Analyzing Rating Transitions and Rating Drift with Continuous Observations

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Motivation

- Credit rating transitions are crucial in credit risk management.
- Traditional methods use discrete-time cohort models, which have limitations,

$$\hat{p}_{ij} = \frac{N_{ij}}{N_i}, \quad i \neq j.$$

N_i firms that started in i-th category.
N_{ij} firms that migratet from i-th to j-th category.

Problem Formulation

- Rating agencies or banks have continuous-time data.
- 'rare event'
- Other advantages
  - Non-Markov type behavior
  - The dependence on external covariates
  - Censoring
  - Generator of the continuous-time Markov chain

Homogeneous Markov Process Model

- K-state Markov chain
- Transition probability matrix $\boldsymbol{P}(t)$ with elemets $p_{ij}(t)$
- Generator matrix $\boldsymbol{\Lambda}$ such that

$$\boldsymbol{P}(t) = exp(\boldsymbol{\Lambda} t), \quad t \geq 0.$$

- Maximum likelihood estimation

Generator

- $\lambda_{ij} \neq 0, \lambda_{ii} = -\sum_{j \neq 1} \lambda_{ij}$
- Matrix exponential $\sum_{k=1}^{\infty} \frac{\mathbf{\Lambda}^k t^k}{k!}$
- $\hat{\lambda}_{ij} = \frac{N_{ij}(T)}{\int_0^T Y_i(s)\, ds}$
  (Kuchler and Sørensen, 1997)

Example

$$\hat{\boldsymbol{\Lambda}} = \begin{bmatrix} 0.10084 & 0.10084 & 0 \\ 0.10909 & -0.21818 & 0.10909 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\boldsymbol{P}}_{\boldsymbol{G}}(t) = \begin{bmatrix} 0.90887 & 0.08618 & 0.00495 \\ 0.09323 & 0.80858 & 0.09819 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{\boldsymbol{P}}(t) = \begin{bmatrix} 0.90 & 0.10 & 0 \\ 0.10 & 0.80 & 0.10 \\ 0 & 0 & 1 \end{bmatrix}, \ (cohort \ metod).$$

Data

- Standard and Poor's data set
- 10 years of rating history in the S&P system starting on 1. January 1988.
- 6669 firms were rated at some point
- 8 categories (AAA, AA, A, BBB, BB, B, CCC, D)
- The exact transition date is recorded

Homogeneous Markov Process with data

	NR	AAA	AA	A	BBB	BB	B	CCC	D
NR	0.9939	0.0001	0.0000	0.0001	0.0006	0.0006	0.0004	0.0000	0.0044
AAA	0.0266	0.9040	0.0607	0.0070	0.0000	0.0016	0.0000	0.0000	**0.0000**
AA	0.0302	0.0054	0.8786	0.0791	0.0039	0.0006	0.0008	0.0000	**0.0000**
A	0.0401	0.0004	0.0157	0.8903	0.0445	0.0068	0.0017	0.0001	0.0003
BBB	0.0583	0.0001	0.0028	0.0519	0.8375	0.0388	0.0068	0.0018	0.0018
BB	0.0906	0.0000	0.0003	0.0051	0.0795	0.7452	0.0587	0.0110	0.0095
B	0.1268	0.0000	0.0008	0.0015	0.0050	0.0730	0.7081	0.0326	0.0500
CCC	0.1658	0.0020	0.0000	0.0061	0.0089	0.0279	0.1003	0.4842	**0.2048**
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Table 1: This shows the average of 10 one-year transition matrices, each estimated using a cohort method in the period 1988–1998. (Lando, D., & Skødeberg, T. M. (2002))

