NMSA 332 Suficientné štatistiky vzorové: 25) 29) 32) 34)

25) $X_i \sim p(1-p)^k \quad k \in N_0 \qquad S = \sum X_i \sim$ neg. bin $= \binom{n+s-1}{s} p^n (1-p)^s$

a) $P(\mathbb{X}=\mathbb{x} \mid S=s) = \dfrac{P(S=s \mid \mathbb{X}=\mathbb{x})\, P(\mathbb{X}=\mathbb{x})}{P(S=s)} = \dfrac{I[\sum x_i = s]\, p^n (1-p)^{\sum x_i}}{\binom{n+s-1}{s} p^n (1-p)^s}$

$= \begin{cases} 1/\binom{n+s-1}{s} & \text{ak } \sum x_i = s \\ 0 & \text{inak.} \end{cases} \Rightarrow$ sufic.

pre dané s ide o rovnomerné rozdelenie na všetkých n-ticiach v N_0 takých že $\sum_{i=1}^{n} x_i = s$.

b) $f(x;p) = p^n (1-p)^{\sum x_i} \Rightarrow S = \sum X_i$ je suf.

26) a) $X_i \sim Po(\lambda) \qquad S = \sum X_i \sim Po(n\lambda)$

$P(\mathbb{X}=\mathbb{x} \mid S=s) = I[\sum x_i = s] \dfrac{e^{-n\lambda} \lambda^{\sum x_i} / (\prod x_i!)}{e^{-n\lambda} (n\lambda)^s / s!} = \begin{cases} \binom{s}{x_1 \dots x_n} \left(\frac{1}{n}\right)^{x_1} \cdots \left(\frac{1}{n}\right)^{x_n} & \text{ak } \sum x_i = s \\ 0 & \text{inak} \end{cases}$

$\Rightarrow$ sufic a ide pre dané s o rozdelenie $M(s; \frac{1}{n}, \dots, \frac{1}{n})$

b) $P(\mathbb{X}=\mathbb{x}) = \underbrace{e^{-n\lambda} \lambda^{\sum x_i}}_{g(\sum x_i, \lambda)} \underbrace{1/\prod(x_i!)}_{h(x)}$

27) $X \sim R\{1 \dots M\} \quad M \in N \quad S = \max_{1 \le i \le n} \{X_i\} \quad P(S \le s) = P(X_i \le s)^n = \left(\frac{s}{M}\right)^n$

$P(S=s) = P(S \le s) - P(S \le s-1) = \left(\frac{s}{M}\right)^n - \left(\frac{s-1}{M}\right)^n$ pre $s \in \{1 \dots M\}$

a) $P(\mathbb{X}=\mathbb{x} \mid S=s) = I[\max x_i = s] \dfrac{(1/M)^n}{(s^n - (s-1)^n)/M^n} = \begin{cases} 1/(s^n - (s-1)^n) & \text{ak } \max x_i = s \\ 0 & \text{inak} \end{cases}$

$\Rightarrow$ S je sufic

b) $P(X=x) = \left(\frac{1}{M}\right)^n \prod I[1 \le x_i \le M] = \left(\frac{1}{M}\right)^n I[\max x_i \le M]$

28) $f(x;\sigma^2) = \dfrac{c}{\sigma^n} \exp\left\{-\frac{1}{2\sigma^2} \sum x_i^2\right\}$ z Lehman-Scheffé je $\sum x_i^2$ minimálna suf $\Downarrow$ $\sum X_i^2$ je funkcion $\forall$ suf. štatistiky

i) $\mathbb{X}$ je suf (viem z nej dostať $\sum X_i^2$, prípadne $\mathbb{X} \mid \mathbb{X} \sim \delta_{\mathbb{x}}$ nezávisí na σ^2)

ii) $(|X_1|, \dots, |X_n|)'$ je suf ($\sum x_i^2 = \sum |x_i|^2$)

iii) $\sum X_i$ nie je suf ($\sum x_i^2$ nie je fce $\sum x_i$, ak $\sum x_i = k$ neviem určiť $\sum x_i^2$)

iv) nie $\sum |X_i|$, ako v iii)

v) $\sum X_i^2$ je suf

vi) $\sum X_i^2 / n$ je suf

vii) $\left(\frac{1}{n} \sum_{i=1}^{n-1} X_i^2, X_n^2\right)$ je suf

ale $\sum X_i^2$ nie je fcion $\sum X_i$
($X_1 = -1$, $X_2 = 1$: $\sum X_i = 0$ a $\sum X_i^2 = 2$
ale $X_1 = -2$, $X_2 = 2$: $\sum x_i = 0$ a $\sum x_i^2 = 4$)
$\Rightarrow \sum X_i^2$ nie je fce $\sum X_i$)

