NMSA 332 2. cvičení – Rao–Cramér 9, morony 8, 10, 11, 12, 13

(9) 1, $\binom{X}{Y} \sim N_2\left(\binom{\theta}{\theta}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ $\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$ $|\Sigma| = 1-\rho^2$

a, $f_{\binom{X}{Y}}(x,y) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\left(\binom{x}{y}-\binom{\theta}{\theta}\right)'\Sigma^{-1}\left(\binom{x}{y}-\binom{\theta}{\theta}\right)\right\} = \frac{c'}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}\left[\frac{(x-\theta)^2+(y-\theta)^2-2\rho(x-\theta)(y-\theta)}{1-\rho^2}\right]\right\}$

$\frac{1}{1-\rho^2}\begin{pmatrix} x-\theta & y-\theta \end{pmatrix}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} x-\theta \\ y-\theta \end{pmatrix} = \frac{1}{1-\rho^2}\left((x-\theta)^2+(y-\theta)^2-2\rho(x-\theta)(y-\theta)\right)$

$\ell_{\binom{x}{y}}(\theta) = c - \frac{1}{2(1-\rho^2)}\left((x-\theta)^2+(y-\theta)^2-2\rho(x-\theta)(y-\theta)\right)$

$\frac{\partial \ell}{\partial \theta}(\theta) = -\frac{1}{\cancel{2}(1-\rho^2)}\left(-\cancel{2}(x-\theta)-\cancel{2}(y-\theta)+\cancel{2}\rho[(y-\theta)+(x-\theta)]\right)$

$\frac{\partial^2 \ell}{\partial \theta^2}(\theta) = -\frac{1}{1-\rho^2}(2-2\rho) = -\frac{2(1-\rho)}{1-\rho^2} = -2/(1+\rho) \Rightarrow J(\theta) = 2/(1+\rho)$

b, $\bar{X} \sim N(\theta, 1/n)$ $\operatorname{var} \bar{X} = 1/n$ R-C: $\frac{1}{n\left(\frac{2}{1+\rho}\right)}$ $\cancel{1/(2n(1+\rho))} = (1+\rho)/2n$

nestranný $1/\cancel{n} > (1+\rho)/(2\cancel{n}) \rightarrow$ nedosahuje R-C mezdu až $\rho = 1$ dosahuje

c, $Z = (X+Y)/2 \sim N\left(\theta, \frac{1}{4}+\frac{1}{4}+\frac{2\rho}{4}\right) = N\left(\theta, \frac{1+\rho}{2}\right)$

$\bar{Z} \sim N\left(\theta, \frac{1+\rho}{2n}\right)$ $\operatorname{var} \bar{Z} = \frac{1+\rho}{2n} = $ R-C: $\frac{1+\rho}{2n} \Rightarrow$ dosahuje R-C mezdu

→ (10) 2, $X \sim Po(\lambda)$

a, $f(x) = e^{-\lambda}\lambda^x/x!$ $x \in N_0$
$\ell(\lambda) = -\lambda + x\log\lambda + c$
$\partial\ell/\partial\lambda = -1 + x/\lambda$
$\partial^2\ell/\partial^2\lambda = -x/\lambda^2$ $J(\lambda) = E\, X/\lambda^2 = 1/\lambda$ b, $J_n(\lambda) = n/\lambda$

c, $g(\lambda) = 2\lambda$ $Y = \sum X_i \sim Po(n\lambda)$ $EY = n\lambda \Rightarrow T := \frac{2\sum X_i}{n}$
$\operatorname{var} T = \frac{4}{n^2} n \operatorname{var} X_1 = \frac{4\lambda}{n} = $ R-C: $\frac{4\lambda}{n} \Rightarrow$ eficientní odhad
$g'(\lambda) = 2$

d, $T = \left(1-\frac{1}{n}\right)^{\sum X_i}$ $ET^k = \sum_{j=0}^{\infty}\left(1-\frac{1}{n}\right)^{j\cdot k} e^{-n\lambda}\frac{(n\lambda)^j}{j!} = e^{-n\lambda}e^{n\lambda\cdot(1-\frac{1}{n})^k} = \cancel{\exp\{(1-\frac{1}{n})^k\}}$

