MATEMATICKO-FYZIKÁLNÍ FAKULTA
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Input-output modeling in the business analysis

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Main topics

- Production process
- Input-output modeling
- Discrete dynamic deterministic programming
- Linear programming
- Application in the alumunium industry

Input - output modeling

- Production process
  - Trivial
  - Simple
  - Complex
- Description of the process
  - Process has n production centers (sectors)
  - Each production center produces a single product
  - Each product can be used as intermediate output or can be sold on the market
- Structure of the process can be represented by the matrix P, with the elements p_{ij} $i, j = 1, \ldots, n$. If the i-th and j-th sector are connected, $p_{i,j} = 1$ and $p_{i,j} = 0$ otherwise

I-O model of W. Leontief (1936)

- I-O models have been widely used for macroeconomic analysis
- Enterprise I-O models
- Assumptions
  - Process has n production centers (sectors)
  - Each production center produces a single product
  - Each product can be used as an intermediate output or can be sold on the market
  - Each sector uses only one production process
  - Each product is produced in one sector only

Enterprise I-O models

- The basic I-O relations in the enterprise are expressed as follows

$$x = Ax + q,$$
$$x^m = A^m x.$$

equivalently can be expressed as

$$x = (I - A)^{-1} q,$$
$$q = x - Ax.$$

Gozinto model

- can be used, if the simple production process is analyzed, but can be also extended for analysis of the complex production process

$$g_{ij} = a_{i1}g_{1j} + a_{i2}g_{2j} + \cdots + a_{in}g_{nj} + \delta_{ij},$$

$$G = AG + I = (I - A)^{-1}.$$

Gozinto model is equivalent to the Leontief's model

$$x_i = g_{ij}q_j,$$

$$x = Gq,$$

$$x^m = G^m q.$$

Two-stage business optimization

- First stage
  - Discrete dynamic deterministic programming
  - Output: Optimal production quantities of the main product

- Second stage
  - Input-output analysis
  - Linear programming
  - Output: Optimal purchasing, production and selling quantities of other elements with maximal contribution to cover fixed costs

Optimizing in every time period $t = 1, \ldots, T$

Discrete dynamic deterministic programming

$$S_t(y_t) = \min_{x_t \in \mathbb{R}} \{s(y_{t-1}) + s(x_t) + S_{t-1}(y_{t-1})\},$$

subject to

$$\begin{aligned}
y_0 &= a, \\
y_t &= y_{t-1} + x_t - d_t,\ t = 1, \ldots, T, \\
y_T &= b, \\
y_t &\in E,\ t = 1, \ldots, T, \\
x_t &\in F,\ t = 1, \ldots, T, \\
S_0 &= 0.
\end{aligned}$$

$$\max_{q_t \in \mathbb{R}^n} \left\{ z_t^T q_t \right\},$$

subject to

$$\begin{aligned} x_t &= (I - A)^{-1} q_t \leq c_t, \\ x_t^m &= A^m (I - A)^{-1} q_t \leq b_t \\ p_m^T A^m (I - A)^{-1} q_t &\leq C_t, \\ m_{lt} \leq q_t &\leq m_{ut}. \end{aligned}$$

Linear programming and I-O approach

$$\max_{\hat{q}_t \in \mathbb{R}^{n-1}} \left\{ \hat{z}_t^T \hat{q}_t \right\},$$

subject to

$$\begin{aligned} \hat{x}_t &= (I - \hat{A})^{-1} \hat{q}_t \leq \hat{c}_t, \\ x_t^m &= \hat{A}^m (I - \hat{A})^{-1} \hat{q}_t \leq \hat{b}_t \\ p_m^T \hat{A}^m (I - \hat{A})^{-1} \hat{q}_t &\leq \hat{C}_t, \\ \hat{m}_{lt} \leq \hat{q}_t &\leq \hat{m}_{ut}. \end{aligned}$$

where

$$\begin{aligned} \hat{c}_t &= c_t - (I - A)^{-1} q_{ti}^{opt}, \\ \hat{b}_t &= b_t - A^m (I - A)^{-1} q_{ti}^{opt}, \\ \hat{C}_t &= C_t - p_m^T A^m (I - A)^{-1} q_{ti}^{opt}. \end{aligned}$$

Application in the aluminium industry

- Consider two main products of the process: aluminium, aluminia
- Alumuminium plant in Kidricevo (Slovenia) has the capacity
  - 40 000 tons of raw aluminium per year
  - 120 000 tons of aluminia per year
- Simplified process can be decomposed into production, purchasing and selling activities

Baoxite ore $\Rightarrow$ Alumina $\Rightarrow$ Aluminium

- Elements of business process: purchasing elements (= inputs), working hours, intermediate outputs, production capacities and outputs

