BACHELOR THESIS

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Tarski axioms of Euclidean geometry

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Abstract: Tarski's axioms of geometry are a first-order axiomatization of elementary Euclidean geometry. Following Schwabhäuser, Szmielew and Tarski, and similarly to Hilbert, the axiomatic development of geometry inside the theory is explored, up to building an algebraic field inside a geometric space. The theory of real-closed fields (RCF) is introduced as a suitable algebraic structure for studying geometry. We prove that RCF has elimination of quantifiers, thus its decidability and completeness can be established. Using the Tarski's representation theorem, decidability and completeness then applies for Tarski's axiomatization as well.

Keywords: Euclidean geometry, Tarski, Hilbert, the theory RCF

Název práce: Tarského axiomatika eukleidovské geometrie

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Abstrakt: Tarského axiomatizace geometrie je prvořádovou teorií elementární eukleidovské geometrie. Věnujeme se axiomatické výstavbě geometrie uvnitř této teorie až k výstavbě algebraického tělesa uvnitř geometrického prostoru, následujeme tak Schwabhäusera, Szmielew a Tarského, výstabu provádíme obdobně jako Hilbert. Teorie reálně uzavřených těles (RCF) je představena jako vhodná algebraická struktura pro zkoumání geometrie. V práci je ukázáno, že teorie RCF má eliminaci kvantifikátorů, z čehož vyvozujeme rozhodnutelnost a úplnost této teorie. Užitím Tarského věty o reprezentaci můžeme výsledky o rozhodnutelnosti a úplnosti přenést i na Tarského axiomatizaci.

Klíčová slova: eukleidovská geometrie, Tarski, Hilbert, teorie RCF

Contents

Introduction

Since antiquity, studying geometry revealed deep mathematical, or better *metamathematical*, questions, many of them logical in nature. Euclid's *Elements* proposed the use of axiomatic method in geometry, i.e., to deductively infer theorems from a small set of basic truths (axioms). From modern perspective, the treatise in Elements was flawed in a number of respects, such as non-trivial hidden presumptions, which complicated the work of mathematicians in the next two millennia. (Of course we cannot blame Euclid and his colleagues, since they considered the axioms as self-evident truths, the validity of which is based in "reality", and thus it was natural to hide some even more self-evident truths.) The famous problem of *independence of Fifth Euclid's postulate* arose from these gaps in reasoning. Its solution in the 19th century, among other, sparked the study of (modern) mathematical logic.

Arguably the greatest contribution to the axiomatic method and to its use in geometry was brought by David Hilbert in *Grundlagen der Geometrie* (translation in [Hil05]). First, he argued that his theory is consistent, i.e. logical contradiction cannot be deduced from them in finite number of steps. Contrary to mathematical tradition at that time (voiced perhaps the most by Frege in the *Frege-Hilbert controversy*, see [Bla24]), the proof of consistency was done by constructing a model of the theory in the field of real numbers (*arithmetic*). Thus consistency of geometry was reduced to consistency of arithmetic. Second, to once and for all settle the problem of independence of axioms, Hilbert constructed his axiomatization in such a way, that the independence of each axiom from the others can be *proven* by proving the consistency of the set of other axioms with the negation of axiom in question, again by constructing a model.

Thus Hilbert's treatise of geometry and his other works significantly contributed to the modern understanding of notions like consistency and independence (establishing model theory). Furthermore, his work lead directly to the modern understanding of syntactic notions, such as formal proofs and decidability (establishing proof theory) [Dea20].

This thesis focuses on so-called *Tarski's axioms of geometry*, or *axiomatization of geometry in Tarski's style*, introduced by Alfred Tarski. This approach is different from many others in being formulated in purely first-order logic[1], and thus more involved metamathematical methods can be used.

Numerous similar axiomatizations of geometry in Tarski's style have been developed, many of them by Tarski himself. One of the distinctive features is that the theory is *one-sorted*, i.e. there is the single primitive notion of *points* and all other geometrical structures are derived from them. This is contrary to Euclid's many primitive notions, including segments and circles, which come from compass and straightedge constructions, and also contrary to Hilbert's three primitive notions, where lines and planes are added. But limiting the primitive notions also provides more flexibility, namely to formulate the geometry in arbitrary dimensions. Most of axiomatizations share the same relational language as well: a ternary predicate *betweenness* and a quaternary predicate *equidistance.*

Our goal will be to unite the geometry developed axiomatically (*synthetic geometry*)

[1] Second-order logic can be used, though, to formulate one axiom instead of an axiom schema (Co). This yields a more powerful theory. As we are mainly interested in first-order logic, we won't consider this variant here. For details see [TG99].

with geometry interpreted algebraically (*analytic geometry*). This connection is known since the time of René Descartes (see [SL54]), but it is fruitful to explore this again, as the first-order grounding provides us with powerful results concerning completeness and decidability.

Note that this thesis focuses solely on Euclidean geometry (as opposed to non-Euclidean ones such as hyperbolic geometry). For comparison of Euclidean and non-Euclidean geometries, see [Kůr23].

In the logical definitions and notation, we mostly follow [Hod93] and [Šve02]. Note that the term *language* is used in the sense of a set of non-logical symbols (and thus replaces *signature*). We will work in first-order logic with equality, so equality is considered as a logical symbol and thus always available.

1 The geometry

Chapter 1 focuses on Tarski's geometry in itself, its axioms and developing it in an "algebraic direction". Treatise on *Tarski's axiomatics* in [Kůr23, pp. 46–52] can serve as an overview of this chapter.

The original version of axiomatization by Tarski can be traced to 1920s and contains twenty axioms plus one axiom schema. Subsequent work by Tarski, his students (most notably Haragauri Narayan Gupta) reduced the amount of necessary axioms to ten (again plus the one axiom schema) [TG99].

More recently, a further simplification was made, leading to nine axioms and one axiom schema[Mak13]. We will use this axiomatization as a base for our thesis.

Reducing the number of axioms is not the only goal for further development of the system. Inquiries have been done into constructive[1] axiomatizations with respect to automated proof checking. Beeson has been exploring this area extensively, arriving at several possible axiomatization (some of them formulated in non-classical logic) [Bee15]. Even more recently, a truly independent axiomatization (in the sense that any axiom is independent of the rest of axioms), with the independence mechanically proven, was proposed in [BKS23].

Models of geometry, which are referred to as *geometric spaces*, will be denoted by cursive uppercase letters (e.g. $\mathscr{M}$) and their domains by the same letter in uppercase italic (e.g. M).

[1]The term *constructivity* is intentionally ambiguous here. Beeson[Bee15, p. 2] explains the different meanings concisely: "Constructivity, in this context, refers to a theory of geometry whose axioms and language are closely related to ruler and compass constructions. It may also refer to the use of intuitionistic (or constructive) logic, but the reader who is interested in ruler and compass geometry but not in constructive logic, will still find this [Beeson's] work of interest. Euclid's reasoning is essentially constructive (in both senses). Tarski's elegant and concise first-order theory of Euclidean geometry, on the other hand, is essentially non-constructive (in both senses), (...)"

1.1 Axioms

Tarski considered geometry as a first-order theory. Accordingly, all theories here will be in first-order logic. Each theory is formulated in a (fixed) *language*, i.e., a set of non-logical (functional or relational) symbols together with their arity.

Consider an arbitrary language L. From symbols of L, together with logical symbols, *L-formulas* can be built. (First-order) *L-theory* is then a set of *L-sentences*, i.e. L-formulas with no free variables. See [Hod93] or [Šve02] for more details.

While for now we will only consider one fixed language, later, in section 2.5, we will explore metamathematical results for L-theories formulated in arbitrary language L.

All of the theories in this section are be formulated in the *geometric language* $\{\overline{-\cdot-}, \simeq\}$. We will refer to (ternary) $\overline{-\cdot-}$ as *betweenness* and to (quaternary) $\simeq$ as *equidistance.* Betweenness expresses that three points lie on a line in the given order (i.e. $\overline{a\dot{b}c}$ is supposed to mean there is a line, upon which a, b, c lie in this order. Equidistance stipulates that the segment given by the first and second point and the segment given by the third and fourth have the same length.[2]

Note that while Hilbert (e.g. in [Hil05]) considers betweenness strictly (i.e., if $a = b$, then $\overline{a\dot{b}c}$ does not hold), betweenness is presumed to be non-strict in Tarski's axioms. The non-strict betweenness is convenient, since degenerate cases of some axioms were used to deduce a few of the axioms used in the older axiomatization, thus reducing the number of axioms.

As mentioned previously, the definitions of theories in this section mostly follow [Mak13] (with small simplifications), while the dimension axioms are from [SST83].

Definition 1.1.1. The *theory of absolute geometry* (denoted as A) has the axioms

(TE) $\forall a, b, p, q, r, s\ (ab \simeq pq \wedge ab \simeq rs \to pq \simeq rs)$ (transitivity of equidistance),

(IE) $\forall a, b, c\ (ab \simeq cc \to a = b)$ (identity of equidistance),

(SC) $\forall a, b, c, q\ \exists x\ (\overline{q\dot{a}x} \wedge ax \simeq bc)$ (segment construction),

(FS) $$\forall \begin{matrix} a, & b, & c, & d, \\ a', & b', & c', & d', \end{matrix} \left(\begin{matrix} a \neq b & ab \simeq a'b' \\ \overline{a\dot{b}c} & bc \simeq b'c' \\ \overline{a'\dot{b'}c'} & ad \simeq a'd' \\ & bd \simeq b'd' \end{matrix} \to dc \simeq d'c' \right)$$ (five-segment axiom),

(IB) $\forall a, b\ (\overline{a\dot{b}a} \to a = b)$ (identity of betweenness),

(Pa) $\forall a, b, c, p, q\ (\overline{a\dot{p}c} \wedge \overline{b\dot{q}c} \to \exists x\ (\overline{p\dot{x}b} \wedge \overline{q\dot{x}a}))$ (axiom of Pasch).

[2]We will briefly touch upon the problem of *independence of primitive notions.* It turns out, that over the axioms presented below, the notion of betweenness is definable in terms of equidistance, while equidistance being undefinable in terms of betweenness. Thus a simpler (in the sense of smaller language) axiomatic system with just one predicate of equidistance may be considered. We, together with Tarski, Makarios or other authors, will not use this simplification, mostly because it would make the axioms unreadable and unintuitive geometrically. See [TG99, p. 200]. Furthermore, since betweenness may be viewed as the *affine* predicate (as opposed to the *metric* equidistance), it can be used solely for constructing an affine geometry. See [SST83].

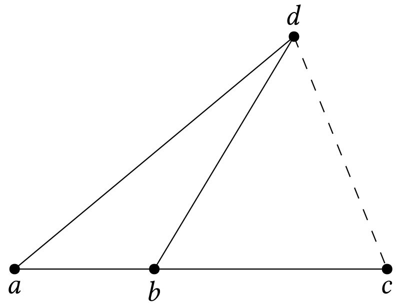

Figure 1.1 Five-segment axiom (FS)

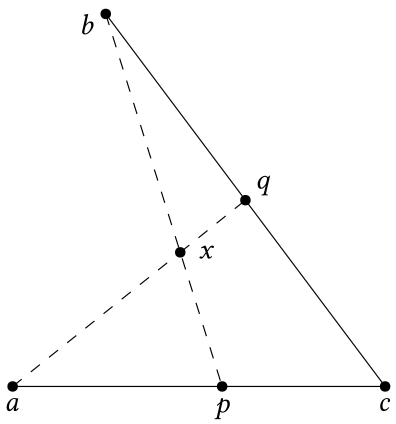

Figure 1.2 Pasch axiom (Pa)

Definition 1.1.2. Let $1 \leq n \leq m$ be given. The *theory of Euclidean geometry* (denoted as CE_n^m) is the extension of the theory of A with additional axioms

$$(\mathrm{Lo}_n)\ \ \exists o, s, u_1, \dots, u_n\ (o \neq s \wedge \overline{u_1}\dot{o}\overline{s} \wedge \textstyle\bigwedge_{i=1}^{n} ou_i \simeq os \wedge \bigwedge_{1\leq i<j\leq n} u_iu_j \simeq su_2)$$
(lower dimension is n),

$$(\mathrm{Up}_m)\ \ \neg\mathrm{Lo}_{m+1}$$
(upper dimension is m),

$$(\mathrm{Eu})\ \ \forall a, b, c, d, t\ (\overline{a}\dot{d}\overline{t} \wedge \overline{b}\dot{d}\overline{c} \wedge a \neq d \to \exists x, y\ (\overline{a}\dot{b}\overline{x} \wedge \overline{a}\dot{c}\overline{y} \wedge \overline{x}\dot{t}\overline{y}))$$
(Euclidean axiom),

and one axiom schema for arbitrary formulas $\varphi(x)$ and $\psi(y)$, s.t. variables a, b, y are not free in $\varphi(z)$ and a, b, x are not free in $\psi(y)$[3]

$$(\mathrm{Co})\ \ (\exists a\ \forall x, y\ (\varphi(x) \wedge \psi(y) \to \overline{a}\dot{x}\overline{y})) \to (\exists b\ \forall x, y\ (\varphi(x) \wedge \psi(y) \to \overline{x}\dot{b}\overline{y}))$$
(schema of continuity).

