Many-valued logics

Adam Přenosil

Institute of Computer Science, Czech Academy of Sciences

Based on:

J. Marcos, A. Přenosil, P. Egré: *Many-Valued Logic*.

Entry for the Stanford Encyclopedia of Philosophy.

https://plato.stanford.edu/entries/logic-manyvalued

Boolean algebra $\mathbf{B}_2$

$\wedge$	t	f
t	t	f
f	f	f

$\vee$	t	f
t	t	t
f	t	f

$\neg$	
t	f
f	t

$$\begin{array}{c} \mathsf{t} \\ | \\ \mathsf{f} \end{array}$$

Atomic valuation $=$ map $v\colon \mathit{Var} \to \mathbf{B}_2$

Valuation $=$ homomorphism $v\colon \mathbf{Fm} \to \mathbf{B}_2$

$$v(A \wedge B) = v(A) \wedge^{\mathbf{B}_2} v(B)$$

$$v(A \vee B) = v(A) \vee^{\mathbf{B}_2} v(B)$$

$$v(\neg A) = \neg^{\mathbf{B}_2} v(A)$$

Boolean algebra $\mathbf{B}_2$

$\wedge$	t	f
t	t	f
f	f	f

$\vee$	t	f
t	t	t
f	t	f

$\neg$	
t	f
f	t

Multiple equivalent ways to define $\Gamma \vdash_{\mathrm{CL}} A$:

Preserving truth:

For each valuation v on $\mathbf{B}_2$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

Preserving non-falsity:

For each valuation v on $\mathbf{B}_2$: if $v(C) > \mathsf{f}$ for all $C \in \Gamma$, then $v(A) > \mathsf{f}$

Preserving order:

For each valuation v on $\mathbf{B}_2$: $v(C_1) \wedge \cdots \wedge v(C_n) \leq v(A)$ for $\Gamma = \{C_1, \ldots, C_n\}$

Third value?

Third value?

- future contingents

Third value?

- future contingents Łukasiewicz 1920

Third value?

- future contingents Łukasiewicz 1920
- neither true nor false (n)

Third value?

- future contingents — Łukasiewicz 1920
- neither true nor false (n) — Kleene 1952

Third value?

- future contingents — Łukasiewicz 1920
- neither true nor false (n) — Kleene 1952
- both true and false (b)

Third value?

- future contingents — Łukasiewicz 1920
- neither true nor false (n) — Kleene 1952
- both true and false (b) — Asenjo 1966, Priest 1979

Third value?

- future contingents — Łukasiewicz 1920
- neither true nor false (n) — Kleene 1952
- both true and false (b) — Asenjo 1966, Priest 1979
- non-sense ($*$)

Third value?

- future contingents — Łukasiewicz 1920
- neither true nor false (n) — Kleene 1952
- both true and false (b) — Asenjo 1966, Priest 1979
- non-sense ($*$) — Bochvar 1938, Halldén 1949

Third value?

• future contingents	Łukasiewicz 1920
• neither true nor false (n)	Kleene 1952
• both true and false (b)	Asenjo 1966, Priest 1979
• non-sense ($*$)	Bochvar 1938, Halldén 1949
• computation did not terminate (ε)	

Third value?

• future contingents	Łukasiewicz 1920
• neither true nor false (n)	Kleene 1952
• both true and false (b)	Asenjo 1966, Priest 1979
• non-sense ($*$)	Bochvar 1938, Halldén 1949
• computation did not terminate (ε)	McCarthy 1959

Third value?

- future contingents	Łukasiewicz 1920
- neither true nor false (n)	Kleene 1952
- both true and false (b)	Asenjo 1966, Priest 1979
- non-sense ($*$)	Bochvar 1938, Halldén 1949
- computation did not terminate (ε)	McCarthy 1959
- …	

Kleene algebra K_3

$\wedge$	t	f	n
t	t	f	n
f	f	f	f
n	n	f	n

$\vee$	t	f	n
t	t	t	t
f	t	f	n
n	t	n	n

$\neg$	
t	f
f	t
n	n

$$\begin{array}{c} \mathsf{t} \\ | \\ \mathsf{n} \\ | \\ \mathsf{f} \end{array}$$

Kleene algebra $\mathbf{K}_3$

$\wedge$	t	f	n
t	t	f	n
f	f	f	f
n	n	f	n

$\vee$	t	f	n
t	t	t	t
f	t	f	n
n	t	n	n

$\neg$	
t	f
f	t
n	n

$$\begin{array}{c} \mathsf{t} \\ | \\ \mathsf{n} \\ | \\ \mathsf{f} \end{array}$$

Strong Kleene logic: $\Gamma \vdash_{\mathrm{KL}} A$

For each valuation v on $\mathbf{K}_3$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

Kleene algebra K_3

$\wedge$	t	f	n
t	t	f	n
f	f	f	f
n	n	f	n

$\vee$	t	f	n
t	t	t	t
f	t	f	n
n	t	n	n

$\neg$	
t	f
f	t
n	n

$$\begin{array}{c} \mathsf{t} \\ | \\ \mathsf{n} \\ | \\ \mathsf{f} \end{array}$$

