Logic and Foundations of Mathematics

Set Theory and Type Theory

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Introduction

Background

- Historically, mathematics was considered as the science of magnitudes, i.e. science of measurements
- Arithmetic was the theory of discrete magnitudes, and geometry was the theory of continuous magnitudes.
- This view was popular at least since the 17th century, until the 19th century.

Background

In 1637, Descartes writes about geometry:

> *Geometry [is] the science which furnishes a general knowledge of the magnitude [measure] of all bodies... (Descartes 1637/1954, 316/43)*

Background

In 1796, Leonhard Euler refers to mathematics as the science of magnitudes.

> *First, everything will be said to be a magnitude, which is capable of increase or diminution, or to which something may be added or substracted ... mathematics is nothing more than the science of magnitudes [Wissenschaft der Grössen], which finds methods by which they can be measured. (Euler 1796, 9; quoted from Ferreiros 2008, 42)*

Background

A little later in 1803, Georg Simon Klügel repeats the same idea

> *Magnitude [Grösse] (Quantitas, Quantum) is that which is compound of homogeneous parts ...*[*everything, in reality or in imagination, that possesses the property of being such a compound is an object of mathematics.*] *Mathematics ... is therefore quite appropriately called the science of magnitudes. (Klügel 1803/08, vol. 2, 649, quoted from Ferreiros 2008, 42)*

Crisis in Mathematics

- In the 19th-century, there was an interest to rigorise fundamental notions (such as natural numbers, real numbers, continuous functions etc.) used in mathematics.
- Dedekind (1872) defined real numbers using the sets (classes) of rational numbers.
- Dedekind (1888) showed that there are infinite sets, such as the set (system) of natural numbers, using sets and functions.

Today, set theory is considered as a foundation of mathematics.
In what sense is set theory a foundation?
Is set theory *the* foundation of mathematics?

Goals

- Explain, in what sense, set theory is considered as a foundation historically and philosophically.
- Introduce type theory as an alternative foundation.
- Compare set theory and type theory as foundational theories of mathematics.

The Origin of Set Theory

Cantor's Naive Set Theory

- In 1874, Cantor showed how we can talk about transfinite numbers.
- He also showed that the set $\mathbb{R}$ of real numbers and the set $\mathbb{N}$ of natural numbers cannot have the same size.

Definition

Two sets, A and B, have the same size (or cardinality), if there is a bijection between them (i.e. there is a one-to-one correspondence). We write $|A| = |B|$.

Theorem (Cantor's theorem)

There is no bijection between $\mathbb{N}$ and $\mathbb{R}$.

Continuum Hypothesis

Definition (Continuum Hypothesis)

There is no set X whose size is strictly between the real numbers and the natural numbers. i.e. there is no X such that $|\mathbb{N}| < |X| < |\mathbb{R}|$.

- David Hilbert (1900) considered this to be one of 23 most important problems in mathematics (Hilbert's first problem)
- Since then, set theory stood as a branch of mathematics on its own.

Unrestricted Comprehension

Implicitly assumed, both by Dedekind and Cantor, was the Unrestricted Comprehension.

Definition (Unrestricted Comprehension)

For any formula $\varphi(x)$, there is a set S such that

$$S := \{x \mid \varphi(x)\}.$$

So for any property, expressed by the formula φ, there is a set that contains only the elements which satisfy φ.

Set Theory and Paradoxes

Unrestricted Comprehension gives rise to Russell's Paradox.

- Known to Cantor in 1890s (as he communicated this to Hilbert and Dedekind)
- Ernest Zermelo in 1899
- Bertrand Russell discovered it in 1901 (published in 1903).

Definition (Russell set)

$R := \{x | x \notin x\}$.

Question: Is R an element of R?

Proof.

If $R \in R$, then by the definition of R, $R \notin R$. CONTRADICTION.
If $R \notin R$, then by the definition of R, $R \in R$. CONTRADICTION.
Thus naive set theory is not consistent. i.e. there is a proposition that is both true and false in naive set theory. □

Solution to Paradox

Zermelo (1908) offers to resolve the paradox by offering axioms of set theory (and also aims to show that CH can be proven from such axioms).

Definition (Restricted Comprehension Axiom)

For any set A and a formula φ, there is a set S such that

$$S := \{x \in A \mid \varphi(x)\}.$$

Instead of an arbitrary collection of elements satisfying φ forming a set, only a subset of an existing set A, could be collected as a set.
By offering different axioms for set theory, paradoxes could be avoided.