[3]Let us mention explicitly that formulas $\varphi(x)$ and $\psi(y)$ may have other free variables apart from x and y, respectively. These additional free variables can be considered as *parameters*.

For simplicity we do not universally quantify over these free variables, so in order for an instance of (Co) to be a sentence (i.e. in the same form as every other axiom in this thesis, which is "properly" quantified), one should use its universal closure.

An alternative axiom, albeit a significantly weaker, to the schema of continuity can be given, using the *circle axiom* (Ci).

Definition 1.1.3. Let $1 \leq n < m$ be given. The *theory of geometry of elementary constructions* (denoted as GEC_n^m) is the theory with axioms (TE), (IE), (SC), (FS), (IB), (Pa), (Lo_n), (Up_m), (Eu) and an addition axiom

$$\text{(Ci)} \quad \forall a,b,c,p,q,r \left(\begin{matrix} \overline{c}\dot{p}\overline{q} & ca \simeq cp \\ \overline{c}\dot{q}\overline{r} & cb \simeq cr \end{matrix} \rightarrow \exists x \left(cx \simeq cq \wedge \overline{a}\dot{x}\overline{b} \right) \right) \qquad \text{(circle axiom)}$$

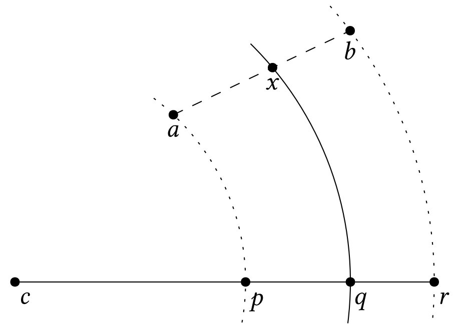

Figure 1.3 Circle axiom (Ci)

If $m = n$, we will simply denote the theories as CE_n and GEC_n, respectively.

Remark 1.1.4. In [Szm70] the Pasch axiom (Pa) was derived as a theorem in a variant of GEC (without the use of (Pa)). We do not know whether this holds in our axiomatization of GEC as well. ([Mak13] suggests that (Pa), or at least it's degenerate case, is used in establishing the equivalence of A and the axioms of absolute geometry presented in [SST83]. Indeed, [BKS23] proposes splitting the Pasch axiom into its degenerate and non-degenerate case to achieve an independent system.)

1.2 Properties of geometric spaces

Following [SST83], we will illustrate the development of geometry in Tarski's style by defining parallelism. There are some classic (in the context of [SST83]) constructions to be observed, namely defining relations, showing they are equivalences and defining objects as the equivalence classes.

Many notions in [SST83] are formulated in a fragment of set theory, e.g. line as an arbitrary point set that has every three points collinear. It should be noted that these formulation do not prohibit working only in first-order logic (as is done in this thesis). We have reformulated some of these definitions into a more first-order style, while leaving out constructions such as equivalence classes, as they are more naturally formulated using (in our case definable) sets.

In this section we will reason inside CE_n. Moreover, we do not need to consider the dimension of the space (similarly to [SST83], where majority of notions is dimension-free). However it may be useful to consider at least 2-dimensional space, since in preparation for section 1.3, we will work with parallel lines.

1.2.1 Collinearity and lines

Lemma 1.2.1. *Let a, b, c be given. Then the following holds.*

(i) $\bar{a}\dot{b}\bar{b}$,

(ii) $\bar{a}\dot{b}\bar{c} \to \bar{c}\dot{b}\bar{a}$.

Proof.

(i) We can use (SC) on points b, b, b and a, then there exists x, s.t. $\bar{a}\dot{b}\bar{x}$ and $bx \simeq bb$. Axiom (IE) then implies $b = x$, thus $\bar{a}\dot{b}\bar{b}$ holds.

(ii) Assume $\bar{a}\dot{b}\bar{c}$. From (i), we also obtain $\bar{b}\dot{c}\bar{c}$. By (Pa), there is a point x satisfying $\bar{b}\dot{x}\bar{b}$ and $\bar{c}\dot{x}\bar{a}$. Axiom (IB) implies $b = x$, and so $\bar{c}\dot{b}\bar{a}$ holds. □

Definition 1.2.2. We say that points a, b, c are *collinear* (denoted as $\mathrm{Col}(a, b, c)$), if either $\bar{a}\dot{b}\bar{c}$, $\bar{b}\dot{c}\bar{a}$ or $\bar{c}\dot{a}\bar{b}$ holds.

Corollary 1.2.3. *Collinearity is invariant under permutation of its arguments. That is, if points a, b, c satisfy* $\mathrm{Col}(a, b, c)$*, then also* $\mathrm{Col}(a, c, b)$, $\mathrm{Col}(b, a, c)$, $\mathrm{Col}(b, c, a)$, $\mathrm{Col}(c, a, b)$, $\mathrm{Col}(c, b, a)$.

Remark 1.2.4. Using collinearity we can define lines as points collinear with two (distinct) points using formula with one free variable. We will refer to the line given by two points p, q simply as pq.

For example, let $p \neq q$ be two points, then the formula $L_{pq}(x) = \mathrm{Col}(p, q, x)$ is valid only for points incident with line pq.

1.2.2 Planes

Definition 1.2.5. Let points a, b, p, q be given and let $p \neq q$. The points a, b *lie on the opposite sides* of line pq, if

(i) $\neg\operatorname{Col}(p,q,a)$, $\neg\operatorname{Col}(p,q,b)$ and

(ii) there is a point x satisfying $\overline{a\dot{x}b}$ and $\operatorname{Col}(p,q,x)$.

If furthermore there is a point c lying on the opposite side of pq from b, then the points a, c *lie on the same side* of line pq.

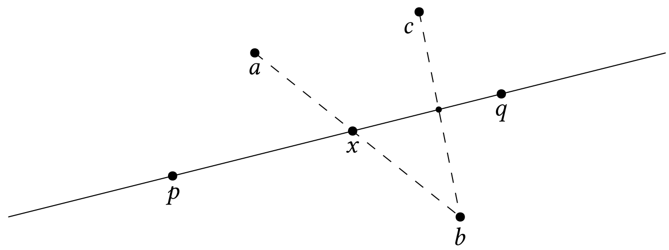

Figure 1.4 Points lying on the opposite sides from a line

Theorem 1.2.6 ([SST83, p. 72]). *Lying on the same side of a fixed line pq is an equivalence.*

Definition 1.2.7. Let a line pq be given. The equivalence classes given by relation *lie on the same side of pq* are called *half-planes* with division line pq.

A *plane* is then a union of the opposite half-planes (i.e., those half planes, which have points lying on the opposite sides of pq) with the division line pq. A point *lies in a plane*, if it is an element of it.

Definition 1.2.8. Two lines pq and rs are said to be *coplanar*, if every point x on the line rs lies in the plane given by line pq and point r.

1.2.3 Parallelism

Definition 1.2.9. The pq and rs are said to be *parallel* (denoted $pq \parallel rs$), if either

(i) they coincide (every point x on pq lies on rs and every point y on rs lies on pq), or

(ii) they are coplanar and they do not intersect (i.e., no point is collinear with p, q and r, s).

Definition 1.2.10. Let p, q, r be points with $p \neq q$. We say that the point s *lies on the parallel line* to pq through r (alternatively *is constructed by parallel projection*), denoted as $\operatorname{Pj}(p,q,r,s)$, if either $pq \parallel rs$ or $r = s$.

Lemma 1.2.11. *For any points a, b, c, d the following holds*

$$ab \parallel cd \iff ba \parallel cd \iff cd \parallel ab.$$

Theorem 1.2.12 (Pappus-Pascal)**.** *Let o, a, b, c, a', b', c' be points, s.t.*

$$\neg\,\mathrm{Col}(o,a,a') \wedge \begin{array}{c}\mathrm{Col}(o,a,b)\\ \mathrm{Col}(o,a,c)\\ b \neq o\\ c \neq o\end{array} \wedge \begin{array}{c}\mathrm{Col}(o,a',b')\\ \mathrm{Col}(o,a',c')\\ b' \neq o\\ c' \neq o\end{array} \wedge \begin{array}{c} bc' \parallel cb'\\ ca' \parallel ac'\end{array}.$$

Then $ab' \parallel ba'$.

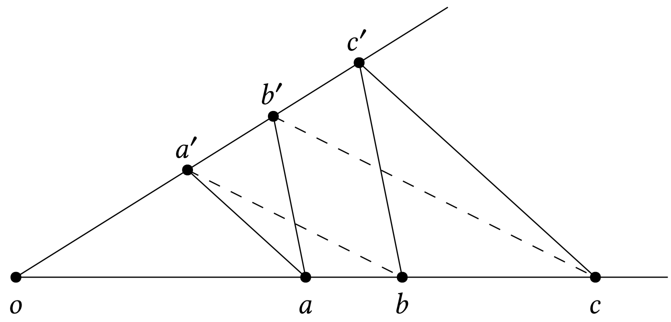

Figure 1.5 Pappus-Pascal theorem

1.2.4 Right angle

Definition 1.2.13. The points a, b, c form a *right angle* (with vertex at b):

$$\mathrm{R}(a,b,c) = \exists c'\ (\overline{cbc'} \wedge bc \simeq bc' \wedge ac \simeq ac').$$

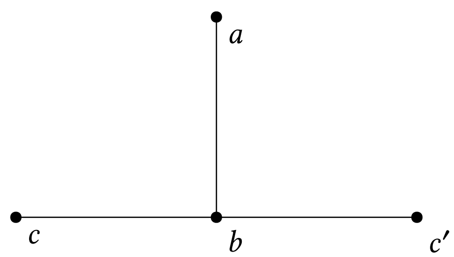

Figure 1.6 Points a, b, c form a right angle

Definition 1.2.14. We say the lines pq and rs are *perpendicular*:

$$pq \perp rs = \exists x\ (\mathrm{Col}(p,q,x) \wedge \mathrm{Col}(r,s,x) \wedge \forall u,v\ (\mathrm{Col}(p,q,u) \wedge \mathrm{Col}(r,s,v) \to \mathrm{R}(u,x,v)).$$

1.3 Number lines and coordinate systems

Given a model of CE_n^m, we can construct a number line and define operations on it, such that it forms a field (see definition 2.1.2). This number line then serves as one axis in defining a coordinate system inside the geometric space. Our ultimate goal is to give coordinates to every point in the entire space, which leads to theorem 3.3.1. However, these constructions require three non-collinear points, therefore the space has to have at least dimension 2.

The following constructions are known since the times of Descartes (see [SL54]) and were used by Hilbert in [Hil05]. We will follow the treatise in [SST83], the reader is referred there for details.

The points o and e, which will be used throughout this section, are the basis for the *number line*, we are constructing. We are going to use an auxiliary e', the choice of which does not matter, but the point is useful for defining operations (and possibly necessary).

Definition 1.3.1. We say that points $a_1, \dots, a_n$ lie on

- the *number line* given by points o, e:

$$\mathrm{Ar}_{oe}(\bar{a}) = o \neq e \wedge \bigwedge_{i=1}^{n} \mathrm{Col}(o, e, a_i),$$

- the *number line* given by points o, e and auxiliary e':

$$\mathrm{Ar}_{oe}^{e'}(\bar{a}) = o \neq e \wedge \neg\,\mathrm{Col}(o, e, e') \wedge \bigwedge_{i=1}^{N} \mathrm{Col}(o, e, a_i).$$

Definition 1.3.2. The point c is called a *geometric sum* of points a, b with respect to o, e, e':

$$\mathrm{Sum}_{oe}^{e'}(a, b, c) = \mathrm{Ar}_{oe}^{e'}(a, b, c) \wedge \exists a', b' \left(\begin{array}{c} \mathrm{Pj}(e, e', a, a') \wedge \mathrm{Col}(o, e', a') \\ \mathrm{Pj}(a, b, a', b') \wedge \mathrm{Pj}(o, a', b, b') \wedge \mathrm{Pj}(a', a, b', c) \end{array} \right).$$

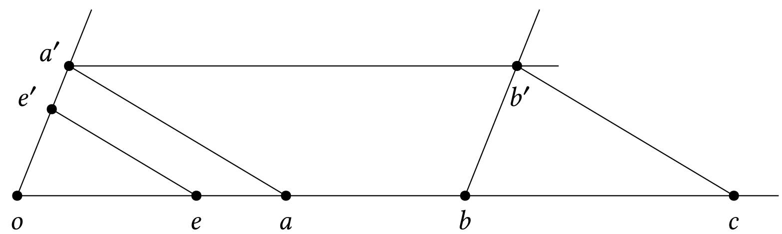

Figure 1.7 Geometric sum

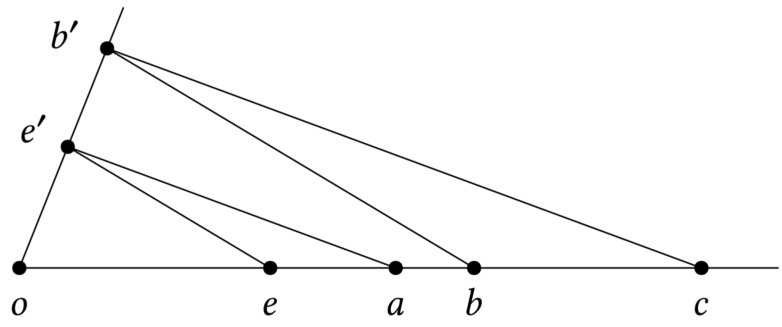

Figure 1.8 Geometric product

Definition 1.3.3. The point c is called a *geometric product* of points a, b with respect to o, e, e':

$$\mathrm{Prod}_{oe}^{e'}(a,b,c) = \mathrm{Ar}_{oe}^{e'}(a,b,c) \wedge \exists b' \, (\mathrm{Pj}(e,e',b,b') \wedge \mathrm{Col}(o,e',b') \wedge \mathrm{Pj}(e',a,b',c)).$$

The following propositions are proven in [SST83].