Strong Kleene logic: $\Gamma \vdash_{\mathrm{KL}} A$

For each valuation v on $\mathbf{K}_3$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

Logic of Paradox: $\Gamma \vdash_{\mathrm{LP}} A$

For each valuation v on $\mathbf{K}_3$: if $v(C) > \mathsf{f}$ for all $C \in \Gamma$, then $v(A) > \mathsf{f}$

Kleene algebra $\mathbf{K}_3$

$\wedge$	t	f	n
t	t	f	n
f	f	f	f
n	n	f	n

$\vee$	t	f	n
t	t	t	t
f	t	f	n
n	t	n	n

$\neg$	
t	f
f	t
n	n

$$\begin{array}{c} \mathsf{t} \\ | \\ \mathsf{n} \\ | \\ \mathsf{f} \end{array}$$

Strong Kleene logic: $\Gamma \vdash_{\mathrm{KL}} A$

For each valuation v on $\mathbf{K}_3$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

Logic of Paradox: $\Gamma \vdash_{\mathrm{LP}} A$

For each valuation v on $\mathbf{K}_3$: if $v(C) > \mathsf{f}$ for all $C \in \Gamma$, then $v(A) > \mathsf{f}$

Kleene's Logic of Order: $\Gamma \vdash_{\mathrm{KO}} A$

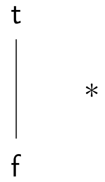
For each valuation v on $\mathbf{K}_3$: $v(C_1) \wedge \cdots \wedge v(C_n) \leq v(A)$ for $\Gamma = \{C_1, \ldots, C_n\}$

Weak Kleene algebra WK_3

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$*$
$*$	$*$	$*$	$*$

$\vee$	t	f	$*$
t	t	t	$*$
f	t	f	$*$
$*$	$*$	$*$	$*$

$\neg$	
t	f
f	t
$*$	$*$

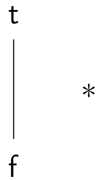


Weak Kleene algebra $\mathbf{WK}_3$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$*$
$*$	$*$	$*$	$*$

$\vee$	t	f	$*$
t	t	t	$*$
f	t	f	$*$
$*$	$*$	$*$	$*$

$\neg$	
t	f
f	t
$*$	$*$



Bochvar Logic: $\Gamma \vdash_{\mathrm{B}} A$

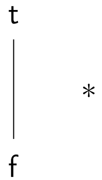
For each valuation v on $\mathbf{WK}_3$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

Weak Kleene algebra $\mathbf{WK}_3$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$*$
$*$	$*$	$*$	$*$

$\vee$	t	f	$*$
t	t	t	$*$
f	t	f	$*$
$*$	$*$	$*$	$*$

$\neg$	
t	f
f	t
$*$	$*$



Bochvar Logic: $\Gamma \vdash_{\mathrm{B}} A$

For each valuation v on $\mathbf{WK}_3$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

Paraconsistent Weak Kleene: $\Gamma \vdash_{\mathrm{PWK}} A$

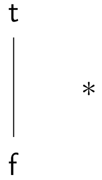
For each valuation v on $\mathbf{K}_3$: if $v(C) \neq \mathsf{f}$ for all $C \in \Gamma$, then $v(A) \neq \mathsf{f}$

Weak Kleene algebra WK_3

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$*$
$*$	$*$	$*$	$*$

$\vee$	t	f	$*$
t	t	t	$*$
f	t	f	$*$
$*$	$*$	$*$	$*$

$\neg$	
t	f
f	t
$*$	$*$



These are related to classical logic by **variable inclusion** conditions:

$$\Gamma \vdash_{\mathrm{B}} A \iff \text{ either } \Gamma \vdash_{\mathrm{CL}} A \text{ and } \mathrm{Var}(A) \subseteq \mathrm{Var}(\Gamma), \text{ or } \Gamma \vdash_{\mathrm{CL}} \bot.$$

$$\Gamma \vdash_{\mathrm{PWK}} A \iff \Delta \vdash_{\mathrm{CL}} A \text{ for some } \Delta \subseteq \Gamma \text{ with } \mathrm{Var}(\Delta) \subseteq \mathrm{Var}(A).$$

McCarthy algebra $\mathbf{M}_3$

$\wedge$	t	f	ε
t	t	f	ε
f	f	f	f
ε	ε	ε	ε

$\vee$	t	f	ε
t	t	t	t
f	t	f	ε
ε	ε	ε	ε

$\neg$	
t	f
f	t
ε	ε

McCarthy algebra $\mathbf{M}_3$

$\wedge$	t	f	ε
t	t	f	ε
f	f	f	f
ε	ε	ε	ε

$\vee$	t	f	ε
t	t	t	t
f	t	f	ε
ε	ε	ε	ε

$\neg$	
t	f
f	t
ε	ε

McCarthy Logic: $\Gamma \vdash_{\mathrm{M}} A$

For each valuation v on $\mathbf{WK}_3$: if $v(C) = \mathsf{t}$ for all $C \in \Gamma$, then $v(A) = \mathsf{t}$

What kind of thing is the relation $\Gamma \vdash A$?

What kind of thing is the relation $\Gamma \vdash A$?