Set Theory and Axiomatisation

- Starting with Zermelo (1908), set theory became axiomatised (following the Hilbert School approach).
- Zermelo's set theory consists of 7 axioms: extensionality, 'empty set', separation, power set, union, infinity, and choice.
- Skolem and Fraenkel (1922) introduced the axiom of replacement to strengthen the theory.
- Today we call these axioms ZFC (Zermelo-Fraenkel set theory with Choice).
- Kurt Gödel (1940) showed that $\neg$CH cannot be proven in ZFC.
- Paul Cohen (1969) showed that CH cannot be proven in ZFC.
- Today, 'axiomatic set theory' refers to the ZFC axioms of set theory (and their extensions), which is a first-order theory with the binary relation $\in$ for the membership relation.

Empty set and Extensionality Axioms

ZFC axioms of set theory characterise what kinds of sets exist. These axioms (and axiom schemes) are as follows:

Definition (Empty set)

There is an empty set $\emptyset$ which has no elements.

Definition (Extensionality)

If any two sets A and B have the same elements, they are equal.

Example

By using Extensionality, we can show that the empty set is unique. Let A and B both be empty sets. Then since A is empty, all elements of A (i.e., none) are elements of B, which is also empty. Thus, A and B have the same elements. By the axiom of Extensionality, A and B are equal. Hence, the empty set is unique.

Pairing Axiom

Definition (Pairing)

Given any sets A and B, there exists a pair set $\{A, B\}$ which contains A and B as its only elements.

Example (Singleton set)

Let $\emptyset$ be the empty set. Then there is a pair set $\{\emptyset, \emptyset\}$ which is the singleton set $\{\emptyset\}$.

Example (Ordered Pair)

Let A and B be sets. By the pairng axiom, we have $\{A\}$ and $\{A, B\}$. Then, we can pair $\{A\}$ and $\{A, B\}$ to have the set $\{\{A\}, \{A, B\}\}$. So we define an ordered pair (A, B) of sets A and B, as the set $\{\{A\}, \{A, B\}\}$.

Power set Axiom

Definition (Power set)

For every set A, there exists the power set of A, denotes $\mathcal{P}(A)$, whose elements are all the subsets of A. i.e., $\mathcal{P}(A)$ is the set of all subsets of A. B is a subset of A ($B \subseteq A$) if all elements of B are elements of A.

Example

Consider the set $A := \{\emptyset\}$. Then the powerset $\mathcal{P}(A)$ is $\{\emptyset, \{\emptyset\}\}$.

In set theory, we denote the natural numbers $0 := \emptyset$, $1 := \{\emptyset\}$, i.e. $\{0\}$, $2 := \{\emptyset, \{\emptyset\}\}$, i.e. $\{0, 1\}$, and $3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, i.e. $\{0, 1, 2\}$, ...

Example

The power set of 3 is

$$\{0, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

Union Axiom

Definition (Union)

For every set A, there is a set $\bigcup A$, called the *union of A*, whose elements are all the elements of the elements of A.

Example

Recall that $3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. The union of 3 is

$$\bigcup 3 := \{\emptyset, \{\emptyset\}\}.$$

And by the axiom of extensionality $\bigcup 3 = 2$.

Example

Recall that $\mathcal{P}(3) = \{0, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.
Then

$$\bigcup(\mathcal{P}(3)) = \{0, 1, 2\} = 3$$

Infinity Axiom

Definition (Infinity)

There is an infinite set.

Example

Let ω denote the set of natural numbers:

$$\{0, 1, 2, 3, \ldots\}.$$

It can also be characterised as follows:

- $\emptyset \in \omega$
- If $x \in \omega$, then $\bigcup\{x, \{x\}\} \in \omega$.

Comprehension Axiom Scheme

Definition (Unrestricted Comprehension)

Given a set A and a set-theoretic formula $\varphi(x)$, the following set exists

$$\{x \in A \mid \varphi(x)\}.$$

Example

Let A be a set, and consider $R := \{x \in A \mid x \notin x\}$. By the above axiom, such set R exists, and it is empty.

Replacement Axiom Scheme

Definition (Replacement)

Given a function f whose domain is the set A, there is a set whose elements are the values of the function. i.e. there is a set

$$f[A] := \{x \mid x = f(a), \text{for some } a \in A.\}$$

Example

Let s be the successor function on natural numbers, so $s(n) = n + 1$. Then by Replacement, there is a set $s[\omega] := \{1, 2, 3, \ldots\}$.

Foundation Axiom

Definition

For any non-empty set A, there is an $\in$-minimal element. i.e. there is an element such that no elements of A belongs to it.