Lemma 1.3.4. *Let a, b be points, s.t.* $\mathrm{Ar}_{oe}(a,b)$*. Then there exist unique points s, p satisfying*

$$\mathrm{Sum}_{oe}^{e'}(a,b,s) \quad \textit{and} \quad \mathrm{Prod}_{oe}^{e'}(a,b,p).$$

Moreover there exists a unique point i satisfying

$$\mathrm{Sum}_{oe}^{e'}(a,i,o)$$

(i.e. an inverse element to a w.r.t. addition).

Thus geometric addition and multiplication can be viewed as operations on points of line oe.

Lemma 1.3.5. *Let e'' be a point with* $\neg\,\mathrm{Col}(o,e,e')$*. Then for any three points a, b, c we have*

$$\begin{aligned} \mathrm{Sum}_{oe}^{e'}(a,b,c) &\iff \mathrm{Sum}_{oe}^{e''}(a,b,c), \\ \mathrm{Prod}\, oe^{e'}(a,b,c) &\iff \mathrm{Prod}\, oe^{e''}(a,b,c). \end{aligned}$$

I.e., the auxiliary point can be chosen arbitrarily. We can now leave the choice of auxiliary from our notation.

Definition 1.3.6. Denote L_{oe} the set of points collinear with points o, e (i.e. the line oe as a point set). We define the ordering on L_{oe} as

$$\begin{aligned} o <_{oe} a &\iff o \neq a \wedge (\overline{o}\dot{a}\overline{e} \vee \overline{o}\dot{e}\overline{a}), \\ a <_{oe} b &\iff a \neq b \wedge \exists c, d \, (\mathrm{Sum}(a,c,0) \wedge \mathrm{Sum}(b,c,d) \wedge 0 <_{oe} d). \end{aligned}$$

The structure $\mathscr{L}_{oe} = (L_{oe}, \mathrm{Sum}_{oe}, \mathrm{Prod}_{oe}, o, e, <_{oe})$ is called a *number line* given by points o, e.

Since we have successfully constructed a number line, we can now continue with introducing Cartesian coordinates.

Definition 1.3.7. Points $s, u_1, \dots, u_n$ together with points o, e constitute an n-dimensional *Cartesian coordinate system*:

$$\mathrm{Cs}_{oe}(s, \bar{u}) = o \neq e \wedge \bigwedge_{i=1}^{n} su_i \simeq oe \wedge \bigwedge_{1 \leq i < j \leq n} \mathrm{R}(u_i, s, u_j).$$

The point s serves as the *origin* and lines oe_i as the *axes* of the coordinate system.

The coordinates of a point are then obtained by (orthogonally) projecting a point onto each of the axes.

Definition 1.3.8. Let a Cartesian coordinate system determined by o, e, s, $\bar{u}$ be given. Point p has coordinates $\bar{x}$:

$$\mathrm{Cd}_{oe}^{s\bar{u}}(p, \bar{x}) = \mathrm{Cs}_{oe}(s, \bar{u}) \wedge \exists \bar{p} \bigwedge_{i=1}^{n} \left(\begin{array}{c} \mathrm{Col}(s, u_i, p_i) \\ (p_i = p \vee pp_i \perp su_i) \end{array} \wedge \begin{array}{c} ox_i \simeq sp_i \\ ex_i \simeq x_i p_i \end{array} \right).$$

2 The algebra

To explore connection between geometry and algebra, we have to recall some algebraic notions (sections 2.1 and 2.2) and establish results about them (sections 2.2, 2.3, 2.5 and 2.6). Furthermore we will employ a few model-theoretic instruments, namely elimination of quantifiers (section 2.5).

Similarly to geometric models, we shall denote various algebraic structures by uppercase letters in bold (e.g. $\mathbf{F}$) and their domains by the same letter in uppercase italic (e.g. F). If $\mathbf{B}$ is a structure and $A \subseteq B$, then $\mathbf{A} = \langle A \rangle_{\mathbf{B}}$ is the substructure generated by $\mathbf{A}$. When using this notation, we assume A is already closed under the operations in $\mathbf{B}$.

2.1 Rings, fields and real-closed fields

Here we will introduce theories of five classes of structures: rings, ordered rings, fields, ordered fields and real-closed fields. We will only consider commutative rings (commutative fields, etc.), and will just refer to them as "rings" (fields etc. respectively).

As for the language, we will use either $\{+, \cdot, -, 0, 1\}$ for unordered structures, or $\{+, \cdot, -, 0, 1, <\}$ for the ordered ones. We will refer to $+$ as *addition*, and to $\cdot$ as *multiplication*.

Definition 2.1.1. The *theory of rings* has the following axioms:

(i) $\forall x, y, z\, ((x + y) + z = x + (y + z))$ (additive associativity),

(ii) $\forall x\, (x + 0 = 0 + x = x)$ (additive neutral element),

(iii) $\forall x\, (x + (-x) = 0)$ (additive inverse elements),

(iv) $\forall x, y, z\, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$ (multiplicative associativity),

(v) $\forall x\, (x \cdot 1 = x)$ (multiplicative neutral element),

(vi) $\forall x, y\, (x \cdot y = y \cdot x)$ (multiplicative commutativity),

(vii) $\forall x, y, z\, ((x + y) \cdot z = x \cdot z + y \cdot z)$ (multiplication distributes over addition).

The *theory of ordered rings* has the axioms of the theory of rings together with:

(viii) $\forall x, y\, (x < y \to y \not< x)$ (antisymmetry),

(ix) $\forall x, y, z\, (x < y \wedge y < z \to x < z)$ (transitivity),

(x) $\forall x, y\, (x < y \vee y < x \vee x = y)$ (trichotomy),

(xi) $\forall x, y, z\, (x < y \to x + z < y + z)$ (additive monotonicity),

(xii) $\forall x, y, z\, (0 < z \wedge x < y \to x \cdot z < y \cdot z)$ (multiplicative monotonicity).

As is customary, multiplication may be written without the infix dot, exponentiation is shorthand for repeated multiplication, and we define further symbols for ordering.

$$xy = x \cdot y \qquad\qquad x > y \iff y < x$$
$$x^n = \underbrace{x \cdot x \cdots x}_{n\text{-times}} \qquad\qquad x \leq y \iff x < y \vee x = y$$
$$x \geq y \iff y < x \vee x = y$$

Definition 2.1.2. The *theory of fields* has the axioms of the theory of rings together with:

(xiii) $\forall x\, (x \neq 0 \to \exists y\, (xy = 1))$ (multiplicative inverse elements).

The *theory of ordered fields* has the axioms of the theory of ordered rings together with the axiom (xiii).

Definition 2.1.3. The *theory of real-closed fields* (denoted as RCF) has the axioms of the theory of ordered fields together with:

(xiv) schema $\forall x_0, \dots, x_{d-1}\, \exists y\, (y^d + x_{d-1}y^{d-1} + \dots + x_1 y + x_0 = 0)$ for all natural odd d,

(xv) $\forall x\, (x > 0 \to \exists y\, (y^2 = x))$.

In literature (e.g. [Hod93]) we can find axiomatizations of real-closed fields without the ordering, but with some other axioms using which the order can be induced. However, since the ordering is vital for the proof of decidability of RCF (see [Mar02, p. 93]), we shall use the ordering from the beginning.

Lemma 2.1.4. *The following sentences are provable:*

- *in the theory of rings:*

  (i) $\forall x\, (0 \cdot x = 0)$,

  (ii) $\forall x\, ((-1) \cdot x = -x)$,

  (iii) $\forall x, y\, (x + y = y + x)$ *(additive commutativity),*

  (iv) $\forall x, y, z\, (xy = 1 \land xz = 1 \to y = z)$ *(multiplicative inverses are unique),*

  (v) $\forall x\, ((-x)^2 = x^2)$,

- *in the theory of ordered rings:*

  (vi) $\forall x\, (x \not< x)$ *(irreflexivity),*

- *in the theory of ordered fields:*

  (vii) $\forall x\, (x^2 \geq 0)$,

  (viii) $\forall x, y\, \exists z\, (x < y \to x < z \land z < y)$ *(density),*

  (ix) $\forall x_1, \dots, x_n\, (x_1^2 + \dots + x_n^2 \neq -1)$ *(ordered fields are formally real).*

Proof. Omitted. □

As for the standard model for the theory of real-closed fields, the reader should think about the structure of real numbers $\mathbb{R} = (\mathrm{R}, +, \cdot, 0, 1, <)$ (the signature may vary depending on context). We will return to this in theorem 2.4.1.

Remark 2.1.5. Let us talk briefly about the choice of language. We could have chosen smaller language like $\{+, \cdot\}$, or more expressive one like $\{+, \cdot, -, ^{-1}, 0, 1\}$ (since our main object of interest are real-closed fields, almost all elements have inverses).

Since we will work with arbitrary substructures of real-closed fields, by the choice of language these will be subrings and not subfields. This complicates the situation, because most propositions in commutative algebra are formulated for fields, so we will need to generalize these results.

But the main benefit is that any term is equal (over the ring axioms) to a polynomial. Furthermore, atomic formula can up to ring-equivalence be only $f(\bar{x}) = 0$, or $f(\bar{x}) > 0$ in the ordered case, where f is a polynomial (in possibly many variables).

Remark 2.1.6. We will use the term *polynomial* quite vaguely. In the more precise sense, polynomial is a formal sum with indeterminates, i.e. a (logical) term in the language of rings (usually with one free variable). This has the immediate expressivity drawback, that only "formal integers" (i.e. terms like $1 + 1 + \cdots + 1$) can be polynomial coefficients.

In the looser sense, but more customary in algebra, polynomial over a ring is a finite sequence of elements of the ring, representing coefficients of the polynomial. Polynomials then themselves form a ring. We will mostly use this interpretation in the following sections, and return to the logical one in section 2.5.

2.2 Algebraic closures

2.2.1 Relative algebraic closures

Definition 2.2.1. Let $\mathbf{F}$ be a field and $\mathbf{D}$ its subring.

- An element $\alpha \in F$ is *algebraic* over $\mathbf{D}$ if there is a non-zero polynomial f with coefficients from D such that $f(\alpha) = 0$.
- If $\mathbf{E}$ is a subring of $\mathbf{F}$ and every $\alpha \in E$ is algebraic over $\mathbf{D}$, then $\mathbf{E}$ is called an *algebraic extension* of $\mathbf{D}$.
- The set of all elements of F algebraic over $\mathbf{D}$ is called the *(relative) algebraic closure* of $\mathbf{D}$ in $\mathbf{F}$.

Note that these definitions diverge slightly from the standard ones in (commutative) algebra, namely we consider not only fields, but also rings as algebraic extensions or structures to be extended. See remark 2.1.5.

We shall review some properties of relative algebraic closures.

Lemma 2.2.2 ([DF04, Thm 20, p. 527])**.** *Let $\mathbf{D}$ be a ring and $\mathbf{E}, \mathbf{F}$ fields, s.t. $\mathbf{D} \leq \mathbf{E} \leq \mathbf{F}$, $\mathbf{F}$ is an algebraic extension of $\mathbf{E}$ and $\mathbf{E}$ is an algebraic extension of $\mathbf{D}$. Then $\mathbf{F}$ is an algebraic extension of $\mathbf{D}$.*

Proof. Omitted. □

Lemma 2.2.3 ([DF04, Cor 19, p. 527])**.** *Let $\mathbf{F}$ be a field and $\mathbf{D}$ its subfield. Denote E the relative algebraic closure of $\mathbf{D}$ in $\mathbf{F}$. Then $\mathbf{E}$ is a subfield of $\mathbf{F}$.*

Corollary 2.2.4. *The previous theorem remains valid even if $\mathbf{D}$ is just a subring.*

Proof. Denote $C = \{c \in F : a = b \cdot c \text{ with } a, b \in D\}$. One should see the elements of C as the “fractions over $\mathbf{D}$”. Since $\mathbf{D}$ is a subring of a field $\mathbf{F}$, substructure $\mathbf{C}$ forms a subfield of $\mathbf{F}$. By applying lemma 2.2.3 to $\mathbf{C}$ and $\mathbf{F}$, we obtain subfield $\mathbf{E}$ as the relative algebraic closure of $\mathbf{C}$ in $\mathbf{F}$.