A **structural consequence relation** over some set of formulas:

- reflexive: if $A \in \Gamma$, then $\Gamma \vdash A$
- monotonic: if $\Gamma \vdash A$, then $\Gamma, \Delta \vdash A$,
- transitive: if $\Phi \vdash C$ for each $C \in \Gamma$ and $\Gamma \vdash A$, then $\Phi \vdash A$
- structural: if $\Gamma \vdash A$, then $\sigma[\Gamma] \vdash \sigma(A)$ for each substitution σ

We will simply call such relations **logics**.

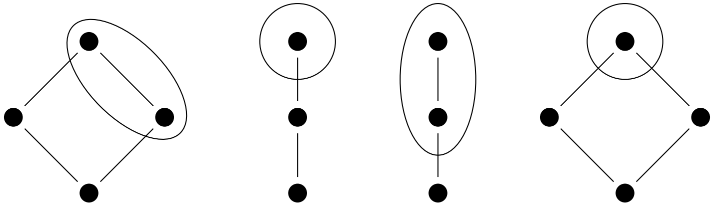
What kind of structures provide semantics for logics?

What kind of structures provide semantics for logics?

A **logical matrix** is a pair $\langle \mathbf{A}, F \rangle$ consisting of an algebra $\mathbf{A}$ (a set with some finitary operations) and a set of **designated values** $F \subseteq A$.

What kind of structures provide semantics for logics?

A **logical matrix** is a pair $\langle \mathbf{A}, F\rangle$ consisting of an algebra $\mathbf{A}$ (a set with some finitary operations) and a set of **designated values** $F \subseteq A$.



A rule $\Gamma \vdash A$ **holds** in a matrix $\langle \mathbf{A}, F \rangle$ if

$$\text{for each valuation } v \text{ on } \mathbf{A}, \text{ if } v[\Gamma] \subseteq F, \text{ then } v(A) \in F.$$

A rule $\Gamma \vdash A$ **holds** in a matrix $\langle \mathbf{A}, F \rangle$ if

$$\text{for each valuation } v \text{ on } \mathbf{A}, \text{ if } v[\Gamma] \subseteq F, \text{ then } v(A) \in F.$$

Each class of matrices K determines a logic $\mathrm{L} = \mathrm{Log}\,\mathsf{K}$:

$$\Gamma \vdash_{\mathsf{L}} A \iff \Gamma \vdash A \text{ holds in each matrix } \langle \mathbf{A}, F \rangle \in \mathsf{K}.$$

A rule $\Gamma \vdash A$ **holds** in a matrix $\langle \mathbf{A}, F \rangle$ if

$$\text{for each valuation } v \text{ on } \mathbf{A}, \text{ if } v[\Gamma] \subseteq F, \text{ then } v(A) \in F.$$

Each class of matrices K determines a logic $\mathrm{L} = \mathrm{Log}\,\mathsf{K}$:

$$\Gamma \vdash_{\mathsf{L}} A \iff \Gamma \vdash A \text{ holds in each matrix } \langle \mathbf{A}, F \rangle \in \mathsf{K}.$$

A matrix $\langle \mathbf{A}, F \rangle$ is a **model** of a logic L if it validates each rule $\Gamma \vdash_{\mathrm{L}} A$ of L.

A rule $\Gamma \vdash A$ **holds** in a matrix $\langle \mathbf{A}, F \rangle$ if

$$\text{for each valuation } v \text{ on } \mathbf{A}, \text{ if } v[\Gamma] \subseteq F, \text{ then } v(A) \in F.$$

Each class of matrices K determines a logic $\mathrm{L} = \mathrm{Log}\,\mathsf{K}$:

$$\Gamma \vdash_{\mathsf{L}} A \iff \Gamma \vdash A \text{ holds in each matrix } \langle \mathbf{A}, F \rangle \in \mathsf{K}.$$

A matrix $\langle \mathbf{A}, F \rangle$ is a **model** of a logic L if it validates each rule $\Gamma \vdash_{\mathrm{L}} A$ of L.

Theorem (Wójcicki). Each logic is determined by some set of matrices.

A rule $\Gamma \vdash A$ **holds** in a matrix $\langle \mathbf{A}, F \rangle$ if

$$\text{for each valuation } v \text{ on } \mathbf{A}, \text{ if } v[\Gamma] \subseteq F, \text{ then } v(A) \in F.$$

Each class of matrices K determines a logic $\mathrm{L} = \mathrm{Log}\,\mathsf{K}$:

$$\Gamma \vdash_{\mathsf{L}} A \iff \Gamma \vdash A \text{ holds in each matrix } \langle \mathbf{A}, F \rangle \in \mathsf{K}.$$

A matrix $\langle \mathbf{A}, F \rangle$ is a **model** of a logic L if it validates each rule $\Gamma \vdash_{\mathrm{L}} A$ of L.

Theorem (Wójcicki). Each logic is determined by some set of matrices.

Equivalently, $\mathrm{L} = \mathrm{Log}(\mathrm{Mod}\,\mathrm{L})$ where $\mathrm{Mod}\,\mathrm{L}$ is the class of all models of L.