Homogeneous Markov Process with data

	NR	AAA	AA	A	BBB	BB	B	CCC	D
NR	0.9935	0.0000	0.0001	0.0003	0.0006	0.0009	0.0003	0.0000	0.0043
AAA	0.0248	0.8995	0.0640	0.0091	0.0005	0.0020	0.0001	0.0000	**0.0001**
AA	0.0321	0.0061	0.8788	0.0761	0.0057	0.0006	0.0004	0.0000	**0.0001**
A	0.0424	0.0004	0.0129	0.8944	0.0436	0.0047	0.0011	0.0002	0.0002
BBB	0.0545	0.0003	0.0023	0.0479	0.8479	0.0393	0.0063	0.0008	0.0008
BB	0.0965	0.0000	0.0012	0.0090	0.0869	0.7303	0.0612	0.0084	0.0065
B	0.1518	0.0001	0.0022	0.0024	0.0084	0.0643	0.6734	0.0534	0.0440
CCC	0.1429	0.0025	0.0003	0.0053	0.0017	0.0215	0.0674	0.3824	**0.3760**
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Table 2: The one-year transition matrix estimated from continuous-time data over the period 1988–1998 as the matrix exponential of the maximum-likelihood estimator of the generator. (Lando, D., & Skødeberg, T. M. (2002))

Non-homogeneous Markov Process

- continuous-time non-homogeneous Markov chain η
- transition matrix $\mathbf{P}(s,t)$
- $p_{ij}(s,t) = P(\eta_t = i, \eta_s = j)$, s<t
- $\lambda_{ij}(t) = \lim_{h \to 0^+} p_{ij}(t, t+h)/h$
- $\mathbf{A}_{ij}(t) = \int_0^t \lambda_{ij}(s)\, ds$
- $\mathbf{P}(s,t) = \prod_{[s,t]}(\mathbf{I} + d\mathbf{A}) \equiv$ $\lim_{\max|t_i - t_{i-1}| \to 0^+} \prod_i (I + \mathbf{A}(t_i) - \mathbf{A}(t_{i-1}))$

Non-homogeneous Markov Process Model

- Consider a non-homogeneous, continuous-time Markov process
- Time-dependent transition matrix $\boldsymbol{P}(s, t)$
- $p_{ij}(s, t)$, from state i on date s to state j on date t
- Aalen–Johansen estimator, for the transition probabilities $\mathbf{P}(s, t)$

$$\hat{\mathbf{P}}(s, t) = \prod_{i=1}^{m} (\mathbf{I} + \boldsymbol{\Delta}\hat{\mathbf{A}}(T_i)).$$

Non-homogeneous Markov Process Model

$$\mathbf{\Delta\hat{A}}(T_i) = \begin{bmatrix} -\frac{\Delta N_1(T_i)}{Y_1(T_i)} & \frac{\Delta N_{12}(T_i)}{Y_1(T_i)} & \frac{\Delta N_{13}(T_i)}{Y_1(T_i)} & \cdots & \frac{\Delta N_{1p}(T_i)}{Y_1(T_i)} \\ \frac{\Delta N_{21}(T_i)}{Y_2(T_i)} & -\frac{\Delta N_2(T_i)}{Y_2(T_i)} & \frac{\Delta N_{23}(T_i)}{Y_2(T_i)} & \cdots & \frac{\Delta N_{2p}(T_i)}{Y_2(T_i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\Delta N_{p-1,1}(T_i)}{Y_{p-1}(T_i)} & \frac{\Delta N_{p-1,2}(T_i)}{Y_{p-1}(T_i)} & \cdots & -\frac{\Delta N_{p-1}(T_i)}{Y_{p-1}(T_i)} & \frac{\Delta N_{p-1,p}(T_i)}{Y_{p-1}(T_i)} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$\Delta N_{hj}(T_i)$ is number of transitions from state h to j in time T_i.
$\Delta N_k(T_i)$ is total number of transitions from state k.
$Y_k(T_i)$ is number of firms in state k right before date T_i.

Example

$$\hat{\mathbf{P}}(0,1) = \begin{bmatrix} 0.90909 & 0.08181 & 0.00909 \\ 0.09091 & 0.81818 & 0.09091 \\ 0 & 0 & 1 \end{bmatrix}, \ (non - homogenous \ \ time),$$

$$\hat{\boldsymbol{P}_{\boldsymbol{G}}}(t) = \begin{bmatrix} 0.90887 & 0.08618 & 0.00495 \\ 0.09323 & 0.80858 & 0.09819 \\ 0 & 0 & 1 \end{bmatrix}, \ (homogenous \ \ time).$$

