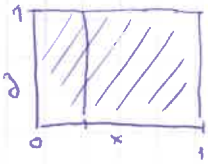
NMSA 332 1. cvičení – Podmíněné rozdělení

2 písemky 11–12 cvičení

1) $f(x,y) = (x+y) I_M \qquad M = \{0<x<1,\ 0<y<1\}$



a) $E_{Y|X}[XY \mid X = x] = x\, E[Y \mid X = x] = \ast$

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{(x+y) I_M}{\int_0^1 (x+y)\,dy} = \frac{(x+y) I_M}{x + 1/2} \qquad x \in (0,1) \quad y \in (0,1)$$

$$E[Y \mid X = x] = \int_0^1 y \frac{(x+y) I_M}{x + 1/2}\, dy = \frac{x/2}{x+1/2} + \frac{1/3}{x+1/2} = \cancel{\frac{x}{x+1/2}(5/6)}$$

$$\ast = x \cdot \left(\frac{x}{2x+1} + \frac{1}{3x+3/2} \right), \quad x \in (0,1)$$

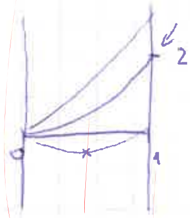
b) $E[XY \mid X] = X \left(\frac{X}{2X+1} + \frac{1}{3X+3/2} \right)$

c) $E[XY^2 \mid X] = X \int_0^1 y^2 \frac{(X+y)}{X+1/2}\, dy = \frac{X}{X+1/2}\left[\frac{X}{3} + \frac{1}{4} \right]$

Úmluva (V 3.24).
$EY^2 < \infty$, $\binom{Y}{Z}$ má libovolné rozdělení
$E(Y - f(Z))^2 \geq E(Y - E(Y|Z))^2$
pro libovolné f měřitelné.
s rovností $\iff f(Z) = E(Y|Z)$ s.j.

podmíněný rozptyl:
$\operatorname{var}(Y|X) := E((Y - E(Y|X))^2 \mid X)$

2) $Y|X \sim N(2X^3, 3X^2) \qquad X \sim R(0,1)$



$E\,Y|X$

a) $E\left[\frac{Y}{X^2} \mid X\right] = \frac{1}{X^2} E[Y \mid X] = \frac{2X^3}{X^2} = 2X$

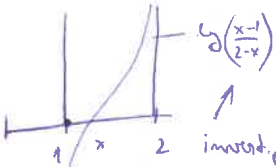
$$E[Y|X] = \int_{\mathbb{R}} y\, f_{Y|X}(x,y)\, dy = \int_{\mathbb{R}} y \text{ "}N(2x^3, 3x^2)\text{" } dy = 2X^3$$

b) $E \frac{Y}{X^2} = E\left(E\left[\frac{Y}{X^2} \mid X\right]\right) = E\, 2X = 2 \cdot 1/2 = 1$

c) $EY = E(E[Y|X]) = E\, 2X^3 = 2 \int_0^1 x^3\, dx = 2 \cdot \frac{1}{4} = \frac{1}{2}$

d) $\operatorname{var} Y = E \operatorname{var}(Y|X) + \operatorname{var} E(Y|X) = E\, 3X^2 + \operatorname{var}(2X^3) = 3 \cdot 1/3 + 4\left(\frac{1}{7} - \left(\frac{1}{4}\right)^2\right) =$
$= 1 + 4/7 - 1/4$

3) $f(x,y) = \frac{1}{x} e^{-y/x} \qquad x \in (1,2),\ y > 0$



$$f_X(x) = \int_0^\infty \frac{1}{x} e^{-y/x}\, dy = \left[\frac{1}{x} e^{-y/x} \cdot \left(-\frac{1}{x^{-1}}\right) \right]_{y=0}^{\infty} = 1 \qquad x \in (1,2)$$

$f_{Y|X}(x,y) = f_{Y|X}(y|x) = \frac{1}{x} e^{-y/x}, \ y \geq 0 \quad Y|X \sim Exp\left(\frac{1}{x}\right) \quad X \sim R(1,2)$

a) $E[Y \mid X = t] = t \quad t \in (1,2), \quad E[Y|X] = X$

b) $E\left[Y \mid \lg\left(\frac{X-1}{2-X}\right) = t\right] = \frac{2e^t + 1}{e^t + 1}$

$$E\Big[Y \mid \underbrace{\lg\left(\frac{X-1}{2-X}\right)}_{T}\Big] = \frac{2e^T + 1}{e^T + 1} = X$$

$\lg \frac{x-1}{2-x} = t$

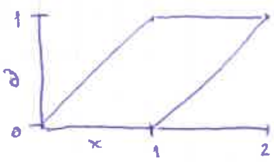
$\frac{x-1}{2-x} = e^t$

$x(1 + e^t) = 2e^t + 1$

$x = (2e^t + 1)/(e^t + 1)$

c) $E\left[\frac{Y}{X^6} \mid \lg\left(\frac{X-1}{2-X}\right)\right] = \frac{1}{X^5}$

4) $f(x,y) = c y\, I_M(x,y) \qquad M = \{ y \in (0,1),\ y \leq x \leq y+1 \} \quad \to c = 2$



$$\int_0^1 \int_y^{y+1} c y\, dx\, dy = c \int_0^1 y\, dy = c \cdot 1/2$$

a) $E\left[911 X - \lg \frac{Y}{1-Y} \mid Y\right] = 911\, E[X|Y] - \lg \frac{Y}{1-Y} = \ast$