A rule $\Gamma \vdash A$ **holds** in a matrix $\langle \mathbf{A}, F \rangle$ if

$$\text{for each valuation } v \text{ on } \mathbf{A}, \text{ if } v[\Gamma] \subseteq F, \text{ then } v(A) \in F.$$

Each class of matrices K determines a logic $\mathrm{L} = \operatorname{Log} \mathsf{K}$:

$$\Gamma \vdash_{\mathsf{L}} A \iff \Gamma \vdash A \text{ holds in each matrix } \langle \mathbf{A}, F \rangle \in \mathsf{K}.$$

A matrix $\langle \mathbf{A}, F \rangle$ is a **model** of a logic L if it validates each rule $\Gamma \vdash_{\mathrm{L}} A$ of L.

Theorem (Wójcicki). Each logic is determined by some set of matrices.

Equivalently, $\mathrm{L} = \operatorname{Log}(\operatorname{Mod} \mathrm{L})$ where $\operatorname{Mod} \mathrm{L}$ is the class of all models of L.

Proof. Take K to be the set of all matrices of the form $\langle \mathbf{Fm}, T \rangle$, where T ranges over all **theories** of L (sets $T \subseteq Fm$ such that $T \vdash_{\mathrm{L}} A$ implies $A \in T$).

Example. Kleene's Logic of Order is determined by $\{\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle, \langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle\}$.

Example. Kleene's Logic of Order is determined by $\{\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle, \langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle\}$.

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \cap \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \cap \mathrm{L}_2} A \iff \Gamma \vdash_{\mathrm{L}_1} A \text{ and } \Gamma \vdash_{\mathrm{L}_2} A.$$

Example. Kleene's Logic of Order is determined by $\{\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle, \langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle\}$.

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \cap \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \cap \mathrm{L}_2} A \iff \Gamma \vdash_{\mathrm{L}_1} A \text{ and } \Gamma \vdash_{\mathrm{L}_2} A.$$

If $\mathrm{L}_1 = \mathrm{Log}\,\mathsf{K}_1$ and $\mathrm{L}_2 = \mathrm{Log}\,\mathsf{K}_2$, then $\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{Log}(\mathsf{K}_1 \cup \mathsf{K}_2)$.

Example. Kleene's Logic of Order is determined by $\{\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle, \langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle\}$.

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \cap \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \cap \mathrm{L}_2} A \iff \Gamma \vdash_{\mathrm{L}_1} A \text{ and } \Gamma \vdash_{\mathrm{L}_2} A.$$

If $\mathrm{L}_1 = \operatorname{Log} \mathsf{K}_1$ and $\mathrm{L}_2 = \operatorname{Log} \mathsf{K}_2$, then $\mathrm{L}_1 \cap \mathrm{L}_2 = \operatorname{Log}(\mathsf{K}_1 \cup \mathsf{K}_2)$.

(Careful, $\operatorname{Mod}(\mathrm{L}_1 \cap \mathrm{L}_2) \supsetneq \operatorname{Mod} \mathrm{L}_1 \cup \operatorname{Mod} \mathrm{L}_2$, except in some trivial cases.)

Example. Kleene's Logic of Order is determined by $\{\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle, \langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle\}$.

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \cap \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \cap \mathrm{L}_2} A \iff \Gamma \vdash_{\mathrm{L}_1} A \text{ and } \Gamma \vdash_{\mathrm{L}_2} A.$$

If $\mathrm{L}_1 = \operatorname{Log} \mathsf{K}_1$ and $\mathrm{L}_2 = \operatorname{Log} \mathsf{K}_2$, then $\mathrm{L}_1 \cap \mathrm{L}_2 = \operatorname{Log}(\mathsf{K}_1 \cup \mathsf{K}_2)$.

(Careful, $\operatorname{Mod}(\mathrm{L}_1 \cap \mathrm{L}_2) \supsetneq \operatorname{Mod} \mathrm{L}_1 \cup \operatorname{Mod} \mathrm{L}_2$, except in some trivial cases.)

Example. $\mathrm{KO} = \mathrm{KL} \cap \mathrm{LP}$, where $\mathrm{KL} = \operatorname{Log}\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ and $\mathrm{LP} = \operatorname{Log}\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle$.

A logic L is **axiomatized** by a set of rules $R = \{\Gamma_i \vdash A_i \mid i \in I\}$, symbolically $\mathrm{L} = \mathrm{Ax}R$, if it is the smallest logic which contains (validates) these rules.

A logic L is **axiomatized** by a set of rules $R = \{\Gamma_i \vdash A_i \mid i \in I\}$, symbolically $\mathrm{L} = \mathrm{Ax}\,R$, if it is the smallest logic which contains (validates) these rules.

Equivalently, $\Gamma \vdash_{\mathrm{L}} A$ iff A can be proved from Γ via instances of these rules, in the normal sense that it **has a Hilbert-style proof** using the rules.

A logic L is **axiomatized** by a set of rules $R = \{\Gamma_i \vdash A_i \mid i \in I\}$, symbolically $\mathrm{L} = \mathrm{Ax}R$, if it is the smallest logic which contains (validates) these rules.

Equivalently, $\Gamma \vdash_{\mathrm{L}} A$ iff A can be proved from Γ via instances of these rules, in the normal sense that it **has a Hilbert-style proof** using the rules.