Non-homogenous Mrakov Process with data

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.95912	0.03982	0.00096	0.00010	0.00000	0.00000	0.00000	0.00000
AA	0.01249	0.93689	0.04519	0.00524	0.00015	0.00004	0.00000	0.00000
A	0.00011	0.01666	0.93097	0.04906	0.00274	0.00042	0.00001	0.00003
BBB	0.00002	0.00253	0.03635	0.90603	0.03955	0.01398	0.00030	0.00125
BB	0.00000	0.00012	0.00318	0.07866	0.85980	0.05411	0.00317	0.00096
B	0.00000	0.00005	0.00495	0.00385	0.07029	0.87618	0.02941	0.01527
CCC	0.00000	0.00004	0.00091	0.02523	0.02890	0.11823	0.52289	0.30380
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 3: One-year transition probability matrix estimated for the year 1997 using the maximum-likelihood estimator based on continuous observations. (Lando, D., & Skødeberg, T. M. (2002))

Non-homogenous Mrakov Process with data

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.95866	0.03926	0.00184	0.00023	0.00001	0.00000	0.00000	0.00000
AA	0.01273	0.93714	0.04440	0.00544	0.00022	0.00007	0.00000	0.00000
A	0.00010	0.01682	0.93088	0.04880	0.00278	0.00061	0.00000	0.00001
BBB	0.00002	0.00252	0.03632	0.90736	0.03888	0.01353	0.00009	0.00128
BB	0.00000	0.00016	0.00347	0.07905	0.86016	0.05342	0.00294	0.00079
B	0.00000	0.00005	0.00507	0.00378	0.07066	0.87599	0.02797	0.01647
CCC	0.00000	0.00000	0.00045	0.02302	0.03104	0.12522	0.51784	0.30242
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 4: Aalen–Johansen estimator for the year 1997. A non-parametric estimator of the one-year transition probability matrix. (Lando, D., & Skødeberg, T. M. (2002))

Covariates

- Testing for non-Markov behavior
- Dependence on previous rating
- Waiting-time effects

$$\lambda_{hji}(t) = Y_{hi}(t)\alpha_{hji}(t, Z_i(t)),$$
$$\alpha_{hji}(t, Z_i(t)) = \alpha_{hj0} \exp\{\beta_{hj} Z_i(t)\}.$$

$$Y_{hi}(t) = \begin{cases} 1, & \text{if firm } i \text{ is in state } h \text{ at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Covariates

- $$Z_i(t) = \begin{cases} 1, & \text{i was upgraded to the present rating class,} \\ 0, & \text{otherwise.} \end{cases}$$

  The statistical test for the hypothesis of no rating drift
  $H_0 : \boldsymbol{\beta} = 0$.
- $\tilde{Z}_i(t) =$ "time since last entry into the present state".

Test results for rating drift

From	To	$\hat{\beta}$	std($\hat{\beta}$)	n1	n2	p
AA+	AA	0.897	0.281	149	65	<0.01
AA	AA-	0.936	0.211	314	100	<0.01
AA-	A+	0.871	0.172	490	162	<0.01
A+	A	0.582	0.147	663	198	<0.01
A	A-	0.868	0.160	842	193	<0.01
A-	BBB+	1.180	0.196	780	161	<0.01
BBB+	BBB	0.714	0.168	721	180	<0.01
BBB	BBB-	1.180	0.222	712	140	<0.01
BBB-	BB+	1.090	0.241	641	95	<0.01
BB+	BB	0.970	0.303	513	59	<0.01
BB	BB-	0.144	0.227	571	82	0.53
BB-	B+	0.858	0.253	522	74	<0.01
B+	B	1.010	0.282	575	87	<0.01
B	B-	0.541	0.457	437	43	<0.01
B-	CCC+	2.030	1.040	271	28	<0.01
CCC+	CCC	6.170	23.5	194	15	0.20
CCC	CCC-	-0.929	0.873	150	18	0.32

Table 5: Results for the test of an effect of a previous downgrade on the intensity of a downgrade to a neighboring state.
(Lando, D., & Skødeberg, T. M. (2002))