Example

Consider the set ω. The $\in$-minimal element is $\emptyset$. $\emptyset \in \omega$, and there is no element of A belonging to $\emptyset$.

Axiom of Choice

Definition (Axiom of Choice)

For every set A which doesn't contain the empty set, there is a function f which contains exactly one element from each set $B \in A$.

Theorem (Well-Ordering Theorem)

Every set A can be well-ordered. i.e. there is a strict-order $<$ on X such that every non-empty subset of X has a $<$-least element.

Set Theory and Foundation of Mathematics

Question: in what sense is set theory a foundation of mathematics?

Answer 1 Sets (as manifolds) are the basic objects for mathematics

Answer 2 All mathematical concepts can be defined as sets.

Answer 3 Foundational goals/aims of set theory.

Sets are the basic objects for mathematics (historically)

- It goes back to Riemann (1854), where he considered 'manifolds' as the fundamental objects of mathematics.
- Manifolds were generalisations of magnitudes, so studying manifolds meant studying arbitrary magnitudes.
- Cantor's early work on set theory (1878 - 1890) refers to sets as 'manifolds' [*Mannigfaltigkeiten*], and Cantor (1878) makes a direct connection to Riemann.

Sets as Manifolds

Riemann's 'manifold' is a generalisation of the notion of 'magnitude'.

Example

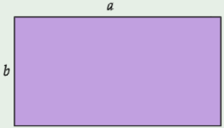
Consider a line denoting the magnitude (i.e. measurement) a, and a line denoting the magnitude b.

Instead of a fixed magnitude a, we can think of the line to be representing a *collection* of magnitudes between 0 and a (and also a *collection* of magnitudes between 0 and b).

Sets as Manifolds

Example

A *manifold* refers to the collection of magnitudes. The following depicts a manifold of 2 dimensions, which is a product of two manifolds of 1 dimension.



It can be further generalised to an n-dimensional manifold with n magnitudes.

Sets as Manifolds

- Manifold, in Riemannian sense, is a collection of arbitrary magnitudes, i.e. domain of magnitudes.
- Historically speaking, set theory was a foundation of mathematics, as it studied the general notion of manifold
- Riemann (1854) considered his work to be foundational, as he considered 'manifolds' to be the generalisation of magnitudes
- We will see later that this idea comes back in type theory!

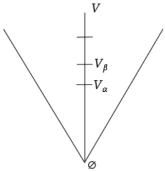
But what do we think sets are in axiomatic set theory today?

Sets Today

Today, sets are not considered as domains, but objects which appear in the universe of sets V. In particular, every set appears at some level V_α in the universe.

Definition (von Neumann Universe)

- $V_0 := \emptyset$
- $V_{\alpha+1} := \mathcal{P}(V_\alpha)$
- $V_\beta := \bigcup_{\gamma<\beta} V_\gamma$



Hence the Riemannian/Cantorian answer does not apply to today's set theory.

Mathematical Concepts and Sets

Answer 2 Every mathematical concept can be defined as a set.

This idea goes back to Zermelo (1908), and continues on in contemporary set theory.

Mathematical Concepts and Sets

For Zermelo (1908), all mathematical concepts in arithmetic and analysis could be defined in terms of sets.

> *Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions 'number', 'order', and 'function', taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis. (Zermelo 1908, 200)*

Mathematical Concepts and Sets

Ken Kunen (1980) makes a similar remark about set theory in his textbook on forcing, for **all mathematics**

> *Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate [...] axioms about these primitive notions [...]. From such axioms, all known mathematics may be derived. (Kunen 1980, xi)*

Mathematical Concepts and Sets

In another textbook, Enderton (1977) shares Kunen's view and adds further comments on roles of axioms

> *It is sometimes said that 'mathematics can be embedded in set theory'. This means that mathematical objects (such as numbers and differentiable functions) can be defined to be certain sets. And the theorems of mathematics (such as the fundamental theorem of calculus) then can be viewed as statements about sets. Furthermore, these theorems will be provable from our axioms. Hence our axioms provide a sufficient collection of assumptions for the development of the whole of mathematics – a remarkable fact. (Enderton 1977, pp. 10–11)*

Mathematical Concepts and Sets

Example

Natural numbers are expressed as sets in the following way:

- $0 := \emptyset$
- For any natural number n, $n + 1 := \bigcup\{n, \{n\}\}$.