Elements of the business process, purchasing and selling activities

Input elements		Unit	Purch. activity	Purch. price	Max. purch. quantity
E1	Prebaked anodes	Ton	Z1	240	
E2	Electric energy	Kwh	Z2	0.03	
E3	Labour	Hour	Z3	0.50	
E4	Bauxite ore	Ton	Z4	30	
E5	Metal suplements	Ton	Z5	3200	
E6	Dissolving device	Hours	Z6		252
E7	Electrolyc cells	Hours	Z7		252
E8	Cast-machine 1	Hours	Z8		252
E9	Cast-machine 2	Hours	Z9		252
E10	Rolling machine	Hours	Z10		252

Elements of the business process, purchasing and selling activities

Intermediate outputs		Unit	Selling activity	Selling price	Max. sell. quantity
D2	Molten Al	Ton			
D3	Remnant 1	Ton			
D4	Remnant 2	Ton			
Output elements					
P1	Alumina	Ton	Y1	300	4000
P2	Vanadium salt	Ton	Y2	1000	∞
P3	Alloyed ingots	Ton	Y3	2200	200
P4	Pure ingots	Ton	Y4	1900	800
P5	Al blocks	Ton	Y5	2000	0
P6	Al foil	Ton	Y6	2950	0

Production and set-up costs

Alumina	0	1	2	3	4	5	6	7
Production cost	70	360	710	1035	1340	1625	1932	2240
Aluminium	0	0.5	1	1.5	2	2.5		
Production cost	240	1100	1750	2475	2900	3125		

Tabulka: Production costs (USD) and possible produced quantities (1000 tons) of alumina and aluminium in every time period

Time period t	**1**	**2**	**3**	**4**
The demand for **alumina**	4	1	2	3
The demand for **aluminium**	1	2	3.5	3

Tabulka: The demand for alumina and aluminia (1000 tons)

Storage costs

Alumina	1	2	3	4	5
Storage cost	24	30	38	45	51
Aluminium	0	0.5	1	1.5	2
Storage cost	24	33	55	150	250

Tabulka: Storage costs (USD) for possible storage quantities (1000 tons) of alumina and aluminium in every time period

First stage of the recursion equations

$$S_t(y(1)_t, y(2)_t) = \min_{x(1)_t, x(2)_t} \{s(y(1)_{t-1}) + y(2)_{t-1}) + s(x(1)_t) + s(x(2)_t) + S_{t-1}(y(1)_{t-1}, y(2)_{t-1})\}$$

subject to

$$y(1)_0 = 1$$
$$y(2)_0 = 1$$
$$y(1)_4 = 0$$
$$y(2)_4 = 0.5$$
$$y(1)_t = y(1)_{t-1} + x(1)_t - d(1)_t - 2x(2)_t$$
$$y(2)_t = y(2)_{t-1} + x(2)_t - d(2)_t$$

$$y(1)_t \in \{1, 2, 3, 4, 5\}$$
$$x(1)_t \in \{0, 1, 2, 3, 4, 5, 6, 7\}$$
$$y(2)_t \in \{0, 0.5, 1, 1.5, 2\}$$
$$x(2)_t \in \{0, 0.5, 1, 1.5, 2, 2.5\}$$

where $t = 1,\ 2,\ 3,\ 4$ and $S_0(y(1)_0, y(2)_0) = 0$

Result after the first stage of the optimization

	y_{t-1}		x_t		d_t		y_t		$S_t(y_t)$
t	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1) + (2)
1	1	1	7	1.5	4	1	1	1.5	4794
2	1	1.5	6	2.5	1	2	1	2	10025
3	1	2	7	2.5	2	3.5	1	1	15664
4	1	1	7	2.5	3	3	0	0.5	21108

Optimal solution for time period t=1

Element	Opt. purchasing quantity	Element	Opt. producing quantity	Element	Opt. selling quantity
E1	750	D2	1500	P1	4000
E2	27 927 270	D3	0	P2	350
E3	2 485 455	D4	0	P3	200
E4	17 500	D1	9800	P4	800
E5	18.18	P1	7000	P5	0
E6	252	P2	350	P6	0
E7	165	P3	200		
E8	88	P4	800		
E9	52.73	P5	0		
E10	0	P6	0		

Conclusion

- What if demand changes?
- Optimization must be performed for each time period
- Time period was considered one month
- We obtained optimal producing, purchasing and selling quantities of elements in the process

References

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- Artenjak, J., Tominc, P. (1999). Two-stage optimisation of the multiphase production. Mathematical Communications, 4(1999), 43-51