~~$Y \sim R(0,1)$~~
~~$X|Y \sim R(Y, Y+1)$~~

$Y \sim 2y \quad y \in (0,1)$
$X|Y \sim R(Y, Y+1)$

$$f_{X|Y}(x|y) = \frac{2y\, I_M(x,y)}{\int_y^{y+1} 2y\, dx} = I_M(x,y)$$

$Y = y$ fixed

$$E[X|Y] = \int x\, I_M(x,y)\, dx = \int_y^{y+1} x\, dx = \left[\frac{x^2}{2}\right]_y^{y+1} = \frac{(y+1)^2 - y^2}{2} = y + \frac{1}{2}$$

$\ast = 911 \cdot (Y + 1/2) - \lg\left(\frac{Y}{1-Y}\right)$

b) $E[\sin X \mid Y] = \int_y^{y+1} \sin x \, dx = [-\cos x]_y^{y+1} = \cos y - \cos(y+1)$

5) a) $E[X+Y \mid X] = X + E[Y \mid X] \underset{\text{nezáv.}}{=} X + EY$

b) $E[X+Y \mid X] = X + E[Y \mid X]$

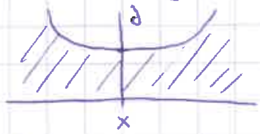
c) $Z = X+Y$

$E[X \mid Z] = E[Z - Y \mid Z] = Z - E[Y \mid Z] \overset{\text{zameniteľnosť}}{=} Z - E[X \mid Z]$

$E[X \mid Z] = Z/2 \Rightarrow E[X \mid X+Y] = \frac{X+Y}{2}$

6) $Y \mid X \sim R(0, x^2+1)$

$X \sim N(0,1)$



$EX^4 = 3!! = 3.1 = 3$

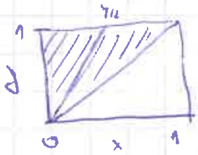
$EX^p = \sigma^p (p-1)!!$ p sudé

a) $E[Y \mid e^X] \underset{\substack{Z = e^X \\ X = \log Z}}{=} E[Y \mid Z] = \frac{(\log Z)^2 + 1}{2} = \frac{X^2+1}{2}$

b) $EY = EE[Y \mid X] = E\frac{X^2+1}{2} = 1$ $\qquad \operatorname{var} R(a,b) = (b-a)^2/12$

c) $\operatorname{var} Y = \operatorname{var}(E[Y \mid X]) + E(\operatorname{var}(Y \mid X)) = \operatorname{var}\frac{X^2+1}{2} + E\frac{(X^2+1)^2}{12} = \frac{3}{4} - \frac{1}{4} + \frac{1}{12}[3 + 2 + 1] = 1$

7) $(X,Y)^T \sim U(M)$ $\qquad M = \{0 < x < 1, x < y\}$



$f_{X,Y}(x,y) = 2 I_M$

$f_{X|Y}(x|y) = \frac{2 I_M}{\int_0^y 2\,dx} = \frac{I_M}{y}$

a) $E[\log X \mid Y] = \int_0^y \frac{\log x}{y} dx = \frac{1}{y}\left([x \log x]_0^y - \int_0^y 1\,dx\right) = \log y - 1$

b) $E[X \mid \log Y] = E[X \mid Y] = \int_0^y \frac{x}{y} dx = \frac{y}{2}$

invertná fce rovnaká

c) $E[\log X \mid \log Y] = \log Y - 1$

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15) $X \sim Exp(\lambda)$ $\qquad g(\lambda) = 1/\lambda^2 \quad [g'(\lambda)]^2 = 4/\lambda^6$ $\qquad EX = \frac{1}{\lambda} \quad \operatorname{var} X = \frac{1}{\lambda^2} \quad EX^2 = \frac{2}{\lambda^2}$

i) $ET = c\,m\,EX_1^2 = c\,m\,2/\lambda^2 \Rightarrow c = 1/2m$

ii) $\operatorname{var} T = \frac{m}{4m^2} \operatorname{var} X_1^2 = 5/\lambda^4 m$ $\qquad EX^k = \int_0^\infty \lambda e^{-\lambda x} x^k dx \overset{\lambda x = t}{=} \int_0^\infty e^{-t} t^k dt / \lambda^k = \frac{k!}{\lambda^k}$

$L(\lambda) = \lambda e^{-\lambda x} \rightsquigarrow \ell = \log \lambda - \lambda x \rightsquigarrow \ell' = 1/\lambda - x \rightsquigarrow \ell'' = -1/\lambda^2 \rightsquigarrow J(\lambda) = 1/\lambda^2$

RC: $\frac{\lambda^2}{m} \cdot \frac{4}{\lambda^6} = \frac{4}{m\lambda^4} < \operatorname{var} T = \frac{5}{\lambda^4 m}$

16) v normálnom modeli sú S^2 a $\bar{X}$ nezávislé $\Rightarrow$ T_1 a T_2 sú nezávislé

ďalej $\quad S^2(m-1) \sim \sigma^2 \chi^2_{m-1}$

$\sqrt{S^2(m-1)} \sim \sigma \chi_{m-1}$

spracovanie Mathematica skripty