Let $e \in E$ be arbitrary, then there is a polynomial $f \in C[x]$ satisfying $f(e) = 0$. The polynomial has the following form

$$f(x) = \frac{a_n}{b_n}x^n + \cdots + \frac{a_1}{b_1}x + \frac{a_0}{b_0},$$

where $a_i, b_i \in D$. If we multiply all coefficients by $b_0 \cdot b_1 \cdots b_n$ (all of which are clearly non-zero), we obtain a polynomial $g \in D[x]$, which also satisfies $g(e) = 0$, so e is algebraic over $\mathbf{D}$. Conversely from $D[x] \subseteq C[x]$, every algebraic element over $\mathbf{D}$ is algebraic over $\mathbf{C}$. Therefore E is the relative algebraic closure of $\mathbf{D}$. □

Theorem 2.2.5. *Let $\mathbf{F}$ be a field, $\mathbf{C}$ its subring and $\mathbf{D}, \mathbf{E}$ its subfields, s.t. D is the relative algebraic closure of $\mathbf{C}$ and E is the relative algebraic closure of $\mathbf{D}$ in $\mathbf{F}$. Then $\mathbf{D} = \mathbf{E}$.*

Proof. Since $\mathbf{C} \leq \mathbf{D} \leq \mathbf{E}$, subfield $\mathbf{D}$ is an algebraic extension of $\mathbf{C}$ and subfield $\mathbf{E}$ is an algebraic extension of $\mathbf{D}$, from lemma 2.2.2 $\mathbf{E}$ is an algebraic extension of $\mathbf{C}$, i.e. all elements of $\mathbf{E}$ are algebraic over $\mathbf{C}$. But $D = \{\alpha \in F : \alpha \text{ is algebraic over } \mathbf{C}\}$, so $E \subseteq D$. From $\mathbf{D} \leq \mathbf{E}$ we have $D \subseteq E$, therefore $D = E$, so finally $\mathbf{D} = \mathbf{E}$. □

2.2.2 Real closures

Definition 2.2.6. Let $\mathbf{D}$ be an ordered ring. If $\mathbf{E}$ satisfies the following:

(i) $\mathbf{E}$ is a real-closed field,

(ii) $\mathbf{D}$ is a substructure of $\mathbf{E}$,

(iii) $\mathbf{E}$ is an algebraic extension of $\mathbf{D}$,

then $\mathbf{E}$ is called a *real closure* of $\mathbf{D}$.

Theorem 2.2.7. *Let* $\mathbf{F}$ *a be real-closed field,* $\mathbf{D}$ *its ordered subring and* $\mathbf{E}$ *an algebraic closure of* $\mathbf{D}$ *in* $\mathbf{F}$*. Then* $\mathbf{E}$ *is a real closure of* $\mathbf{D}$*.*

Proof. Clearly $\mathbf{D}$ is a substructure of $\mathbf{E}$ and $\mathbf{E}$ is an algebraic extension of $\mathbf{D}$. We just need to prove that $\mathbf{E}$ is a real-closed field.

(i) $\mathbf{E}$ is a field from lemma corollary 2.2.4.

(ii) Let $f \in E[x]$ be an odd-degree polynomial. Since $\mathbf{E} \leq \mathbf{F}$, the polynomial f is over $\mathbf{F}$, so it has a root $\alpha \in F$, but this α is algebraic over $\mathbf{E}$, from lemma 2.2.2 it must be algebraic over $\mathbf{D}$ as well, so $\alpha \in E$.

(iii) Let $\alpha \in E$ be s.t. $\alpha > 0$. From $\mathbf{F}$ being a real-closed field there is $\beta \in F$ satisfying $\beta^2 = \alpha$. But this β must also be a root of polynomial $f(x) = x^2 - \alpha$. The polynomial f is over $\mathbf{E}$, so β is algebraic over $\mathbf{E}$. Similarly to previous case, β is also algebraic over $\mathbf{D}$, so $\beta \in E$. □

Theorem 2.2.8 ([Kne72])**.** *Suppose* $\mathbf{D}$ *is an ordered ring and* $\mathbf{E}, \mathbf{F}$ *are its real closures. Then* $\mathbf{E}$ *and* $\mathbf{F}$ *are isomorphic and the isomorphism is the identity function on elements of D.*

2.3 Polynomials over ordered and real-closed fields

This section establishes a few properties of polynomials, which are to be expected in the field of real numbers, but are true more generally.

In all propositions let $\mathbf{F}$ be an ordered field and $f \in F[x]$ a polynomial over it.

Lemma 2.3.1. *Let $\alpha \in F$ be such that $f(\alpha) > 0$. Then there exists $\delta \in F$, $\delta > 0$,*

$$\forall x\, (\alpha - \delta < x < \alpha + \delta \to f(x) > 0).$$

Lemma 2.3.2. *Let f be a monic polynomial of odd degree. Then there are $\lambda, \mu \in F$, such that*

$$x < \lambda \to f(x) < 0,$$
$$x > \mu \to f(x) > 0.$$

Theorem 2.3.3 ([War65, Thm 44.4, p. 492])**.** *In a real-closed field the only irreducible polynomials over the field are linear polynomials and quadratic polynomials $\alpha x^2 + \beta x + \gamma$, where $\beta^2 - 4\alpha\gamma < 0$.*

Corollary 2.3.4. *Let $\mathbf{F}$ be a real-closed field. Then there exists $\gamma \in F$, linear polynomials $\ell_i \in F[x]$ and irreducible quadratic polynomials $q_j \in F[x]$, s.t.*

$$f(x) = \gamma \cdot \prod_i \ell_i(x) \cdot \prod_j q_j(x).$$

Proof. Since a polynomial ring over a field is Gaussian, any polynomial has a unique factorization containing only irreducible polynomials. But from theorem 2.3.3 the only irreducible polynomials over real-closed field are linear and quadratic. □

2.4 Examples of real-closed fields

Let's start by proving the motivating fact: the field of real numbers is real-closed.

Theorem 2.4.1. $\mathbb{R} \models \mathsf{RCF}$.

Proof. It is taken for granted that $\mathbb{R}$ is an ordered field.

Let n be an odd natural number and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ a polynomial, which can WLOG[1] be assumed to be monic. (The leading coefficient merely switches signs.) By lemma 2.3.2, there are $\alpha, \beta \in \mathrm{R}$ satisfying $f(\alpha) < 0 < f(\beta)$. Since polynomials are continuous, using the intermediate value theorem we obtain at least one root of f.

Since function $f(x) = x^2$ is continuous and increasing on $(0, \infty)$ with image $(0, \infty)$, it has a continuous inverse function $g(x) = \sqrt{x}$ with domain $(0, \infty)$. Thus any positive number has a square root. □

We may ask whether there are countable real-closed fields. Since we can construct real closures of subrings of a real-closed field, the answer is the algebraic closure of the rational numbers inside the real numbers. (We shall denote this structure as $\mathbb{A}$.)

Corollary 2.4.2. $\mathbb{A} \models \mathsf{RCF}$. *Furthermore,* $\mathbb{A}$ *is countable.*

Proof. $\mathbb{Q}$ is an ordered subring of $\mathbb{R}$ and $\mathbb{A}$ is the relative algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. By theorem 2.2.7, the structure $\mathbb{A}$ is a real closure of $\mathbb{Q}$, so $\mathbb{A}$ is by definition a real-closed field.

Every element of $\mathbb{A}$ is a root of some polynomial over $\mathbb{Q}$, but since $\mathbb{Q}$ is countable, there are countably many polynomials over $\mathbb{Q}$, and every polynomial has only finitely many roots, the polynomials can have only countably-many roots. So $\mathbb{A}$ is countable. □

2.4.1 Hyperreal numbers

In this short section we will introduce a non-standard model for the theory of real-closed fields: the hyperreal numbers. They are constructed from the real numbers via the *ultrapower construction* (as in [Kei07, p. 23]), which can be better understood with the *voting analogy* from [Tao07].

Our basic object will be an ω-long sequence of reals, which we shall view as a situation where countably many agents are choosing a number and they collectively have to decide about properties of their choice. It's simple if everyone chooses the same number. (The properties of their choice are the same as the properties of the one chosen number.) If there is one agent choosing different one, it is natural to disregard his choice and conform with the majority. With the same logic the voice of finitely many agents can be outvoted. However, when there are two infinite groups of agents disagreeing about the choice, which one should prevail?

This is where the notion of *(free) ultrafilter*[Kei07, p. 24] comes into play. It tells us which groups (subsets of ω) are the deciding ones and which can be disregarded. We also require the ultrafilter to be free, i.e., no finite subset is important. Naturally, this ultrafilter induces a partition of sequences of reals, where two sequences are equivalent if enough agents voted for the same number in both elections.

[1] without loss of generality

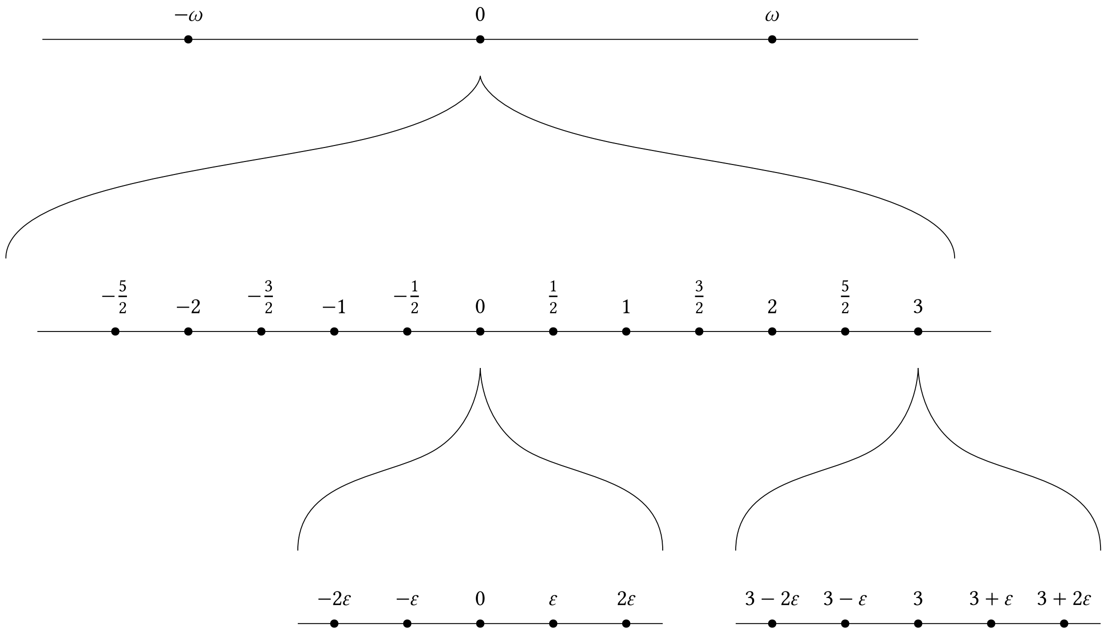

Figure 2.1 Sketch of the hyperreal numbers

Definition 2.4.3. Let U be a free ultrafilter on ω. Then by *hyperreal numbers* we mean the ultrapower $\mathbb{H} = \mathbb{R}^{\omega}/U$. We shall denote its elements as $[s]_U$ for a given sentence $s \in \mathrm{R}^{\omega}$.

The hyperreal numbers contain real numbers: $r \in \mathrm{R}$ is represented as

$$\mathbf{r} = [(r, r, r, \dots)]_U.$$

We can also introduce infinitesimals and infinites: the number

$$\varepsilon := \left[\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)\right]_U$$

is less than any positive real (because the set of indices, where the real is smaller, is finite), while greater than 0 (because on all indices is ε greater than 0), similarly the number $[(1, 2, 3, \dots)]_U$ will be greater than any positive real number.

Note that the choice of the ultrafilter was arbitrary, so with different choices of the ultrafilter we could get different hyperreals. This won't concern us, we refer the reader to [Kei07, p. 194]. But keep in mind, that properties of some numbers, e.g. $[(0, 1, 0, 1, 0, 1, \dots)]_U$, can't be established without specifying the ultrafilter.

Historically, hyperreal numbers were explored as a tool in calculus, but constructions built on limits prevailed as they were more rigorous. Only in the 20th century were the methods of model theory developed enough to construct hyperreal numbers formally and, subsequently, infinitesimal analysis could be rigorously developed. (See [Kei00].)

Why are we interested in the hyperreals? Because they show us the limit of what the theory of real-closed fields can describe.

Theorem 2.4.4. $\mathbb{H} \vDash \mathsf{RCF}$.