The results of rating drift

From	To	$\hat{\beta}$	std($\hat{\beta}$)	n1	n2	p
AA+	AAA	-0.106	0.525	149	15	0.84
AA	AA+	-0.011	0.545	314	14	0.98
AA-	AA	-0.132	0.268	490	56	0.62
A+	AA-	0.337	0.233	663	85	0.14
A	A+	0.449	0.190	842	116	0.02
A-	A	0.261	0.151	780	177	0.08
BBB+	A-	0.720	0.168	721	153	<0.01
BBB	BBB+	0.508	0.173	712	137	<0.01
BBB-	BBB	0.143	0.173	641	144	0.405
BB+	BBB-	0.535	0.174	513	152	<0.01
BB	BB+	-0.100	0.187	571	122	0.60
BB-	BB	0.1947	0.190	522	114	0.315
B+	BB-	0.667	0.214	575	90	<0.01
B	B+	0.560	0.277	437	63	0.05
B-	B	0.490	0.477	271	22	0.31
CCC+	B-	-6.150	25.7	194	17	0.24
CCC	CCC+	-7.280	45.1	150	6	0.25

Table 6: Results for the test of an effect of a previous upgrade on the intensity of an upgrade to a neighboring state.
(Lando, D., & Skødeberg, T. M. (2002))

The results for duration in rating class

From	To	$\hat{\beta}$	std($\hat{\beta}$)	n1	n2	p
AAA	AA+	-0.348	0.114	61	13	<0.01
AA+	AA	-0.405	0.067	149	65	<0.01
AA	AA-	-0.282	0.037	314	100	<0.01
AA-	A+	-0.380	0.041	490	162	<0.01
A+	A	-0.351	0.035	663	198	<0.01
A	A-	-0.547	0.046	842	193	<0.01
A-	BBB+	-0.628	0.064	780	161	<0.01
BBB+	BBB	-0.360	0.047	721	180	<0.01
BBB	BBB-	-0.555	0.056	712	140	<0.01
BBB-	BB+	-0.679	0.095	641	95	<0.01
BB+	BB	-0.708	0.134	513	59	<0.01
BB	BB-	-0.453	0.099	571	82	<0.01
BB-	B+	-0.621	0.110	522	74	<0.01
B+	B	-0.529	0.085	575	87	<0.01
B	B-	-0.683	0.155	437	43	<0.01
B-	CCC+	-0.902	0.216	271	28	<0.01
CCC+	CCC	-2.241	0.690	194	15	<0.01
CCC	CCC-	-0.704	0.259	150	18	<0.01

Table 7: Results for the test of an effect of the waiting time in the initial category listed under 'From' on the intensity of a downgrade to a neighboring state.

(Lando, D., & Skødeberg, T. M. (2002))

The results for a duration in rating class

From	To	$\hat{\beta}$	std($\hat{\beta}$)	n1	n2	p
AA+	AAA	-0.416	0.132	149	15	<0.01
AA	AA+	-0.226	0.096	314	14	<0.01
AA-	AA	-0.360	0.072	490	56	<0.01
A+	AA-	-0.331	0.057	663	85	<0.01
A	A+	-0.329	0.049	842	116	<0.01
A-	A	-0.376	0.045	780	177	<0.01
A-	BBB+	-0.449	0.057	721	153	<0.01
BBB+	BBB	-0.266	0.043	712	137	<0.01
BBB	BBB-	-0.346	0.051	641	144	<0.01
BBB-	BB+	-0.532	0.075	513	152	<0.01
BB+	BB	-0.540	0.085	571	122	<0.01
BB	BB-	-0.537	0.084	522	114	<0.01
BB-	B+	-0.383	0.071	575	90	<0.01
B+	B	-0.359	0.100	437	63	<0.01
B	B-	-0.430	0.189	271	22	0.012
B-	CCC+	-0.507	0.247	194	17	0.016
CCC+	CCC	-0.934	0.631	150	6	0.032

Table 8: Results for the test of an effect of the waiting time in the initial category listed under ‘From’ on the intensity of an upgrade to a neighboring state.

(Lando, D., & Skødeberg, T. M. (2002))

Conclusion

- Importance of estimating transition data based on the full story of rating transitions.
  - MLE in homogenous case
  - Aalen–Johansen estimator in the non-homogeneous case
- Non-Markov behavior in the rating process, especially for downgrades.

References

- Lando, D., & Skødeberg, T. M. (2002). Analyzing rating transitions and rating drift with continuous observations. *Journal of Banking & Finance, 26*(3), 423-444.