Example. CL is axiomatized by the following set of rules ($x \to y := \neg x \vee y$):

$$
\begin{aligned}
& x, x \to y \vdash y \\
& \vdash x \to (y \to x) \\
& \vdash (x \to (y \to z)) \to ((x \to y) \to (x \to z)) \\
& \vdash (\neg x \to \neg y) \to ((\neg x \to y) \to x) \\
& \ldots
\end{aligned}
$$

Example. $\mathrm{LP} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{b}\}\rangle$ is axiomatized by the following rules plus the Law of the Excluded Middle ($\emptyset \vdash p \vee \neg p$):

$$\text{(R1)}\ \frac{p \wedge q}{p} \qquad \text{(R2)}\ \frac{p \wedge q}{q} \qquad \text{(R3)}\ \frac{p \quad q}{p \wedge q}$$

$$\text{(R4)}\ \frac{p}{p \vee q} \qquad \text{(R5)}\ \frac{p \vee q}{q \vee p} \qquad \text{(R6)}\ \frac{p \vee p}{p}$$

$$\text{(R7)}\ \frac{p \vee (q \vee r)}{(p \vee q) \vee r} \qquad \text{(R8)}\ \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)} \qquad \text{(R9)}\ \frac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$$

$$\text{(R10)}\ \frac{p \vee r}{\neg\neg p \vee r} \qquad \text{(R12)}\ \frac{\neg(p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r} \qquad \text{(R14)}\ \frac{\neg(p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r}$$

$$\text{(R11)}\ \frac{\neg\neg p \vee r}{p \vee r} \qquad \text{(R13)}\ \frac{(\neg p \wedge \neg q) \vee r}{\neg(p \vee q) \vee r} \qquad \text{(R15)}\ \frac{(\neg p \vee \neg q) \vee r}{\neg(p \wedge q) \vee r}$$

Also, $\mathsf{K} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ is axiomatized by the above plus MP ($p, \neg p \vee q \vdash q$).

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \vee \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \vee \mathrm{L}_2} A \iff A \text{ is provable from } \Gamma \text{ using the rules of } \mathrm{L}_1 \text{ and } \mathrm{L}_2.$$

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \vee \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \vee \mathrm{L}_2} A \iff A \text{ is provable from } \Gamma \text{ using the rules of } \mathrm{L}_1 \text{ and } \mathrm{L}_2.$$

If $\mathrm{L}_1 = \mathrm{Ax}\, R_1$ and $\mathrm{L}_2 = \mathrm{Ax}\, R_2$, then $\mathrm{L}_1 \vee \mathrm{L}_2 = \mathrm{Ax}(R_1 \cup R_2)$.

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \vee \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \vee \mathrm{L}_2} A \iff A \text{ is provable from } \Gamma \text{ using the rules of } \mathrm{L}_1 \text{ and } \mathrm{L}_2.$$

If $\mathrm{L}_1 = \mathrm{Ax}\, R_1$ and $\mathrm{L}_2 = \mathrm{Ax}\, R_2$, then $\mathrm{L}_1 \vee \mathrm{L}_2 = \mathrm{Ax}(R_1 \cup R_2)$.

A logic L_2 is axiomatized **relative to** L_1 by a set of rules R if $\mathrm{L}_2 = \mathrm{Ax}(\mathrm{L}_1 \cup R)$.

Fact. Given logics L_1 and L_2, there is a logic $\mathrm{L}_1 \vee \mathrm{L}_2$ such that

$$\Gamma \vdash_{\mathrm{L}_1 \vee \mathrm{L}_2} A \iff A \text{ is provable from } \Gamma \text{ using the rules of } \mathrm{L}_1 \text{ and } \mathrm{L}_2.$$

If $\mathrm{L}_1 = \mathrm{Ax}\, R_1$ and $\mathrm{L}_2 = \mathrm{Ax}\, R_2$, then $\mathrm{L}_1 \vee \mathrm{L}_2 = \mathrm{Ax}(R_1 \cup R_2)$.

A logic L_2 is axiomatized **relative to** L_1 by a set of rules R if $\mathrm{L}_2 = \mathrm{Ax}(\mathrm{L}_1 \cup R)$.

Example. $\mathrm{CL} = \mathrm{KL} \vee \mathrm{LP}$, where $\mathrm{KL} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ and $\mathrm{LP} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle$.

Example. $\mathrm{CL} = \mathrm{KL} \vee \mathrm{LP}$, where $\mathrm{KL} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ and $\mathrm{LP} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle$.

Lemma. $\mathrm{KL} \subseteq \mathrm{CL}$ and $\mathrm{LP} \subseteq \mathrm{CL}$ because the matrix $\langle \mathbf{B}_2, \{\mathsf{t}\}\rangle$ is a **submatrix** of $\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ and $\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle$ ($\langle \mathbf{A}, F\rangle \leq \langle \mathbf{B}, G\rangle$ if $\mathbf{A} \leq \mathbf{B}$ and $F = A \cap G$).

Example. $\mathrm{CL} = \mathrm{KL} \vee \mathrm{LP}$, where $\mathrm{KL} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ and $\mathrm{LP} = \mathrm{Log}\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle$.