Proof. From Łoś's theorem[Kei10] a sentence holds in the ultraproduct, if and only if the set of indices of structures where the sentence holds, is in the ultrafilter. Since all of the factor structures are the same (= $\mathbb{R}$), a sentence holds in hyperreals iff it holds in the reals, i.e., hyperreal numbers are elementary equivalent to the real numbers. Since real numbers are a model of RCF, so must the hyperreal numbers. ☐

Theorem 2.4.5. *Hyperreal numbers are not isomorphic to the real numbers.*

Proof. We shall denote hyperreal counterparts to real numbers in bold, e.g. $\mathbf{r} = [(r, r, r, \dots)]_U$.

For the sake of contradiction let $\sigma : \mathrm{R} \to \mathrm{H}$ be such isomorphism.

Since 0 and 1 are constants, $\sigma(0) = \mathbf{0}$ and $\sigma(1) = \mathbf{1}$. As an isomorphism, σ commutes with $+$, so any positive integer $n \in \mathrm{R}$ must be mapped to $\mathbf{n} \in \mathrm{H}$. Furthermore σ commutes with $-$, so any integer $z \in \mathrm{R}$ must be mapped to $\mathbf{z} \in \mathrm{H}$.

Let $a, b \in \mathrm{R}$ be integers, $b \neq 0$, we have $\mathbf{a}, \mathbf{b} \in \mathrm{H}$, $\sigma(a) = \mathbf{a}$, $\sigma(b) = \mathbf{b}$. Then

$$\mathbf{a} = \sigma(a) = \sigma\left(\frac{a}{b} \cdot b\right) = \sigma\left(\frac{a}{b}\right) \cdot \sigma(b) = \sigma\left(\frac{a}{b}\right) \cdot \mathbf{b},$$

since hyperreals form a field, $\sigma\left(\frac{a}{b}\right) = \frac{\mathbf{a}}{\mathbf{b}}$.

So far we know that any rational real number will be mapped to its hyperreal counterpart. Now take the set of all positive rationals P, we have $\inf(P) = 0$, so $\sigma(\inf(P)) = \inf(\sigma[P]) = \mathbf{0}$. But, we know that $\mathbf{0} < \varepsilon < \mathbf{q}$ for any $q \in P$, which contradicts $\mathbf{0}$ being an infimum of $\sigma[P]$. ☐

2.5 Decidability of the theory of real closed fields

In this section we will prove the key result about the theory of real-closed fields – it is complete and decidable. Our main tool will be *quantifier elimination*. The proof of RCF having quantifier elimination will be quite model-theoretic, the important part will be the semantic criterion, which hides the quantifier elimination into a property of models of the theory in question.

This section follows [Jeř24].

Definition 2.5.1. Let T be a L-theory. We say that T has *quantifier elimination*, if for every L-formula $\varphi(x_0, \dots, x_n)$, there exists a quantifier-free formula $\psi(x_0, \dots, x_n)$ satisfying

$$T \vdash \varphi(\bar{x}) \leftrightarrow \psi(\bar{x}).$$

Lemma 2.5.2. *Let T be a L-theory. Assume that every L-formula $\varphi(\bar{x}) = \exists y \bigwedge \theta_i(\bar{x}, y)$, where θ_i is a literal*[2]*, is T-equivalent to a quantifier-free L-formula $\hat{\varphi}(\bar{x})$. Then T has quantifier elimination.*

Proof. We prove that every formula φ is equivalent to a quantifier-free one by induction on complexity of the formula. WLOG let us assume φ contains only $\wedge$, $\vee$ and $\neg$ as connectives and $\exists$ as quantifiers (other connectives and quantifiers, i.e. $\rightarrow$ and $\forall$, can be (logically) equivalently expressed in those).

(i) φ is atomic.
Then φ is already quantifier-free.

(ii) $\varphi = \psi \wedge \chi$.
By induction hypothesis let $\hat{\psi}$ and $\hat{\chi}$ be a quantifier-free T-equivalent formulas to ψ and χ respectively. Then $\hat{\psi} \wedge \hat{\chi}$ is T-equivalent to $\psi \wedge \chi$.

(iii) $\varphi = \psi \vee \chi$ or $\varphi = \neg\psi$.
This case is analogous to (ii).

(iv) $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$.
By induction hypothesis there is a quantifier-free λ satisfying

$$T \vdash \psi(\bar{x}, y) \leftrightarrow \lambda(\bar{x}, y).$$

We can write λ in DNF:

$$\lambda(\bar{x}, y) \equiv \bigvee_j \bigwedge_i \theta_{j,i}(\bar{x}, y),$$

where $\theta_{j,i}(\bar{x}, y)$ is a literal. Because existential quantifier distributes over disjunction, we obtain

$$T \vdash \varphi(\bar{x}) \leftrightarrow \bigvee_j \exists y \bigwedge_i \theta_{j,i}(\bar{x}, y).$$

We can now use the assumption to get a disjunction of quantifier-free formulas, which itself is quantifier-free. □

[2] A *literal* is an atomic formula or negation of an atomic formula.

Let us introduce in accord with [Hod93, p. 16] the notion of a diagram. While we are defining the diagram using the notation for finite sequences (thus we can obtain only a finitely generated structure), one may generalize the sequence notation to arbitrary length (in the same manner Hodges does).

Definition 2.5.3. Suppose that $\mathbf{A}$ is an L-structure generated by elements $\bar{a}$, i.e., $\mathbf{A} = \langle \bar{a} \rangle$. Let L_A be the language $L \cup \{\dot{a}_i : a_i \in \bar{a}\}$ and consider the L_A-expansion $\mathbf{A}_A$ of $\mathbf{A}$, where $\dot{a}_i^{\mathbf{A}_A} = a_i$. The *diagram* of $\mathbf{A}$ is the theory

$$\mathrm{Diag}(\mathbf{A}) = \{\varphi \text{ closed } L_A\text{-literal} : \mathbf{A}_A \models \varphi\}.$$

Remark 2.5.4. Our main use case for the diagrams is the situation where we have a structure $\mathbf{A}$, its diagram $\mathrm{Diag}(\mathbf{A})$, and some other structure $\mathbf{B}$ satisfying $\mathbf{B}_A \models \mathrm{Diag}(\mathbf{A})$, where $\mathbf{B}_A$ is expansion of $\mathbf{B}$ to language $\mathbf{A}$. In such case, we want to say that $\mathbf{A}$ is a substructure of $\mathbf{B}$. This can be said, without loss of generality, from considerations in [Hod93, p. 17] and by identifying the domain with the image via some embedding given by $\mathbf{B}_A \models \mathrm{Diag}(\mathbf{A})$. The reader is referred to Hodges for more details on embeddings and diagrams.

Lemma 2.5.5 (Semantic criterion of quantifier elimination)**.** *Let T be an L-theory for at-most countable L. Then the following are equivalent:*

(i) T has quantifier elimination.

(ii) Let $\mathbf{A}, \mathbf{B}$ be models of T and $\mathbf{C}$ their common substructure. Denote L_C the extended language $L \cup \{\dot{c}_i : c_i \in C\}$, and let $\mathbf{A}_C, \mathbf{B}_C$ be the expansions of $\mathbf{A}, \mathbf{B}$ to L_C, where $\dot{c}_i^{\mathbf{A}_C} = \dot{c}_i^{\mathbf{B}_C} = c_i$. Then $\mathbf{A}_C$ are $\mathbf{B}_C$ elementarily equivalent (denoted $\mathbf{A}_C \equiv \mathbf{B}_C$).

(iii) If

1) $\mathbf{A}, \mathbf{B}$ are models of T,

2) $\mathbf{C}$ is their common substructure,

3) $\{\theta_i : i < n\}$ is a finite set of literals,

4) $\varphi(\bar{x}) = \exists y \bigwedge_{i<n} \theta_i(\bar{x}, y)$,

5) $\bar{c} \in C$ is a tuple of elements from $\mathbf{C}$,

then $\mathbf{A} \models \varphi(\bar{c}) \iff \mathbf{B} \models \varphi(\bar{c})$.

Proof.

(i) $\longrightarrow$ (ii) Let φ be an L_C-sentence valid in $\mathbf{A}_C$. We will transform φ to an L-formula $\psi(\bar{x})$, where every C-constant in L_C is substituted for a (new) variable. So naturally $\mathbf{A} \models \psi(\bar{c})$ for some $\bar{c} \in C$.

Since T has quantifier elimination, there is a quantifier-free L-formula $\lambda(\bar{x})$ such that $T \vdash \psi(\bar{x}) \leftrightarrow \lambda(\bar{x})$. Because $\lambda(\bar{x})$ has no quantifiers, we can infer

$$\mathbf{A} \models \psi(\bar{c}) \implies \mathbf{A} \models \lambda(\bar{c}) \implies \mathbf{C} \models \lambda(\bar{c}) \implies \mathbf{B} \models \lambda(\bar{c}) \implies \mathbf{B} \models \psi(\bar{c}).$$

The C-constants in $\mathbf{A}_C$ and in $\mathbf{B}_C$ are interpreted the same, so from $\mathbf{B} \models \psi(\bar{c})$ we finally obtain $\mathbf{B} \models \varphi$.

(ii) ⟶ (iii) If $\mathbf{A}_C \equiv \mathbf{B}_C$, then their reducts must be elementarily equivalent as well, so $\mathbf{A} \vDash \varphi(\bar{c}) \iff \mathbf{B} \vDash \varphi(\bar{c})$ (for any formula φ).

(iii) ⟶ (i) Suppose $\varphi(\bar{x}) = \exists y \ \bigwedge_{i<n} \theta_i(\bar{x}, y)$ is given, let m be the number of free variables in $\varphi(\bar{x})$. If we can find T-equivalent quantifier-free formula to $\varphi(\bar{x})$, then by lemma 2.5.2 T has quantifier elimination.

Let
$$\Lambda_{\bar{x}} = \{\lambda(\bar{x}) \text{ open } L\text{-formula} : T \vdash \varphi(\bar{x}) \to \lambda(\bar{x})\}.$$

Take m new constants $\bar{c}$ and denote L_C the expansion of the language L with the new constants $\bar{c}$. In the following we will suppose a constant c_i always substitutes for free variable x_i in $\varphi(\bar{x})$ (and vice-versa).

Suppose $T + \Lambda_{\bar{c}} \vdash \varphi(\bar{c})$. Then by finiteness of proofs there is a finite subset $\tilde{\Lambda}_{\bar{c}} \subseteq \Lambda_{\bar{c}}$, s.t. $T + \tilde{\Lambda}_{\bar{c}} \vdash \varphi(\bar{c})$. If we denote $\mu(\bar{x}) = \bigwedge_{\lambda \in \tilde{\Lambda}_{\bar{x}}} \lambda(x)$, deduction theorem gives us
$$T \vdash \mu(\bar{c}) \to \varphi(\bar{c}),$$
and therefore by substituting the same free variables for $\bar{c}$ as in $\varphi(\bar{x})$
$$T \vdash \mu(\bar{x}) \to \varphi(\bar{x}),$$
so with definition of formulas in Λ we obtain
$$T \vdash \varphi(\bar{x}) \leftrightarrow \mu(\bar{x}).$$

Now for the sake of contradiction suppose $T + \Lambda_{\bar{c}} \nvdash \varphi(\bar{c})$. From the completeness theorem let $\mathbf{A}$ be a L_C-structure such that
$$\mathbf{A} \vDash T + \Lambda_{\bar{c}} + \neg\varphi(\bar{c}),$$
and let $\mathbf{C} = \langle \bar{c}^{\mathbf{A}} \rangle \leq \mathbf{A}$ be a finitely generated substructure of $\mathbf{A}$.

We want to obtain a second L_C-structure $\mathbf{B}$ satisfying
$$\mathbf{B} \vDash T + \mathrm{Diag}(\mathbf{C}) + \varphi(\bar{c}).$$

If there was no such structure, this theory would be contradictory, i.e., we would have $T + \mathrm{Diag}(\mathbf{C}) + \varphi(\bar{c}) \vdash \bot$, by finiteness of proofs and by deduction theorem we again obtain a finite $\tilde{\Delta} \subseteq \mathrm{Diag}(\mathbf{C})$ satisfying
$$T \vdash \varphi(\bar{c}) \to \neg \bigwedge \tilde{\Delta}.$$

Consider the formula $\delta(\bar{x})$ created from $\bigwedge \tilde{\Delta}$ by substituting the same free variables for constants $\bar{c}$ as in $\varphi(\bar{x})$. By definition of diagram, we get $\mathbf{C} \vDash \delta(\bar{c})$.

Notice that both $\delta(\bar{c}), \neg\delta(\bar{c})$ are open formulas, so on one hand $\mathbf{A} \vDash \delta(\bar{c})$, but on the other from $T \vdash \varphi(\bar{c}) \to \neg\delta(\bar{c})$, we get $\neg\delta(\bar{c}) \in \Lambda_{\bar{c}}$, so $\mathbf{A} \vDash \neg\delta(\bar{c})$. But this is a contradiction.

Let us continue with the structure $\mathbf{B}$. Since $\mathbf{B} \vDash \mathrm{Diag}(\mathbf{C})$, we can WLOG suppose that $\mathbf{C}$ is a substructure of $\mathbf{B}$. But as we can see, the structures $\mathbf{A}$, $\mathbf{B}$ do not agree on validity of $\varphi(\bar{c})$: while $\mathbf{A} \nvDash \varphi(\bar{c})$, also $\mathbf{B} \vDash \varphi(\bar{c})$, which is in contradiction with (iii).

