# QUANTUM INFORMATION

MFF UK

(0) The Toffoli gate $T : (a, b, c) \mapsto (a, b, c \oplus ab)$ is said to be able to *copy states*, i.e. $T : (a, 1, 0) \mapsto (a, 1, a)$. However, the so-called *no-cloning theorem* states that no unitary operator $U$ acting on, say, $\mathbb{H}^n \otimes \mathbb{H}$, can implement the mapping $|\varphi\rangle \otimes |0\rangle \mapsto |\varphi\rangle \otimes |\varphi\rangle$. Resolve the apparent contradiction.

(1) The natural analogy of states $|+\rangle$ and $|-\rangle$ in a 2-qubit system are the so-called *Bell's states* defined as

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x |1\overline{y}\rangle}{\sqrt{2}},$$

where $x, y$ are $0, 1$ and $\overline{y}$ denotes $1 - y$.

Show that the vectors $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$ form an orthonormal basis of $\mathbb{H}^2$. Express the circuit realizing $|xy\rangle \mapsto |\beta_{xy}\rangle$ as a composition of the single-qubit Hadamard operator $H$ and the $\mathsf{CNOT}$ gate.

(2) (Quantum teleportation) Contrary to the no-cloning theorem, it is possible to *teleport* quantum states using only a classical communication channel. However, the teleported qubit is destroyed for its original owner, so the no-cloning theorem is not violated.

The process can be described as follows. Two parties (Alice and Bob) first create the entangled state $|\beta_{00}\rangle$ (prove that this state is, indeed, entangled) and share it with each other. This formally means that Alice is able to measure $|\beta_{00}\rangle$ (and any other state of the two-qubit system) *the first qubit*, i.e. her measurement has the form $m_1 P_1 + m_2 P_2$, where $P_1$ is a projection operator onto the subspace generated by $|00\rangle$ and $|01\rangle$, and $P_2$ is a projection onto the subspace generated by $|10\rangle$ and $|11\rangle$. Similarly, Bob is able to measure the second qubit.

Alice is then given some arbitrary single-qubit state $|\varphi\rangle$, which she wants to teleport to Bob. She performs a $\mathsf{CNOT}$ operation on her pair of qubits (formally, she implements the mapping $\mathsf{CNOT} \otimes id$ applied to $|\varphi\rangle \otimes |\beta_{00}\rangle$) followed by the Hadamard operator applied to the first qubit (again, we formally mean the operator $H \otimes id \otimes id$).

Finally, she measures both of her qubits separately and sends the results of her measurements via some classical communication channel to Bob. Depending upon the received data, Bob applies Pauli matrices $Z$ and/or $X$ to his single qubit. Recall that $X$ maps $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto |0\rangle$ and $Z$ maps $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto -|1\rangle$.

Finish the description of the protocol, draw its diagram, and verify that $|\varphi\rangle$ is indeed teleported to Bob and destroyed for Alice.

(3) Verify that applying

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

to the first qubit in a 2-qubit system is equivalent to the controlled application of

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

to the second qubit.

Verify the correctness of the circuit that realizes the controlled application of a single-qubit operator $U$ using the $ABC$ decomposition of $U$ discussed in the lecture.

(4) Compute square roots of Pauli matrices $X, Y, Z$ and the Hadamard matrix $H$.
(5) Express the operator acting on a 4-qubit system, which is described as the application of the Hadamard $H$ gate to the fourth qubit controlled by the parity of the first three qubits, as a combination of Toffoli gates and single-controlled single-qubit operators.
(6) Express the two-level operator $M$ defined as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

as a composition of controlled single-qubit operators.

Express an arbitrary two-level operator acting non-identically on $|0000\rangle$ and $|1111\rangle$ as a composition of controlled single-qubit operators.

(7) Decompose the matrix

$$\frac{\sqrt{3}}{3}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

into the composition of two-level operators.