Lemma. $\mathrm{KL} \subseteq \mathrm{CL}$ and $\mathrm{LP} \subseteq \mathrm{CL}$ because the matrix $\langle \mathbf{B}_2, \{\mathsf{t}\}\rangle$ is a **submatrix** of $\langle \mathbf{K}_3, \{\mathsf{t}\}\rangle$ and $\langle \mathbf{K}_3, \{\mathsf{t}, \mathsf{n}\}\rangle$ $\quad$ ($\langle \mathbf{A}, F\rangle \leq \langle \mathbf{B}, G\rangle$ if $\mathbf{A} \leq \mathbf{B}$ and $F = A \cap G$).

Lemma. The rule of Modus Ponens (Disjunctive Syllogism) holds in KL:

$$x, \neg x \vee y \vdash_{\mathrm{KL}} y.$$

Fact. CL is axiomatized relative to LP by Modus Ponens ($x, \neg x \vee y \vdash y$).

Define the **information order** $\sqsubseteq$ on $\mathbf{K}_3$ as: $\mathsf{n} \sqsubseteq \mathsf{t}$, $\mathsf{n} \sqsubseteq \mathsf{f}$.

For valuations v, w on $\mathbf{K}_3$ define

$$v \sqsubseteq w \iff v(x) \sqsubseteq w(x) \text{ for each variable } x.$$

Equivalently, by induction over the complexity of A,

$$v \sqsubseteq w \iff v(A) \sqsubseteq w(A) \text{ for each formula } A.$$

A valuation w on $\mathbf{K}_3$ is **classical** if $w(x) \in \{\mathsf{t}, \mathsf{f}\}$ for each variable x,
or equivalently if $w(A) \in \{\mathsf{t}, \mathsf{f}\}$ for each formula A.
A classical valuation on $\mathbf{K}_3$ is in effect the same as a valuation on $\mathbf{B}_2$.

For each valuation v on $\mathbf{K}_3$ there is a classical valuation w with $v \sqsubseteq w$.

Fact. LP and CL have the same **theorems**:

$$\emptyset \vdash_{\mathrm{CL}} A \iff \emptyset \vdash_{\mathrm{LP}} A.$$

Proof. If $\emptyset \nvdash_{\mathrm{CL}} A$, there is a valuation v on $\mathbf{B}_2$ with $v(A) = \mathsf{f}$, which is also a classical valuation v on $\mathbf{K}_3$ with $v(A) = \mathsf{f}$, so $\emptyset \nvdash_{\mathrm{LP}} A$.

If $\emptyset \nvdash_{\mathrm{LP}} A$, there is a valuation v on $\mathbf{K}_3$ with $v(A) = \mathsf{f}$.
Take a classical valuation w on $\mathbf{K}_3$ with $v \sqsubseteq w$.
Then $v(A) \sqsubseteq w(A)$, so $w(A) = \mathsf{f}$. Thus w witnesses that $\emptyset \nvdash_{\mathrm{CL}} A$.

Corollary. CL $=$ LP $+$ Modus Ponens $(x, \neg x \vee y \vdash y)$.

Proof. By the completeness theorem for CL, CL is axiomatized by some particular set of theorems of CL plus Modus Ponens.

Fact. For all formulas A and B

$$A \vdash_{\mathrm{CL}} B \iff \vdash_{\mathrm{LP}} \neg A \vee B.$$

Proof. If $\vdash_{\mathrm{LP}} \neg A \vee B$, then $\vdash_{\mathrm{CL}} \neg A \vee B$ because $\mathrm{LP} \subseteq \mathrm{CL}$, so $A \vdash_{\mathrm{CL}} B$ by an application of Modus Ponens in CL.

If $A \nvdash_{\mathrm{LP}} \neg A \vee B$, there is a valuation v on $\mathbf{K}_3$ with $v(\neg A \vee B) = \mathsf{f}$.
But $v(\neg A \vee B) = \mathsf{f}$ implies $v(\neg A) = \mathsf{f}$ and $v(B) = \mathsf{f}$, so $v(A) = \mathsf{t}$.
Take a classical valuation w with $v \sqsubseteq w$. Then $w(A) = \mathsf{t}$ and $w(B) = \mathsf{f}$.
Thus w witnesses that $A \nvdash_{\mathrm{CL}} B$.

Corollary. CL = LP + Modus Ponens $(x, \neg x \vee y \vdash y)$.

Proof. If B is provable from A in CL, then $\neg A \vee B$ is a theorem of LP.
Applying Modus Ponens on $A, \neg A \vee B$ now yields B.

Fact. Logics (in a given signature = choice of connectives and variables) form a lattice ordered by $\subseteq$, with meets $L_1 \cap L_2$ and joins $L_1 \vee L_2$.

Fact. Logics (in a given signature = choice of connectives and variables) form a lattice ordered by $\subseteq$, with meets $\mathrm{L}_1 \cap \mathrm{L}_2$ and joins $\mathrm{L}_1 \vee \mathrm{L}_2$.

The extensions of a given logic L form a lattice denoted by $\operatorname{Ext} \mathrm{L}$. The top element of $\operatorname{Ext} \mathrm{L}$ is the **trivial logic** Triv: $\quad \Gamma \vdash_{\mathrm{Triv}} A$ for all Γ and A.