□

This method of proving that a theory has quantifier elimination does not yield an algorithm, so at first glance it would not help us with establishing decidability of the theory. However, for "well-behaved" theories, the existence of the algorithm can be established.

Lemma 2.5.6. *Let T be a recursively axiomatizable theory having quantifier elimination. Then there exists an algorithm which for a given formula in language of T gives a T-equivalent quantifier-free formula.*

Proof. Let the formula $\varphi(\bar{x})$ be given. The algorithm iterates over all proofs from T and checks whether they are proofs of the formula $\varphi(\bar{x}) \leftrightarrow \theta(\bar{x})$ for some quantifier-free $\theta(\bar{x})$. Since there exists quantifier-free $\theta_\varphi(\bar{x})$, s.t. $T \vdash \varphi(\bar{x}) \leftrightarrow \theta_\varphi(\bar{x})$, the proof of $\varphi(\bar{x}) \leftrightarrow \theta_\varphi(\bar{x})$ must be found in a finite time, and so the algorithm terminates. $\square$

Theorem 2.5.7. RCF *has quantifier elimination.*

Proof. We wish to verify the condition (iii) from lemma 2.5.5.

Let $\mathbf{A}, \mathbf{B}$ be two real-closed fields, $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$ their common substructure and $\bar{c} \in C$. Furthermore let $\varphi(\bar{x}) = \exists y \ \bigwedge_{i<n} \theta_i(\bar{x}, y)$ be a RCF-formula, where $\{\theta : i < n\}$ is a finite set of literals.

Every literal $\theta_i(\bar{x}, y)$ is of form

$$t_i = s_i, \quad t_i \neq s_i, \quad t_i < s_i, \text{ or } t_i \not< s_i,$$

$t_i(\bar{x}, y)$, $s_i(\bar{x}, y)$ being terms. The second can be RCF-equivalently rewritten as $t_i < s_i \vee s_i < t_i$, the fourth as $s_i < t_i \vee t_i = s_i$. Moreover every atomic RCF-formula $\psi(\bar{x}, y)$ is RCF-equivalent to

$$0 = f(y), \text{ or } 0 < f(y), \tag{$*$}$$

where f is a polynomial with coefficients being RCF-terms with variables in $\bar{x}$.[3] Using the same manipulation as in proof of lemma 2.5.2 and using the forms in eq. ($*$) we get

$$T \vdash \varphi(\bar{x}) \leftrightarrow \bigvee_j \exists y \bigwedge_k \chi_{j,k}(\bar{x}, y),$$

where $\chi_{j,k}(\bar{x}, y)$ are atomic. So we may WLOG assume that formulas $\theta_i(\bar{x}, y)$ looked like in eq. ($*$) in the first place, i.e., we may suppose

$$\varphi(\bar{x}) = \exists y\, (f_0(\bar{x}, y) = 0 \wedge \ldots \wedge f_{k-1}(\bar{x}, y) = 0 \wedge f_k(\bar{x}, y) > 0 \wedge \ldots \wedge f_{m-1}(\bar{x}, y) > 0),$$

where f_i are polynomials (in the logical sense, see remark 2.1.6.

Denote $\mathbf{D}$ the relative algebraic closure of $\mathbf{C}$ in $\mathbf{A}$, likewise $\mathbf{E}$ the closure in $\mathbf{B}$. Our goal will be to show

$$\mathbf{A} \vDash \varphi(\bar{c}) \implies \mathbf{D} \vDash \varphi(\bar{c}) \implies \mathbf{E} \vDash \varphi(\bar{c}) \implies \mathbf{B} \vDash \varphi(\bar{c}),$$

so let us start by assuming $\mathbf{A} \vDash \varphi(\bar{c})$. In other words there is $\xi \in A$ satisfying $\mathbf{A} \vDash \theta_i(\bar{c}, \xi)$ for all θ_i. Thus by the substitution $\bar{c}$ for $\bar{x}$, polynomials $f_i(\bar{x}, y)$ become $f_i(y)$, where coefficients are elements of C.

[3] By the choice of language, the coefficients are in fact "formal integers", i.e. WLOG either $1+1+\cdots+1$, 0 or $-(1+1+\cdots+1)$.

By corollary 2.3.4 we can factor the polynomials f_i as

$$f_i(y) = \gamma_i \cdot \prod_j (y - \varepsilon_{ij}) \cdot \prod_j ((y - \alpha_{ij})^2 + \beta_{ij}^2),$$

where γ_i, ε_{ij}, α_{ij} and β_{ij} are elements from D. From the irreducibility of quadratic factors we have $\beta \neq 0$ and therefore $((y - \alpha_{ij})^2 + \beta_{ij}^2) > 0$, so for the validity of $f_i(y) = 0$ or $f_i(y) > 0$ the quadratic factors are irrelevant.

Since θ_i is a polynomial equation and ξ is solution to it, only two cases may occur:

(i) There is $\theta_0(\bar{c}, \xi) = (f_0(\xi) = 0)$. Then there is some $\varepsilon_{0j} \in D$ that $\varepsilon_{0j} = \xi$. Thus $\xi \in D$ and we're done.

(ii) All θ_i look like $f_i(\xi) > 0$. Therefore $\varepsilon_{ij} \neq \xi$ for all ε_{ij} (otherwise some polynomial would equal 0). Since the ε_{ij}s divide the order in D (and in A) into finitely many open intervals, let us choose some $\zeta \in D$, s.t. $\xi > \varepsilon_{ij} \iff \zeta \in \varepsilon_{ij}$. As we can see, this ensures $f_i(\xi) > 0 \iff f_i(\zeta) > 0$, by $\zeta \in D$ we're done as well.

So far the first implication has been proven. By theorem 2.2.7 **D** and **E** are real closures of **C** and by theorem 2.2.8 $\mathbf{D} \cong \mathbf{E}$. Thus $\mathbf{D} \vDash \varphi(\bar{c}) \iff \mathbf{E} \vDash \varphi(\bar{c})$.

As for the final implication, since $\varphi(\bar{x})$ is an existential formula, its validity is preserved from substructure to superstructure. So we obtain $\mathbf{E} \vDash \varphi(\bar{c}) \implies \mathbf{B} \vDash \varphi(\bar{c})$.

Therefore $\mathbf{A} \vDash \varphi(\bar{c}) \implies \mathbf{B} \vDash \varphi(\bar{c})$. The other direction could be proven in the exactly same manner. By lemma 2.5.5, RCF has quantifier elimination. □

Theorem 2.5.8. RCF *is complete and decidable.*

Proof. Let φ be a sentence in language of RCF. By quantifier elimination, there is an RCF-equivalent quantifier-free sentence ψ. But quantifier-free sentence is just a boolean combination of relations on constant terms, so ψ must RCF-equivalently consist of atomic formulas

$$t = 0 \quad \text{or} \quad t > 0,$$

where t is a constant term, i.e., WLOG $t = 1 + 1 + \cdots + 1$ or $t = -(1 + 1 + \cdots + 1)$. Provability of these sentences or provability of their negation can be established easily, similarly for their boolean combination, thus RCF is complete.

Since from lemma 2.5.6 there is an algorithm yielding an equivalent quantifier-free formula and the process described above is algorithmic, the theory is decidable as well. □

Corollary 2.5.9. *Every two real-closed fields are elementary equivalent.*

2.6 Undecidability of finite subtheories of RCF

In the previous section we have arrived at an important result: decidability of RCF. It is only natural to ask whether we can claim decidability with a weaker axiomatic system (one may have a problem with the infinite number of axioms arrising from schema (xiv) in definition 2.1.3).

But as Tarski conjectured in [Tar57, p. 55], any finitely axiomatizable theory, which has $\mathbb{R}$ as a model, is undecidable. The conjecture was generalized and proven in [Zie82].

Theorem 2.6.1 (Ziegler). *A finitely axiomatizable theory, which has either an algebraically closed field,* $\mathbb{R}$ *or one of the p-adic fields* $\mathbb{Q}_p$ *as a model, is undecidable.*

We are not intersted in algebraically closed or p-adic fields. However let us define some finite extensions of the theory of fields, which will be useful for modeling geometry.

Definition 2.6.2. The *theory of Pythagorean fields* is a theory of ordered fields with the axiom

(xvi) $\forall x, y \, \exists z \, (z^2 = x^2 + y^2)$.

The *theory of Euclidean fields* is a theory of ordered fields with the axiom (xv) from definition 2.1.3.

Clearly a Euclidean fields is also a Pythagorean field, because over the theory of ordered fields (xv) implies (xvi). Moreover these theories are subtheories of RCF (every sentence provable in these theories is also provable in RCF). Since both of these theories are finitely axiomatized and are valid in $\mathbb{R}$, the theorem 2.6.1 applies.

Corollary 2.6.3. *The theory of Pythagorean and Euclidean fields are undecidable.*

3 The geometry and the algebra

Let us recall the hierarchy of algebraic fields suitable for geometry (sorted from the most general to the most specific):

Pythagorean fields fields closed under the operation $\sqrt{a^2 + b^2}$ for any a, b

Euclidean fields ordered fields containing square roots of all positive elements

real-closed fields Euclidean fields containing a roof of any odd-degree polynomial

In this chapter we will explore the suitability also establish correspondence between these fields and the Tarski geometry.

3.1 Cartesian spaces over ordered fields

This section focuses on algebraization of geometry, exploring the so-called Cartesian spaces over arbitrary fields, which are constructed to be models for Euclidean geometry. They are named after René Descartes, who in 17th century introduced coordinates into geometry and thus founded analytic geometry, in other words, studying geometry using algebraic (or other arithmetic) means (see [SL54]).

We will examine relation between subtheories of CE_n^m and subtheories of RCF. The following definition and theorem follows [SST83, p. 16].

Definition 3.1.1. Let $\mathbf{F} = (F, +, \cdot, 0, 1, <)$ be an ordered field. By *n-dimensional Cartesian space over* $\mathbf{F}$ we mean the structure $\mathscr{C}_n(\mathbf{F}) = (F^n, \beta, \delta)$, where

$$\begin{aligned}\beta(\bar{p}, \bar{q}, \bar{r}) &\iff \exists \lambda\, (0 \leq \lambda \leq 1 \wedge \bar{q} - \bar{p} = \lambda(\bar{r} - \bar{p}))\\ \delta(\bar{p}, \bar{q}, \bar{r}, \bar{s}) &\iff (p_1 - q_1)^2 + \dots + (p_n - q_n)^2 = (r_1 - s_1)^2 + \dots + (r_n - s_n)^2\end{aligned}$$

We will now prove that this structure is a model of axiomatization of geometry presented in this thesis. The formulation of this theorem follows [SST83, p. 16].

Theorem 3.1.2. *Let an ordered field* $\mathbf{F}$ *and a number* $n \geq 1$ *be given.*

(i) The axioms (TE), (IE), (FS), (IB), (Pa) and (Eu) hold in $\mathscr{C}_n(\mathbf{F})$*.*

(ii) If $n = 1$ *or* $\mathbf{F}$ *is Pythagorean, then (SC) holds in* $\mathscr{C}_n(\mathbf{F})$*.*

(iii) (Lo$_k$) holds in $\mathscr{C}_n(\mathbf{F})$*, if and only if* $k \leq n$*.*

(iv) (Up$_k$) holds in $\mathscr{C}_n(\mathbf{F})$*, if and only if* $k \geq n$*.*

(v) If $\mathbf{F}$ *is Euclidean, then (Ci) holds in* $\mathscr{C}_n(\mathbf{F})$*.*

(vi) If $\mathbf{F}$ *if real-closed, then all instances of (Co) hold in* $\mathscr{C}_n(\mathbf{F})$*.*

Proof. Let $\mathbf{F}$ and n be given. We shall prove the theorem for each axiom separately.

(TE) Suppose $\delta(\bar{a}, \bar{b}, \bar{p}, \bar{q})$ and $\delta(\bar{a}, \bar{b}, \bar{r}, \bar{s})$ hold, i.e.,

$$\begin{aligned}(a_1 - b_1)^2 + \dots + (a_n - b_n)^2 &= (p_1 - q_1)^2 + \dots + (p_n - q_n)^2\\ (a_1 - b_1)^2 + \dots + (a_n - b_n)^2 &= (r_1 - s_1)^2 + \dots + (r_n - s_n)^2.\end{aligned}$$

Since the left-hand sides of these equations are equal, the right-hand sides have to be as well, so we have arrived at $\delta(\bar{p}, \bar{q}, \bar{r}, \bar{s})$.

(IE) Suppose $\delta(\bar{a}, \bar{b}, \bar{c}, \bar{c})$, i.e.,

$$(a_1 - b_1)^2 + \dots + (a_n - b_n)^2 = (c_1 - c_1)^2 + \dots + (c_n - c_n)^2 = 0.$$

In an ordered field, the squares are non-negative (lemma 2.1.4), so all summands on the left-hand side have to be zero, i.e. $a_i = b_i$ for all $1 \leq i \leq n$, therefore $\bar{a} = \bar{b}$.

