Y_{+1} - \sum_{i=1}^{I} \frac{\sum_{k \in K_i} k \binom{m_{i1}}{k}\binom{m_{i0}}{Y_{i+}-k} e^{\tau k}}{\sum_{k \in K_i} \binom{m_{i1}}{k}\binom{m_{i0}}{Y_{i+}-k} e^{\tau k}}$$

→ for $\tau_0 = 0$:

$$U_n^{(c)}(0) = Y_{+1} - \sum_{i=1}^{I} \frac{\sum_{k \in K_i} k \binom{m_{i1}}{k}\binom{m_{i0}}{Y_{i+}-k}}{\sum_{k \in K_i} \binom{m_{i1}}{k}\binom{m_{i0}}{Y_{i+}-k}}$$

THE WAY HOW Y_{i+} (PATIENTS) CAN BE CHOSEN FROM TWO GROUPS OF SIZES m_{i1} AND m_{i0}.

$$= \binom{m_{i1} + m_{i0}}{Y_{i+}}$$



$$\rightsquigarrow U_n^{(c)}(0) = Y_{+1} - \sum_{i=1}^{I} \sum_{k \in K_i} k \frac{\binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k}}{\binom{n_{i1}+n_{i0}}{Y_{i+}}}$$

$P(Z_i = k)$

$P_{H_0}(Y_{i1} = k \mid Y_{i+})$

$$\rightsquigarrow U_n^{(c)}(0) = \underbrace{Y_{+1}}_{\text{OBSERVED}} - \underbrace{\sum_{i=1}^{I} \mathbb{E}\, Z_i}_{\text{EXPECTED UNDER } H_0}$$

$\left(\mathbb{E}_{H_0}[Y_{+1} \mid Y_{1+}, \dots, Y_{I+}] \right)$

$$\rightsquigarrow U_n^{(c)}(0) = \sum_{i=1}^{I} \left[Y_{i1} - Y_{i+} \frac{n_{i1}}{n_{i1}+n_{i0}} \right]$$

$$U_n^{(c)}(\tau) = Y_{+1} - \sum_{i=1}^{I} \frac{\sum_{k \in K_i} k \binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k} e^{\tau k}}{\sum_{k \in K_i} \binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k} e^{\tau k}}$$

FURTHER WE NEED:

$$I_n^{(c)}(\tau) = \frac{-1}{n} \frac{\partial^2 \ell_n^{(c)}(\tau)}{\partial \tau^2} = -\frac{1}{n} \frac{\partial U_n^{(c)}(\tau)}{\partial \tau}$$

$$\rightsquigarrow I_n^{(c)}(0) = \frac{1}{n} \sum_{i=1}^{I} \left[\frac{\sum_{k \in K_i} k^2 \binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k}}{\sum_{k \in K_i} \binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k}} - \left\{ \frac{\sum_{k \in K_i} k \binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k}}{\sum_{k \in K_i} \binom{n_{i1}}{k}\binom{n_{i0}}{Y_{i+}-k}} \right\}^2 \right]$$

$$= \frac{1}{n} \sum_{i=1}^{I} \left[\mathbb{E}\, Z_i^2 - \{\mathbb{E}\, Z_i\}^2 \right]$$

$$= \frac{1}{n} \sum_{i=1}^{I} \mathrm{var}(Z_i) \leftarrow \mathrm{var}_{H_0}(Y_{i1} \mid Y_{i+})$$

$$= \frac{1}{n} \sum_{i=1}^{I} Y_{i+} \underbrace{\frac{n_{i1} n_{i0}}{(n_{i1}+n_{i0})^2}}_{\text{ANALOGOUS TO BINOMIAL DISTRIBUTION}} \underbrace{\frac{n_{i1}+n_{i0}-Y_{i+}}{n_{i1}+n_{i0}-1}}_{\text{FINITE SAMPLE CORRECTION}}$$

$$\underset{\sim}{R}^{(c)} = \frac{1}{n} \frac{\left[\underset{\sim}{U}^{(c)}(0)\right]^2}{\underset{\sim}{I}^{(c)}(0)} = \frac{\cancel{\frac{1}{n}} \left(\sum_{i=1}^{I} Y_{i1} - \mathbb{E}_{H_0}(Y_{i1} | Y_{i+})\right)^2}{\cancel{\frac{1}{n}} \sum_{i=1}^{I} \operatorname{var}_{H_0}(Y_{i1} | Y_{i+})}$$

$$= \frac{\left(\sum_{i=1}^{I} Y_{i1} - Y_{i+} \frac{n_{i1}}{n_{i1}+n_{i0}}\right)^2}{\sum_{i=1}^{I} Y_{i+} \frac{n_{i1} n_{i0}}{(n_{i1}+n_{i0})^2} \frac{n_{i1}+n_{i0}-Y_{i+}}{n_{i1}+n_{i0}-1}}$$

$$\underset{\sim}{R}^{(c)} \underset{H_0: \tau = 0}{\overset{as}{\approx}} \chi^2_1 \leftarrow \text{COCHRAN-MANTEL-HAENSZEL TEST}$$