$ET = e^{-\lambda} \Rightarrow$ nestranný $g(\lambda) = e^{-\lambda}$
$\operatorname{var} T = e^{\cancel{-n\lambda}+\cancel{n\lambda}-2\lambda+\lambda/n} - e^{-2\lambda} = e^{-2\lambda}(-1+e^{\lambda/n})$ $g'(\lambda) = -e^{-\lambda}$
R-C: $e^{-2\lambda}\frac{\lambda}{n}$
$\operatorname{var} T = e^{-2\lambda}\sum_{j=1}^{\infty}\left(\frac{\lambda}{n}\right)^j\frac{1}{j!} > e^{-2\lambda}\frac{\lambda}{n} \Rightarrow$ nedosahuje

e, jasné

(11) 3, neregulární systém, support závisí na α

(12) 4, $X \sim N(\theta, \sigma^2)$
a, $L(\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(x-\theta)^2\right\}$ $\ell(\sigma) = c - \log\sigma - \frac{1}{2\sigma^2}(x-\theta)^2$

$\frac{\partial\ell}{\partial\sigma} = -\frac{1}{\sigma} + \frac{(x-\theta)^2}{\sigma^3}$ $\frac{\partial^2\ell}{\partial\sigma^2} = \frac{1}{\sigma^2} - \frac{3(x-\theta)^2}{\sigma^4}$ $J(\sigma) = -\frac{3}{\sigma^2}+\frac{1}{\sigma^2} = \frac{2}{\sigma^2}$

b, $L(\sigma^2) = \ldots$ $\ell(\sigma^2) = c - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(x-\theta)^2$
$J(\sigma^2) = \frac{1}{(\sigma^2)^2} - \frac{1}{2(\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} = \frac{J(\sigma)}{(2\sigma)^2}$

$\frac{\partial\ell}{\partial\sigma^2} = -\frac{1}{2\sigma^2} + \frac{(x-\theta)^2}{2(\sigma^2)^2}$ $\frac{\partial^2\ell}{\partial(\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} - \frac{(x-\theta)^2}{(\sigma^2)^3}$

c) jasné

13) $X \sim N(0,\sigma^2)$

i) mat. statistika I: nestrannosť S_n^2 (strana 2.2.3, veta 2.6)

pre normalitu: (veta 2.8) $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$ $\quad \operatorname{var} \frac{n-1}{\sigma^2} S_n^2 = \operatorname{var} \chi^2_{n-1} = 2(n-1)$

$\Rightarrow \operatorname{var} S_n^2 = \frac{2\sigma^4}{n-1}$

$L(\sigma^2) =$ ako v 12) $\Rightarrow J_n(\sigma^2) = \frac{1 \cdot n}{2(\sigma^2)^2}$ $\quad$ RC: $\frac{2\sigma^4}{n} < \operatorname{var} S_n^2 = \frac{2\sigma^4}{n-1}$

asymptoticky je to ale ekvivalentné

ii) $T_n = \frac{1}{n}\sum X_i^2 \quad ET_n = EX^2 = \sigma^2 \quad \operatorname{var} T_n = \frac{1}{n^2}\sum \operatorname{var} X_i^2 = \frac{1}{n}(EX^4 - \sigma^4) = \frac{3\sigma^4 - \sigma^4}{n} =$

$= \frac{2\sigma^4}{n} =$ RC medza

iii) $\hat\sigma_n = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{n}\sum |X_i| \quad E\hat\sigma_n = \sqrt{\frac{\pi}{2}}\, E|X| = \sqrt{\frac{\pi}{2}} \int_R \frac{|x|}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\,dx = 2\sqrt{\frac{\pi}{2}} \int_0^\infty \frac{x e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx =$ $\quad \frac{x}{\sqrt{2\sigma^2}} = t$

$= \cancel{\sqrt{2\pi}} \int_0^\infty \frac{\sqrt{2\sigma^2}\, t}{\cancel{\sqrt{2\pi\sigma^2}}} e^{-t^2} \sqrt{2\sigma^2}\,dt = 2\sigma \int_0^\infty t e^{-t^2}\,dt \underset{t^2 = s}{=} \sigma \int_0^\infty e^{-s}\,ds = \underline{\underline{\sigma}}$