(FS) Suppose $\beta(\bar{a},\bar{b},\bar{c})$, $\beta(\bar{p},\bar{q},\bar{s})$, $\delta(\bar{a},\bar{b},\bar{p},\bar{q})$, $\delta(\bar{b},\bar{c},\bar{q},\bar{r})$, $\delta(\bar{a},\bar{d},\bar{p},\bar{s})$, $\delta(\bar{b},\bar{d},\bar{q},\bar{s})$.

First let us consider $\bar{a} = \bar{c}$, therefore by (IE), $\bar{p} = \bar{r}$. By simple manipulation

$$\sum_{i=1}^{n}(d_i - c_i)^2 = \sum_{i=1}^{n}(c_i - d_i)^2 = \sum_{i=1}^{n}(a_i - d_i)^2$$

and similarly with $\bar{p}$, $\bar{r}$, $\bar{s}$, we arrive at

$$\delta(\bar{a},\bar{d},\bar{p},\bar{s}) \to \sum_{i=1}^{n}(d_i - c_i)^2 = \sum_{i=1}^{n}(s_i - r_i)^2,$$

and therefore $\delta(\bar{d},\bar{c},\bar{s},\bar{r})$.

Now we will assume $\bar{a} \neq \bar{c}$, therefore $\bar{p} \neq \bar{r}$ (by (IE)). The betweennesses give us $\lambda, \mu \in F$, s.t. $0 < \lambda, \mu \leq 1$, $\bar{b} - \bar{a} = \lambda(\bar{c} - \bar{a})$ and $\bar{q} - \bar{p} = \mu(\bar{r} - \bar{p})$. (The strict positivity is given by $\bar{a} \neq \bar{c}$.) This implies

$$\sum_{i=1}^{n}(a_i - b_i)^2 = \lambda^2 \sum_{i=1}^{n}(a_i - c_i)^2$$

(similarly with $\bar{p}$, $\bar{q}$, $\bar{r}$, μ). But from $\delta(\bar{a},\bar{b},\bar{p},\bar{q})$ and $\delta(\bar{a},\bar{c},\bar{p},\bar{r})$ we obtain $\lambda^2 = \mu^2$, or equivalently $(\lambda - \mu)(\lambda + \mu) = 0$. Since the second term in the product is strictly positive, we conclude $\lambda = \mu$.

The rest of the proof involves rather lengthy algebraic computation, so we will present it only in outline. From $\delta(\bar{a},\bar{b},\bar{p},\bar{q})$, $\delta(\bar{a}.\bar{d},\bar{p},\bar{s})$ and $\delta(\bar{b},\bar{d},\bar{q},\bar{s})$ we obtain

$$\sum_{i=1}^{n}(a_i - d_i)(a_i - b_i) = \sum_{i=1}^{n}(p_i - s_i)(p_i - q_i).$$

Multiplying this equation by $\frac{1}{\lambda}$ (since $\lambda > 0$) and using $\lambda = \mu$ we get

$$\sum_{i=1}^{n}(a_i - d_i)(a_i - c_i) = \sum_{i=1}^{n}(p_i - s_i)(p_i - r_i).$$

Now we continue "in reverse", using $\delta(\bar{b},\bar{c},\bar{q},\bar{r})$, $\delta(\bar{b},\bar{d},\bar{q},\bar{s})$ we arrive at

$$\sum_{i=1}^{n}(c_i - d_i)^2 = \sum_{i=1}^{n}(r_i - s_i)^2,$$

which implies $\delta(\bar{d},\bar{c},\bar{s},\bar{r})$.

(IB) Suppose $\beta(\bar{a},\bar{b},\bar{a})$, i.e., for some $\lambda \in F$ we have $\bar{b} - \bar{a} = \lambda(\bar{a} - \bar{a}) = 0$, thus $\bar{a} = \bar{b}$ is obtained, which was desired.

(Pa) Suppose $\beta(\bar{a},\bar{p},\bar{c})$ and $\beta(\bar{b},\bar{q},\bar{c})$, i.e., there are $\lambda, \mu \in F$ satisfying $0 \leq \lambda, \mu \leq 1$, $\bar{p} - \bar{a} = \lambda(\bar{c} - \bar{a})$ and $\bar{q} - \bar{b} = \mu(\bar{c} - \bar{b})$. We will want to prove there exists $\bar{x}$, s.t. $\beta(\bar{p},\bar{x},\bar{b})$ and $\beta(\bar{q},\bar{x},\bar{a})$. Note that this $\bar{x}$ will be uniquely determined by parameters $\rho, \sigma \in F$, $0 \leq \rho, \sigma \leq 1$, s.t.

$$\bar{x} - \bar{p} = \rho(\bar{b} - \bar{p}), \quad \text{and} \quad \bar{x} - \bar{q} = \sigma(\bar{a} - \bar{q}),$$

so it suffices to establish existence of ρ, σ.

By using $\bar{x} = \bar{x}$, the following transformations can be (equivalently) made

$$\begin{aligned}
0 &= \rho(\bar{b} - \bar{p}) + \bar{p} - \sigma(\bar{a} - \bar{q}) - \bar{q},\\
0 &= \rho(\bar{b} - \bar{c}) + \rho(\bar{c} - \bar{a}) + \rho(\bar{a} - \bar{p})\\
&\quad + \sigma(\bar{q} - \bar{b}) + \sigma(\bar{b} - \bar{c}) + \sigma(\bar{c} - \bar{a})\\
&\quad + (\bar{p} - \bar{a} + \bar{a} - \bar{c} + \bar{c} - \bar{b} + \bar{b} - \bar{q}),\\
0 &= (\bar{c} - \bar{a})(\rho - \lambda\rho + \sigma + \lambda - 1) + (\bar{c} - \bar{b})(-\sigma + \mu\sigma - \rho - \mu + 1).
\end{aligned}$$

If we can find ρ, σ, s.t.

$$\rho - \lambda\rho + \sigma + \lambda - 1 = 0, \quad \text{and} \quad -\sigma + \mu\sigma - \rho - \mu + 1 = 0,$$

we're done. So consider this system of linear equation in its matrix form

$$\begin{pmatrix} 1-\lambda & 1 & 1-\lambda \\ -1 & \mu - 1 & \mu - 1 \end{pmatrix},$$

which clearly has solution for any $0 \le \lambda, \mu \le 1$. Detailed observation reveals that there is always a solution satisfying $0 \le \rho, \sigma \le 1$, which concludes the proof.

(Eu) Suppose $\beta(\bar{a}, \bar{d}, \bar{t})$, $\beta(\bar{b}, \bar{d}, \bar{c})$ and $\bar{a} \neq \bar{d}$. Therefore we have $0 < \lambda \le 1$ satisfying $\bar{d} - \bar{a} = \lambda(\bar{t} - \bar{a})$. We will define

$$\bar{x} = \frac{1}{\lambda}(\bar{b} - \bar{a}) + \bar{a}, \quad \text{and} \quad \bar{y} = \frac{1}{\lambda} \cdot (\bar{c} - \bar{a}) + \bar{a},$$

and prove that these $\bar{x}$, $\bar{y}$ satisfy the required properties. The first two, $\beta(\bar{a}, \bar{b}, \bar{x})$, $\beta(\bar{a}, \bar{c}, \bar{y})$, are quite obvious, so let us concentrate upon $\beta(\bar{x}, \bar{t}, \bar{y})$.

From $\beta(\bar{b}, \bar{d}, \bar{c})$, we obtain $0 \le \mu \le 1$, s.t. $\bar{d} - \bar{b} = \mu(\bar{c} - \bar{b})$. Using straightforwards transformations:

$$\begin{aligned}
(\bar{d} - \bar{a}) - (\bar{b} - \bar{a}) &= \mu((\bar{c} - \bar{a}) - (\bar{b} - \bar{a}))\\
\frac{1}{\lambda}(\bar{d} - \bar{a}) - \frac{1}{\lambda}(\bar{b} - \bar{a}) &= \mu\left(\frac{1}{\lambda}(\bar{c} - \bar{a}) - \frac{1}{\lambda}(\bar{b} - \bar{a})\right)\\
(\bar{t} - \bar{a}) - (\bar{x} - \bar{a}) &= \mu((\bar{y} - \bar{a}) - (\bar{x} - \bar{a}))\\
\bar{t} - \bar{x} &= \mu(\bar{y} - \bar{x}),
\end{aligned}$$

we obtain $\beta(\bar{x}, \bar{t}, \bar{y})$, as required.

(SC) Suppose $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{q}$ are given.

First let us distinguish the case $\bar{a} = \bar{q}$. Then $\beta(\bar{q}, \bar{a}, \bar{x})$ holds for any $\bar{x}$, we will define the $\bar{x}$ as $\bar{a} - \bar{b} + \bar{c}$, i.e., $\bar{x} - \bar{a} = \bar{c} - \bar{b}$, which entails $\delta(\bar{a}, \bar{x}, \bar{b}, \bar{c})$.

Now suppose $\bar{a} \neq \bar{q}$. Since we want $\beta(\bar{q}, \bar{a}, \bar{x})$, we can uniquely determine $\bar{x}$ as

$$\bar{x} = \frac{1}{\lambda}(\bar{a} - \bar{q}) + \bar{q},$$

for $0 < \lambda \le 1$, so we only need to find this λ. Furthermore, this $\bar{x}$ needs to satisfy

$$\sum_{i=1}^{n}(x_i - a_i)^2 = \sum_{i=1}^{n}(c_i - b_i)^2,$$

i.e., for λ

$$\sum_{i=1}^{n}\left(\frac{1}{\lambda}(a_i-q_i)+q_i-a_i\right)^2=\sum_{i=1}^{n}(c_i-b_i)^2.$$

Since $\bar{a}\neq\bar{q}$, this can be simplified to

$$\left(\frac{1}{\lambda}-1\right)^2=\frac{\sum_{i=1}^{n}(c_i-b_i)^2}{\sum_{i=1}^{n}(a_i-q_i)^2}.$$

In the case of $n=1$ (so $\bar{a}$ is just a etc.), this comes down to $\frac{1}{\lambda}-1=\left|\frac{c-b}{a-q}\right|$, which leads to a unique solution satisfying $0<\lambda\leq 1$.

In the case of **F** being Pythagorean, observe that square root of arbitrary sum of squares can be taken (this can be proven by induction on the length of the sum). So we can write

$$\frac{1}{\lambda}-1=\frac{\sqrt{\sum_{i=1}^{n}(c_i-b_i)^2}}{\sqrt{\sum_{i=1}^{n}(a_i-q_i)^2}},$$

and from **F** being Pythagorean, this also leads to a unique solution satisfying $0<\lambda\leq 1$.

(Lo_k) Assuming $k\leq n$, this is rather straightforward: set $\bar{u}_i=(0,\dots,0,1,0,\dots,0)$, where 1 is only on the i-th index, $\bar{s}=-\bar{u}_1$ and $\bar{o}=(0,\dots,0)$. Then $\beta(\bar{u}_1,\bar{o},\bar{s})$ holds, as well as $\delta(\bar{o},\bar{u}_i,\bar{o},\bar{s})$ for $1\leq i\leq k$ and $\delta(\bar{u}_i,\bar{u}_j,\bar{s},\bar{u}_2)$ for $1\leq i<j\leq k$.

The other direction, namely from validity of the lower dimension axiom to determining the dimension of the Cartesian space, is slightly less obvious and we will only sketch the proof using linear algebra. One can view F^n as domain of a vector space over **F**. The lower dimension axiom gives us a linear independent set of size k (given by $\bar{u}_i-\bar{o}$) for the vector space, which implies $k\leq n$.

(Up_k) Follows from the previous case.

(Ci) Suppose **F** is Euclidean, i.e., square root of an arbitrary positive element can be taken, which allows us to solve quadratic equations (with non-negative discriminant). Note as an observation that this ensures quadratic polynomials satisfy the intermediate value theorem (if there are $x,y\in F$, s.t. $p(x)p(y)<0$, then there is a z which $p(z)=0$).

Assume $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{p}$, $\bar{q}$ and $\bar{r}$ are given. Denote the function $h(\bar{y})$ as

$$h(\bar{y})=\sum_{i=1}^{n}(y_i-c_i)^2-\sum_{i=1}^{n}(r_i-c_i)^2,$$

The hypersphere H (generalization of circle) with center c and with r laying on H is then the set $H=\{\bar{y}\in F^n : h(\bar{y})=0\}$.

For $\bar{x}$ to satisfy $\beta(\bar{a},\bar{x},\bar{b})$ we can instead use a parameter $0\leq\lambda\leq 1$ given by the betweenness. If $\bar{x}$ is substituted into $h(\bar{y})$, we obtain a quadratic polynomial in one variable λ. We know $h(\bar{a})\leq 0$ and $h(\bar{b})\geq 0$, so by the observation there is a root λ_0, which determines the point $\bar{x}$ that by the definition of H satisfies the required properties.