29) $X \sim alt(p)$

i) $P(X=x) = p^{\sum x_i} (1-p)^{n - \sum x_i} \Rightarrow \sum X_i$ je suf.

ii) $\dfrac{P(X=x)}{P(X=y)} = p^{\sum x_i - \sum y_i} (1-p)^{\sum y_i - \sum x_i} \Rightarrow \sum X_i$ je minimálna postač.

iii) nech $E_p w(X_1) \overset{!}{=} 0 = p(w(1)) + (1-p) w(0) \quad \forall p \in (0,1)$
$= p(w(1) - w(0)) + w(0) \Rightarrow w(0) = 0$ a $w(1) - w(0) = 0$.
$\Rightarrow$ úplná. Nie je ale postačujúca $\mathbb{X} \mid X_1 \sim (\delta_{x_1}, X_2 \dots X_n)'$ závisí na p

iv) nech $E_p w(\sum X_i) \overset{!}{=} 0 = \sum_{j=0}^{n} \binom{n}{j} \underbrace{p^j (1-p)^{n-j}}_{} w(j)$
$\sum X_i \sim Bi(n,p)$
LNZ fce v p pre $j = 0 \dots n \Rightarrow w(j) = 0 \ \forall j$
a je úplná.

30) [redacted] $X_i \sim N(\mu, \sigma^2)$

$$\frac{f(\mathbf{x})}{f(\mathbf{y})} = \frac{1/\sigma^n \; \exp\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\}}{1/\sigma^n \; \exp\{-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\}} = \exp\left\{-\frac{1}{2\sigma^2}\left[\sum x_i^2 - 2\mu\sum x_i + \cancel{n\mu^2} - \sum y_i^2 + 2\mu\sum y_i - \cancel{n\mu^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[\sum(X_i^2 - Y_i^2) - 2\mu\sum(X_i - Y_i)\right]\right\} \Rightarrow \left(\sum X_i, \sum X_i^2\right)' \text{ je minim. suf.}$$

31) [redacted] $X \sim R(0,\theta)$ $\qquad M = \max_{1\le i\le n}\{X_i\}$ $\qquad P(M \le m) = (m/\theta)^n \Rightarrow f_M(m) = \frac{n\, m^{n-1}}{\theta^n} \quad m \in [0,\theta]$

i) nech $\forall \theta > 0$

$$0 \overset{!}{=} E_\theta\, w(M) = \int_0^\theta \frac{n\, m^{n-1}}{\theta^n} w(m)\, dm$$

potom by $\quad 0 = \int_0^\theta m^{n-1} w(m)\, dm \quad / \partial/\partial\theta$

OPRAVA: DOSAZUJE SE POUZE MEZ, KTERÁ ZÁVISÍ NA θ.

$$0 = \theta^{n-1} w(\theta) - \cancel{0^{n-1} w(0)} \Rightarrow w(\theta) = 0 \;\; \forall\theta \;(s.v.) \Rightarrow \text{úplná}$$

ii) $0 \overset{!}{=} E_\theta\, w(X_1) = \int_0^\theta \frac{w(x)}{\theta}\, dx \quad /\theta / \frac{\partial}{\partial\theta}$

$0 = w(\theta) \quad \forall\theta\, (s.v.) \Rightarrow$ úplná

32) [redacted] $X \sim R(\theta - 1/2, \theta + 1/2)$

i) $f(\mathbf{x}) = 1 \cdot I[x_1 \in (\theta - 1/2, \theta + 1/2)] \dots I[x_n \in (\theta - 1/2, \theta + 1/2)]$

$= I[\theta - 1/2 \le \min X_i \le \max X_i \le \theta + 1/2] \Rightarrow (\min X_i, \max X_i)$ je suf

ii) $E(\max X_i - \min X_i) =$ nebude záviset na θ kvůli invarianci vůči posunutí

$$= \left(\theta + 1/2 - \frac{1}{n+1}\right) - \left(\theta - 1/2 + \frac{1}{n+1}\right) = 1 - \frac{2}{n+1}$$

33) [redacted] $X \sim \text{Pareto}(\alpha, \beta)$

$$f(\mathbf{x}) = \frac{\beta^n \alpha^{n\beta}}{(\prod x_i)^{\beta+1}} I[\min x_i > \alpha] \Rightarrow \text{suf je } (\min X_i, \prod X_i) \text{ nebo } (\min X_i, \sum \log X_i)$$