Fact. Logics (in a given signature = choice of connectives and variables) form a lattice ordered by $\subseteq$, with meets $\mathrm{L}_1 \cap \mathrm{L}_2$ and joins $\mathrm{L}_1 \vee \mathrm{L}_2$.

The extensions of a given logic L form a lattice denoted by $\operatorname{Ext} \mathrm{L}$. The top element of $\operatorname{Ext} \mathrm{L}$ is the **trivial logic** Triv: $\quad \Gamma \vdash_{\mathrm{Triv}} A$ for all Γ and A.

Example. $\operatorname{Ext} \mathrm{CL} = \{\mathrm{CL} < \mathrm{Triv}\}$. That is, CL is a **maximal** logic.

Fact. Logics (in a given signature = choice of connectives and variables) form a lattice ordered by $\subseteq$, with meets $\mathrm{L}_1 \cap \mathrm{L}_2$ and joins $\mathrm{L}_1 \vee \mathrm{L}_2$.

The extensions of a given logic L form a lattice denoted by $\mathrm{Ext}\,\mathrm{L}$. The top element of $\mathrm{Ext}\,\mathrm{L}$ is the **trivial logic** Triv: $\qquad \Gamma \vdash_{\mathrm{Triv}} A$ for all Γ and A.

Example. $\mathrm{Ext}\,\mathrm{CL} = \{\mathrm{CL} < \mathrm{Triv}\}$. That is, CL is a **maximal** logic.

Theorem (Pynko). $\mathrm{Ext}\,\mathrm{KO}$ has exactly 7 elements,* namely KO, LP, KL, CL, Triv, and KO + (ECQ) and LP + (ECQ), where $\qquad$ ECQ: $\; x, \neg x \vdash y$.

* If the constants $\top$ and $\bot$ are present in the signature.

Nuel Belnap, 1977: *How a computer should think*.

Nuel Belnap, 1977: *A useful four-valued logic*.

Nuel Belnap, 1977: *How a computer should think*.

Nuel Belnap, 1977: *A useful four-valued logic*.

Main idea: record truth and falsity (assertion and denial) independently.

Nuel Belnap, 1977: *How a computer should think*.

Nuel Belnap, 1977: *A useful four-valued logic*.

Main idea: record truth and falsity (assertion and denial) independently.

An **atomic Belnap valuation** determines which atomic variables are asserted and which are denied (could be both, could be neither).

Each atomic Belnap valuation extends to a **Belnap valuation**:

$A \wedge B$ is asserted	iff	A is asserted and B is asserted
$A \wedge B$ is denied	iff	A is denied or B is denied
$A \vee B$ is asserted	iff	A is asserted or B is asserted
$A \vee B$ is denied	iff	A is denied and B is denied
$\neg A$ is asserted	iff	A is denied
$\neg A$ is denied	iff	A is asserted

$$\Gamma \vdash_{\mathrm{BD}} A \iff \text{for each Belnap valuation } v$$
$$(\text{each } C \in \Gamma \text{ is asserted in } v \implies A \text{ is asserted in } v)$$

$$\Gamma \vdash_{\mathrm{BD}} A \iff \text{for each Belnap valuation } v$$
$$(\text{each } C \in \Gamma \text{ is not denied in } v \implies A \text{ is not denied in } v)$$

We can treat Belnap valuations as valuations in $\{\mathsf{t}, \mathsf{f}, \mathsf{n}, \mathsf{b}\}$:

$v(A) = \mathsf{t}$ if $v(A)$ is asserted but not denied,
$v(A) = \mathsf{f}$ if $v(A)$ is denied but not asserted,
$v(A) = \mathsf{n}$ if $v(A)$ is neither asserted nor denied,
$v(A) = \mathsf{b}$ if $v(A)$ is both true and denied.

$$\Gamma \vdash_{\mathrm{BD}} A \iff \text{for each valuation } v \text{ on } \{\mathsf{t}, \mathsf{f}, \mathsf{n}, \mathsf{b}\}$$
$$(v[\Gamma] \subseteq \{\mathsf{t}, \mathsf{b}\} \implies v(A) \in \{\mathsf{t}, \mathsf{b}\})$$

$$\Gamma \vdash_{\mathrm{BD}} A \iff \text{for each valuation } v \text{ on } \{\mathsf{t}, \mathsf{f}, \mathsf{n}, \mathsf{b}\}$$
$$(v[\Gamma] \subseteq \{\mathsf{t}, \mathsf{n}\} \implies v(A) \in \{\mathsf{t}, \mathsf{n}\})$$

We can treat Belnap valuations as valuations in $\{\mathsf{t}, \mathsf{f}, \mathsf{n}, \mathsf{b}\}$:

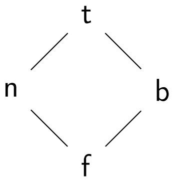
$v(A) = \mathsf{t}$ if $v(A)$ is asserted but not denied,
$v(A) = \mathsf{f}$ if $v(A)$ is denied but not asserted,
$v(A) = \mathsf{n}$ if $v(A)$ is neither asserted nor denied,
$v(A) = \mathsf{b}$ if $v(A)$ is both true and denied.