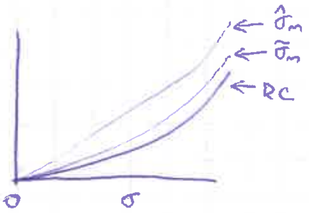
$\operatorname{var} \hat\sigma_n = \frac{\pi}{2n} \operatorname{var} |X_i| = \frac{\pi}{2n}\left(\sigma^2 - \left(\sigma\sqrt{\frac{2}{\pi}}\right)^2\right) = \sigma^2 \frac{\pi}{2n}\left(1 - \frac{2}{\pi}\right) = \sigma^2\left(\frac{\pi}{2n} - \frac{1}{n}\right) = \frac{\sigma^2}{2n}(\pi - 2)$

$RC = \frac{\sigma^2}{2n} < \operatorname{var} \hat\sigma_n = \frac{\sigma^2}{2n}(\pi - 2)$

iv) $\tilde\sigma_n = c\sqrt{\frac{1}{n}\sum X_i^2} \quad E\tilde\sigma_n = c\int_0^\infty \sqrt{\frac{1}{n} y \sigma^2}\, f_{\chi^2_n}(y)\,dy \underset{Y \sim \chi^2_n}{=} \frac{\sigma \cdot c}{\sqrt{n}} E\sqrt{Y} = \frac{\sigma c \sqrt{2}\, \Gamma\left(\frac{1+n}{2}\right)}{\sqrt{n}\; \Gamma\left(\frac{n}{2}\right)}$

$\frac{\sum X_i^2}{\sigma^2} = \sum\left(\frac{X_i}{\sigma}\right)^2 \sim \sum N(0,1)^2 = \chi^2_n \Rightarrow \sum X_i^2 \sim \sigma^2 \chi^2_n$

$c = \sqrt{\frac{n}{2}} \frac{\Gamma(n/2)}{\Gamma\left(\frac{n+1}{2}\right)} \quad \operatorname{var} \tilde\sigma_n = \frac{c^2}{n} E\sum X_i^2 - \sigma^2 = \frac{\cancel{n}}{2\cancel{n}}\left[\frac{\Gamma(n/2)}{\Gamma\left(\frac{n+1}{2}\right)}\right]^2 \sigma^2 n - \sigma^2$



$E\chi_n = \sqrt{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)}$

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx \quad (= (\alpha-1)!$ pre $\alpha \in N)$

14) $X \sim R(0,\theta) \rightarrow$ neregulárny model

i) $2\bar X = \hat\theta_n \quad E\hat\theta_n = 2EX_1 = \theta \qquad P(\max X_i \le t) = \left(\frac{t}{\theta}\right)^n \Rightarrow f_{\max}(t) = \frac{n t^{n-1}}{\theta^n}$

$\tilde\theta_n = \frac{n+1}{n}\max X_i \quad E\tilde\theta_n = \frac{n+1}{n} E\max X_i = \frac{n+1}{n}\int_0^\theta \frac{t\, n\, t^{n-1}}{\theta^n}\,dt = \frac{(n+1)}{\theta^n}\left[\frac{t^{n+1}}{n+1}\right]_0^\theta = \theta$

ii) neregulárny model

15) $X \sim Alt(p)$

i) $\hat p = \bar X$ nestranný $\quad \operatorname{var} \hat p = \frac{p(1-p)}{n} \qquad L(p) = p^{\sum x_i}(1-p)^{n - \sum x_i}$

$\ell(p) = \sum x_i \log p + (n - \sum x_i)\log(1-p)$

$\frac{\partial \ell}{\partial p} = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p} \qquad \frac{\partial^2 \ell}{\partial p^2} = -\frac{\sum x_i}{p^2} - \frac{n - \sum x_i}{(1-p)^2}$

$J_n(p) = \frac{n}{p} + \frac{n}{1-p} = n\left(\frac{1}{p} + \frac{1}{1-p}\right) = \frac{n}{p(1-p)}$ $\qquad$ RC: $\frac{p(1-p)}{n}$ dosahuje RC mezu

ii) jasné