34) [redacted] $X \sim N(\mu, \mu^2)$

i) $$\frac{f(\mathbf{x})}{f(\mathbf{y})} = \exp\left\{-\frac{1}{2\mu^2}\left[\sum x_i^2 - 2\mu\sum x_i + \cancel{n\mu^2} - \sum y_i^2 + 2\mu\sum y_i - \cancel{n\mu^2}\right]\right\} =$$

$$= \exp\left\{-\frac{1}{2\mu^2}\left[\sum(x_i^2 - y_i^2) - 2\mu\sum(x_i - y_i)\right]\right\} \qquad \left(\sum X_i, \sum X_i^2\right)' \text{ je min. suf}$$

ii) $$E_\mu\left[\frac{(\sum X_i)^2}{n+1} - \frac{\sum X_i^2}{2}\right] = 0 \quad \forall\mu$$

$\sum X_i \sim N(n\mu, n\mu^2) \Rightarrow E(\sum X_i)^2 = n\mu^2 + \mu^2 n^2 = \mu^2(n^2+n) = n\mu^2(n+1)$

$E\sum X_i^2 = n\, E X_i^2 = n\mu^2 \cdot 2$

35) [redacted] $X \sim M(n; p_1 \dots p_7)$

i) $$\frac{P(\mathbf{X} = \mathbf{x})}{P(\mathbf{X} = \mathbf{y})} = \frac{\binom{n}{x_1 \dots x_7} \prod p_j^{x_j}}{\binom{n}{y_1 \dots y_7} \prod p_j^{y_j}} = c \cdot \prod p_j^{x_j - y_j} \Rightarrow \mathbf{X} \text{ je min postačující}$$

n pevné $\Rightarrow X_7 = n - \sum_{i=1}^{6} X_i$ a.s.

ii) $= c \cdot p_1^{\sum_1^5 x_j - \sum_1^5 y_j}\, p_6^{\sum_6^7 x_j - \sum_6^7 y_j} \Rightarrow \left(\sum_{i=1}^5 X_i, X_6 + X_7\right)'$ je min. postačující

iii) $= c\; p^{\sum x_i - \sum y_i} \Rightarrow \sum_{i=1}^{7} X_i$ je min. postačující

ale v tomto modelu $1/7$ $p = 1/4$

36) [redacted] $X \sim N(0, \sigma^2)$

i) $\sum X_i \sim N(0, n\sigma^2) \quad E_{\sigma^2} \sum X_i = 0 \quad \forall\sigma^2 \Rightarrow$ není je úplná

ii) $$E_{\sigma^2}(\underbrace{\sin X_1 - 1}_{T}) = \int_R \underbrace{\sin x}_{\text{lichá fce}} \cdot \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dx - 1 = -1$$

$E_{\sigma^2}(T+1) = 0 \quad \forall\sigma^2 \Rightarrow$ není je úplná

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37) [redacted] $X \sim B(a,b)$ $\quad \frac{f(x)}{f(y)} = \frac{(\prod x_i)^{a-1}\prod(1-x_i)^{b-1}}{(\prod y_i)^{a-1}\prod(1-y_i)^{b-1}} = \left(\prod \frac{x_i}{y_i}\right)^{a-1}\left(\prod\frac{1-x_i}{1-y_i}\right)^{b-1}$

$\Rightarrow (\prod X_i, \prod(1-X_i))^T$ alebo $(\sum \log X_i, \sum \log(1-X_i))^T$ sú obe min. suf.

38) [redacted] $X\sim N(\mu_1,\sigma^2)$ $\quad Y\sim N(\mu_2,\sigma^2)$

i) $f(x,y) = \frac{c}{\sigma^{n+m}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum^n x_i^2 - 2\mu_1\sum x_i + n\mu_1^2 - \sum^m y_i^2 + 2\mu_2\sum y_i - m\mu_2^2\right]\right\}$

$\Rightarrow (\sum X_i, \sum X_i^2, \sum Y_i, \sum Y_i^2)^T$ je suf.

ii) $E_{\mu_1,\mu_2,\sigma^2}\, S^2_{n,X} - S^2_{m,Y} = 0 \quad \forall \mu_1,\mu_2,\sigma^2$

39) [redacted] $P(\mathbf{X}=\mathbf{x}) = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x! \cdot C(k,\lambda)^n}\, I[0\le x_1\le k]\dots I[0\le x_n\le k] \Rightarrow (\sum X_i, \max X_i)^T$ je suf.