De Morgan algebra $\mathbf{DM}_4$

$\wedge$	t	f	n	b
t	t	f	n	b
f	f	f	f	f
n	n	f	n	f
b	b	f	f	b

$\vee$	t	f	n	b
t	t	t	t	t
f	t	f	n	b
n	t	n	n	t
b	t	b	t	b

$\neg$	
t	f
f	t
n	n
b	b



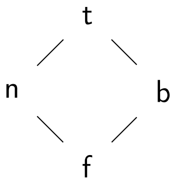
$\mathbf{DM}_4$

$$\Gamma \vdash_{\mathrm{BD}} A \iff \bigwedge v[\Gamma] \leq v(A).$$

$\mathbf{B}_2$

$\mathbf{K}_3$

$\mathbf{P}_3$



$\mathbf{DM}_4$

Fact. BD has the **variable-sharing property**:

$$\text{if } \Gamma \vdash_{\mathrm{BD}} A, \text{ then } \Gamma \text{ and } A \text{ must share at least one variable.}$$

Proof. If not, assign b to all variables in Γ and n to all variables in A.

This is sometimes seen as a minimal condition for **relevance logics**.

Fact. Each formula in BD is equivalent to a formula in conjunctive normal form (or disjunctive normal form), which is **essentially unique**.

Contrast: $(\neg x \vee y) \wedge (\neg y \vee z) \wedge (\neg z \vee y) \quad \equiv_{\mathrm{CL}} \quad (\neg y \vee x) \wedge (\neg z \vee y) \wedge (\neg x \vee z)$.

Gödel algebra $[0,1]_G$

$$\begin{tikzcd} \mathbf{1} \arrow[d, dash] \\ \mathbf{0} \end{tikzcd}$$

$$x \wedge y := \min(x,y), \qquad\qquad x \vee y := \max(x,y),$$

$$x \to y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x \not\leq y. \end{cases}, \qquad\qquad \neg x := x \to 0.$$

$$\Gamma \vdash_{\mathrm{G}} A \iff (v[\Gamma] \subseteq \{1\} \implies v(A) = 1) \text{ for each valuation } v \text{ on } [0,1]_G$$

$$\Gamma \vdash_{\mathrm{G}} A \iff \bigwedge v[\Gamma] \leq v(A)$$

Fact. Gödel logic retains the **deduction theorem** of classical logic:

$$\Gamma, A \vdash_{\mathrm{G}} B \iff \Gamma \vdash_{\mathrm{G}} A \to B.$$

Proof. In the algebra $[0,1]_G$: for all $a, b, c \in [0,1]_G$

$$a \wedge b \leq c \iff a \leq b \to c.$$

Fact. The Law of the Excluded Middle fails in G: $\emptyset \nvdash_{\mathrm{G}} x \vee \neg x$.

But we still have $\emptyset \vdash_{\mathrm{G}} (x \to y) \vee (y \to x)$.

Łukasiewicz algebra $[0,1]_{Ł}$

$$\begin{tikzcd} \mathbf{1} \arrow[d, dash] \\ \mathbf{0} \end{tikzcd}$$

$$x \wedge y := \min(x,y), \qquad x \vee y := \max(x,y), \qquad \neg x := 1 - x,$$

$$x \oplus y := \min(x+y,1), \qquad x \odot y := \neg(\neg x \oplus \neg y) = \max(x+y-1,0).$$

$$x \to y := \neg x \oplus y = \min(1-x+y,1).$$

$$\Gamma \vdash_{Ł} A \iff (v[\Gamma] \subseteq \{1\} \implies v(A) = 1) \text{ for each valution } v \text{ on } [0,1]_{Ł}$$

Recall that

$$x \to y := \neg x \oplus y = \min(1 - x + y, 1).$$

Fact. For all $x, y \in [0,1]_{Ł}$:

$$x \to y = 1 \iff x \leq y$$

Fact. Łukasiewicz logic satisfies Modus Ponens:

$$A, A \to B \vdash_{Ł} B.$$

Fact. Łukasiewicz logic has the following form of the deduction theorem:

$$\Gamma, A \vdash_{Ł} B \iff \Gamma \vdash_{Ł} A^n \to B.$$

Fact. Each logic L determined by a finite set of finite matrices is **finitary**:

$$\Gamma \vdash_{\mathrm{L}} A \iff \Delta \vdash_{\mathrm{L}} A \text{ for some finite } \Delta \subseteq \Gamma.$$

Fact. Gödel logic is finitary. Łukasiewicz logic is not:

$$x \to y,\ x \to y \odot y,\ x \to y \odot y \odot y,\ \cdots \vdash_{\text{Ł}} \neg x \vee y,$$

but

$$x \to y,\ x \to y \odot y,\ x \to y \odot y \odot y,\ \ldots, x \to y^n \nvdash_{\text{Ł}} \neg x \vee y,$$

Further introductory reading:

G. Priest: An Introduction to Non-Classical Logic (2nd ed., 2012)

J. Peregrin: Logika a logiky (2004)
`http://jarda.peregrin.cz/mybibl/PDFTxt/455.pdf`