(Co) In this case we will use an important result of the previous chapter: every two real-closed fields are elementary equivalent (corollary 2.5.9). Furthermore we're just interested in betweenness, which is an affine property. The following WLOG simplifications can thus be made:

- $\mathbf{F} = \mathbb{R}$
- there are $\bar{p}, \bar{q} \in \mathrm{R}^n$ satisfying $\varphi(\bar{p})$, $\psi(\bar{q})$ (otherwise choose arbitrary $\bar{x}$), therefore all points satisfying $\varphi(z)$ or $\psi(z)$ lay on a ray with endpoint $\bar{a}$
- $\bar{a} = (0, \dots, 0)$ and the only $\bar{b}$ satisfying $\varphi(z)$ or $\psi(z)$ are $\bar{b} = (b, 0, \dots, 0)$ for $b \geq 0$; therefore we can disregard the other dimensions and identify the point with its first index

Now since for all $x \in \mathrm{R}$, s.t. $\varphi(x)$, we have $\beta(a, x, q)$, the set $\{x \in \mathrm{R} : \varphi(x)\}$ is upper-bounded, and thus has a supremum s. We want to argue that this s satisfies $\varphi(x) \wedge \psi(y) \rightarrow \beta(x, s, y)$ for arbitrary $x, y \in \mathrm{R}$. This is equivalent to proving $s \leq y$ for any $y \in \mathrm{R}$ with $\psi(y)$.

For the sake of contradiction let $y \in \mathrm{R}$ be, s.t. $\psi(y)$ and $a \leq y < s$. Since s is the least upper bound of all x satisfying $\varphi(x)$, there is a $x_y \in \mathrm{R}$ s.t. $\varphi(x_y)$ and $a \leq y < x_y \leq s$. But then $\beta(a, x_y, y)$ is violated, so we arrive at a contradiction. □

The previous theorem can be stated in a simple form:

Corollary 3.1.3. *Let an ordered field* $\mathbf{F}$ *and a number* $n \geq 1$ *be given.*

(i) $\mathscr{C}_n(\mathbf{F}) \vDash \mathsf{A}$

(ii) If $\mathbf{F}$ *is Euclidean, then* $\mathscr{C}_n(\mathbf{F}) \vDash \mathsf{GEC}_n$.

(iii) If $\mathbf{F}$ *is real-closed, then* $\mathscr{C}_n(\mathbf{F}) \vDash \mathsf{CE}_n$.

3.2 Algebraic properties of the Cartesian coordinate system

In this section we will view the number line and the Cartesian coordinate system constructed in section 1.3 using the algebraic framework built in chapter 2. Thus we will lay groundwork for proving the theorem 3.3.1. Similarly to section 1.3, we assume $n \geq 2$, as we will in the next section.

Theorem 3.2.1. *For any two distinct o, e, the number line $\mathscr{L}_{oe}$ forms a real-closed field.*

Proof. As an example, we're going to prove (i) commutativity of multiplication, (ii) every positive number is a square, and (iii) every polynomial of odd degree has a root.

(i) Suppose we have points a, b, c satisfying and assume $a, b \neq o$. By $\mathrm{Prod}_{oe}^{e'}(a, b, c)$, there is a point b', s.t. $\mathrm{Col}(o, e', b')$ and $\mathrm{Pj}(e, e', b, b')$, similarly construct point a satisfying $\mathrm{Col}(o, e', a')$, $\mathrm{Pj}(e, e', a, a')$.

The $\mathrm{Prod}_{oe}^{e'}(a, b, c)$ gives us $ee' \parallel bb'$ and $ae' \parallel cb'$. Theorem 1.2.12 then establishes $be' \parallel ca'$. Thus from $ee' \parallel aa'$ and $e'b \parallel a'c$ we have $\mathrm{Prod}_{oe}^{e'}(b, a, c)$ and we are done.

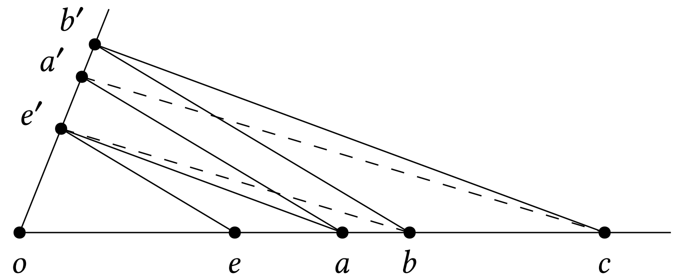

Figure 3.1 Geometric multiplication is multiplicative

(ii) This is easy, as it follows directly from geometric mean theorem.

(iii) Assume we have already proven that $\mathscr{L}_{oe}$ is an ordered field. Let f be a monic polynomial of odd degree. By lemma 2.3.2, there are points λ, μ on the number line oe, s.t. if there is a point $x \in L_{oe}$ with $x < \lambda$, then $f(x) < 0$, and if $x > \mu$, then $f(x) > 0$.

Let us introduce formulas $\varphi(x)$ and $\psi(x)$

$$\varphi(x) = \mathrm{Ar}_{oe}(x) \wedge \forall t\ (t \leq x \rightarrow f(t) < 0),$$
$$\psi(y) = \mathrm{Ar}_{oe}(x) \wedge f(y) > 0.$$

The formula $\exists z\ \forall x, y\ (\varphi(x) \wedge \psi(y) \rightarrow \beta(zxy))$ now holds (e.g. set z as $\lambda - 1$). But from (Co), we obtain a point u, s.t. $\forall x, y\ (\varphi(x) \wedge \psi(y) \rightarrow \beta(xuy))$.

By lemma 2.3.1 for any point x with $f(x) > 0$ there is a point $y > 0$, s.t. $\forall z\ (x - y < z < x + y \rightarrow f(z) > 0)$ (and similarly for $f(x) < 0$). If $f(u) > 0$, we could find some x, y, s.t. $\beta(xuy)$, but $f(x) > 0$, which leads to contradiction. If $f(u) < 0$, we could argue in similar fashion. The only remaining option is $f(u) = 0$, and so we found the root of f, which is also root of f. $\square$

Theorem 3.2.2. *Suppose both (Lo_n) and (Up_n) hold for a fixed $n \geq 2$. Let a Cartesian coordinate system determined by o, e, s, $\bar{u}$ be given. Denote M the set of all points in the space and L_{oe} the line oe (see definition 1.3.6). Then the map $c : M \to L_{oe}^n$ assigning a point its coordinates is a bijection.*

Theorem 3.2.3. *Let a Cartesian coordinate system determined by o, e, s, $u_1, \ldots, u_n$ be given. Let a, b, c, d be points with coordinates $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\delta}$ respectively. Then the following holds*

(i) $ab \simeq cd \iff \sum_{i=1}^{n}(\alpha_i - \beta_i)^2 = \sum_{i=1}^{n}(\gamma_i - \delta_i)^2$,

(ii) $\bar{a}\dot{b}\bar{c} \iff \exists t \; \left(\mathrm{Ar}_{oe}(t) \wedge o \leq t \leq e \wedge \bigwedge_{i=1}^{n} \beta_i - \alpha_i = t \cdot (\gamma_i - \alpha_i)\right)$.

Notice how the definitions of relations on coordinates and definitions of relations in a Cartesian space (definition 3.1.1) are similar. This almost justifies viewing an arbitrary geometric space as a Cartesian space.

The two theorems above together give us an isomorphism between a geometric space and its coordinatization.

Corollary 3.2.4. *Let $\mathsf{CE}_n \models \mathscr{M}$ be a geometric space, $o, e \in M$ two distinct points, and a Cartesian coordinate system determined by o, e, s, $\bar{u}$ be given. Denote $\simeq_{oe}$ and $-\!\cdot\!-_{oe}$ the operations on coordinates (n-tuples of points on number line oe) in the theorem 3.2.3. Then the structures $(M, \simeq, -\!\cdot\!-)$ and $(L_{oe}^n, \simeq_{oe}, -\!\cdot\!-_{oe})$ are isomorphic and the isomorphism is the map assigning a point its coordinates.*

3.3 Representation, completeness and decidability of geometric spaces

In section 3.1, we have shown that Cartesian space over a real-closed field is a model of CE_n. But we can postulate much stronger proposition: that the Cartesian spaces over real-closed fields are, up to an isomorphism, the only models of this geometry. We have prepared for proving this theorem in section 3.2. In this section we will explore further consequences of the theorem.

Theorem 3.3.1 (Tarski representation theorem, [Tar59, p. 21])**.** *For $\mathscr{M}$ to be a model of* CE_n *it is necessary and sufficient that $\mathscr{M}$ be isomorphic with the Cartesian space $\mathscr{C}_n(\mathbf{F})$ over some real-closed field* $\mathbf{F}$*.*

Proof.

($\Leftarrow$) Let $\mathscr{M}$ be isomorphic to the Cartesian space $\mathscr{C}_n(\mathbf{F})$ for a real-closed field $\mathbf{F}$. Then since $\mathscr{C}_n(\mathbf{F})$ is a model of the CE_n (corollary 3.1.3), $\mathscr{M}$ is also a model.

($\Rightarrow$) Let $\mathsf{CE}_n \vDash \mathscr{M}$. From corollary 3.2.4, $\mathscr{M}$ is isomorphic to its coordination, which is in fact a Cartesian space over a real-closed field. □

Thus any proposition about geometric space (in the theory CE_n) can be viewed as an algebraic proposition (in the theory RCF), and vice-versa. Recalling theorem 2.5.8, we arrive at the following theorem.

Theorem 3.3.2. *For any $n \geq 2$, the theory* CE_n *is complete and decidable.*

Remark 3.3.3. Note that the completeness and decidability of the first-order geometric theory is in part given by an infinite axiom schema. If we would try to axiomatize the theory using a finite number of axioms, such as GEC_n, the decidability (and consequently completeness) will inherently fail due to theorem 2.6.1.

Conclusion

In this thesis, we explored Tarski's axiomatization of geometry, and both geometrization of algebra (introducing algebra in geometry) and algebraization of geometry (constructing geometry in algebraic setting).

We have used a recent axiomatization of geometry, simplifying the one developed by Tarski, and introduced some important geometrical notions using it. Contrary to traditional synthetic geometry, we have been constricted by a single primitive notion and first-order logic; but even in this restricted setting, the usual tools such as parallels and right angles can still be used. This was used as a basis for constructing an algebraic field inside the geometrical space, and for introducing coordinates to the space.

Next, we defined real-closed fields (together with ordinary rings and fields) and reviewed results from commutative algebra concerning these structures. We needed to generalize some of the propositions from fields, which are the usual studied structure, to arbitrary rings, as the logical language we were using necessitated that a substructure of a real-closed field is an ordered ring, not an ordered field.

The built theory was then used to prove an important theorem first proved by Tarski: the completeness and decidability of the theory of real-closed fields, as a result of this theory having quantifier elimination. The theorem was proven using a more general model-theoretic method, as opposed to the more direct proof used by Tarski or Švejdar.

The constructions done in geometry and in algebra were then brought together by presenting Tarski's representation theorem, which states that all geometric spaces (in Tarski's axiomatics) are up-to an isomorphism only Cartesian spaces over real-closed fields, and then by concluding that the theory of geometry was complete and decidable as well.

The main contribution of this thesis lies in summarizing and assessing Tarski's axioms of geometry from the axiomatic, geometrical and algebraical standpoints, and in reviewing both classic (due to Tarski and his colleagues) and more recent results in this field.

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List of symbols

WLOG	without loss of generality,	page 25

Geometrical symbols

$ab \parallel cd$	lines ab and cd are parallel,	page 13
$\mathrm{Cd}_{oe}(\bar{a})$	point p has coordinates $\bar{x}$,	page 17
$\mathrm{Cs}_{oe}(s, \bar{u})$	Cartesian coordinate system,	page 17
$\mathscr{L}_{oe}$	(ordered) field of points lying on the number line oe,	page 16
$\mathrm{Ar}_{oe}(\bar{a})$	points $\bar{a}$ lie on a number line oe,	page 15
$\mathrm{R}(a, b, c)$	points a, b, c form a right angle,	page 14
$\overline{a}\dot{b}\overline{c}$	betweenness,	page 9
$ab \simeq cd$	equidistance,	page 9
$\mathrm{Col}(x, y, z)$	points x, y, z are collinear,	page 12
$\mathrm{Pj}(a, b, c, d)$	d is the parallel projection of c onto ab,	page 13
$\mathrm{Prod}_{oe}^{e'}(a, b, c)$	geometric product,	page 16
$\mathrm{Sum}_{oe}^{e'}(a, b, c)$	geometric sum,	page 15

Structures

$\mathbb{A}$	(ordered) field of real algebraic numbers over $\mathbb{Q}$,	page 25
$\mathbb{H}$	(ordered) field of hyperreal numbers,	page 26
$\mathbb{R}$	(ordered) field of real numbers,	page 20

Theories

A	theory of absolute geometry,	page 9
GEC_n^m	theory of geometry of elementary constructions,	page 11
CE_n^m	Tarski's axioms of Euclidean geometry,	page 10
RCF	theory of real-closed fields,	page 20