Vylepšenie sufic. štatistiky $\qquad$ 40,41 uložiť $\qquad$ 46 uložiť

40 [redacted] $X\sim Ge(p) \qquad P(X=x) = p(1-p)^x \quad x\in\mathbb{N}_0$

a) $ET = E\,I[X=0] = p$

b) $(m\neq 1) \Rightarrow S = \sum X_i$ postačujúca $\quad E[T\mid S=n] = E[I[X=0]\mid S=n] = P(X=0\mid S=n) =$

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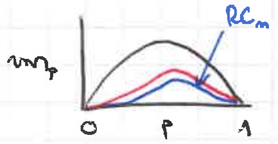
$= \frac{P(S=n\mid X_1=0)\,P(X_1=0)}{P(S=n)} = p\cdot\frac{P(\sum_{i=2}^m X_i = n)}{P(\sum_{i=1}^m X_i=n)} = \frac{p\binom{m+n-2}{n}p^{m-1}(1-p)^n}{\binom{m+n-1}{n}p^m(1-p)^n} =$

$P(\sum_{i=1}^m X_i = n) = \binom{m+n-1}{n}p^m(1-p)^n \quad n\in\mathbb{N}_0$

$= \frac{\frac{(m+n-2)!}{n!\,(m-2)!}}{\frac{(m+n-1)!}{n!\,(m-1)!}} = \frac{m-1}{m+n-1} \qquad RC_m = \frac{p^2q}{m}$



var $T = \frac{p(1-p)}{m}$

var $u(\bar{X})$ je ťažké spočítať

$E[T\mid S] = \frac{m-1}{m+S-1} = \frac{1-1/m}{1+\bar{X}-1/m} = u(\bar{X})$

c) $P(X=x) = p(1-p)^x = \exp\{x\log(1-p)\}\cdot p \Rightarrow \sum X_i = S$ je úplná postač. pre $\theta = \log(1-p)$

$a(\theta) = 1-e^\theta = 1-(1-p) = p \qquad E_\theta T^2 = E_p T^2 < \infty \ \forall p \Rightarrow$ Lehmann-Scheffé $u(\bar{X})$ je najlepší nestr. odhad p

d) $E\,I[X_1=1] = P(X_1=1) = p(1-p) \qquad T = I[X_1=1]$ nestranný odhad.

$E[T\mid S=n] = p(1-p)\cdot\frac{P(\sum_{i=2}^m X_i = n-1)}{P(\sum X_i = n)} = p(1-p)\frac{\binom{m+n-3}{n-1}p^{m-1}(1-p)^{n-1}}{\binom{m+n-1}{n}p^m(1-p)^n} =$

$= \frac{\frac{(m+n-3)!}{(n-1)!\,(m-2)!}}{\frac{(m+n-1)!}{n!\,(m-1)!}} = \frac{(m-1)\,n}{(m+n-1)(m+n-2)}$

$E[T\mid S] = \frac{(m-1)\,S}{(m+S-1)(m+S-2)} = u(S)$

41 [redacted]

$\mathbf{X} \sim M(1;\, p, 1-2p, p)$ s hodnotami $\{-1,0,1\}$

$X=-1 \Leftrightarrow \mathbf{X} = (1,0,0)^T$
$X=0 \Leftrightarrow \mathbf{X} = (0,1,0)^T$
$X=1 \Leftrightarrow \mathbf{X} = (0,1,0)^T$

a) $ET = P(X=1) = p$

b) $P(\mathbf{X} = (x_1,x_2,x_3)^T) = p^{x_1}(1-2p)^{x_2}p^{x_3}$ kde $(x_1,x_2,x_3)^T \in \{0,1\}^3$, $\sum_{i=1}^3 x_i = 1$

počet -1 ... počet $+1$

$= p^{x_1+x_3}(1-2p)^{x_2}$

$= p^{I[X\neq 0]}(1-2p)^{1-I[X\neq 0]}$

c) $E[I[X_1=1]\mid \underbrace{\sum I[X_i\neq 0]}_{S\sim Bi(m,2p)} = n] = p\cdot\frac{P(S=n\mid X_1=1)}{P(S=n)} = p\cdot\frac{\binom{m-1}{n-1}(2p)^{n-1}(1-2p)^{m-n}}{\binom{m}{n}(2p)^n(1-2p)^{m-n}}$