Example

A function $f : X \to Y$ is a set in the following sense: Define $X \times Y$ as the Cartesian product of X and Y. This means that any element $z \in X \times Y$ is of the form (x, y) for $x \in X$ and $y \in Y$. $f \subseteq X \times Y$ such that $f(x) = y$ iff $(x, y) \in f$.

Foundational Goals of Set Theory

Answer 3 Foundational goals/aims of set theory.

Foundational Goals of Set Theory

Set theory offers and achieves certain 'foundational goals' for mathematics.

- **Generous Arena**: 'where all of modern mathematics takes place side-by-side' (Maddy 2019, 298)
- '**Shared Standard** of what counts as a legitimate construction or proof' (Maddy 2019, 298; verbatim Maddy 2017, 296)
- **Meta-mathematical Corral**: 'so that formal techniques can be applied to all of mathematics at once.' (Maddy 2019, 301)

Foundational Goals of Set Theory

One more foundational role of set theory is...

- **Risk Assessment** (Maddy 2017, 2019): set theory offers an order of consistency strengths – allowing us to rank different mathematical theories by their consistency strengths.

  I have not yet [...] been able to prove rigorously that my axioms are consistent, though this is certainly very essential (Zermelo 1908, pp. 200–201)

Risk Assessment

In today's set theory, different **large cardinal axioms** assert that certain cardinals exists (which are larger than what can be proven to exist in ZFC).

Theorem

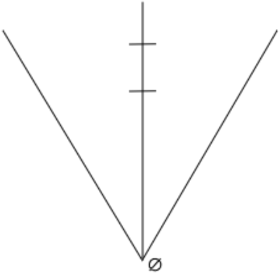
Let P_1 and P_2 be large cardinals axioms. Then one of the following holds:

- $\mathrm{ZFC} + P_1$ *is consistent iff* $\mathrm{ZFC} + P_2$ *is consistent*
- $\mathrm{ZFC} + P_1$ *proves that* $\mathrm{ZFC} + P_2$ *is consistent*
- $\mathrm{ZFC} + P_2$ *proves that* $\mathrm{ZFC} + P_1$ *is consistent*

Risk Assessment

Let V be the universe of set theory. Let α_1 and α_2 be large cardinals such that

- $\mathrm{ZFC} + \alpha_1$ exists proves that $\mathrm{ZFC} + \alpha_2$ is consistent.



An Origin of Type Theory

Type Theory and Paradoxes

- Russell's paradox strikes again!

Another solution to Russell's paradox is to distinguish different kinds of logical/mathematical objects, such as individuals/objects, propositions, relations etc. (Russell 1903)
Given these distinctions, **the Russell set is not even definable!**

Russell's Type Theory and Russell Set (1)

Definition (Russell's Type Hierarchy)

- i is a *type* of **individuals**;
- ( ) is a *type* of **propositions**;
- If $A_1, \ldots, A_n$ are *types*, then $(A_1, \ldots, A_n)$ is the *type* of n-**ary relations** of objects of types $A_1, \ldots, A_n$ respectively.

We are now talking about what *type* a given object has. Any mathematical/logical object or expression is then one (and only one) of *types* of individuals, propositions, or relation.

Type Theory and Russell Set (2)

Example

What *type* does a binary relation $R(x, y)$ have?
(i, i) is the *type* of binary relation of individuals.

Example

What *type* does a binary connective, e.g. $\wedge$, $\vee$, or $\rightarrow$, have?
$((\ ), (\ ))$ is the *type* of binary connectives.

Type Theory and Russell Set (3)

Note that the set $R := \{x \mid x \notin x\}$ can be expressed as a unary predicate R. So the expression $R \in R$ is re-expressed as $R(R)$.
We want to show that $R(R)$ is not expressible in type theory.

Proof.

Note that if $P(a)$ is a proposition, then *its type* must be of the following form: for some *type* A,

- P is of *type* (A), and
- a is of *type* A.

We express the type using superscripts: $P^{(A)}(a)^A$. So the *type* of the expression $R(R)$ must satisfy that R is of the *type* (A) and also of *type* A, for some *type* A. But this is not possible, since no term can be of multiple types. □

Resolving Paradoxes

How are set theory and type theory offering different approaches to solving the paradox?

- In set theory, new axioms were offered: Unrestricted Comprehension was changed to Restricted Comprehension.
- In type theory, the language was modified to ensure that we cannot grammatically express certain paradoxical expressions.

From Russell to Church

- Russell introduced his doctrine of types in *Principle of Mathematics* (1903)
- He (1908-) and Whitehead (1910-) went on to develop ramified type theory – which ranked propositions into levels
- The basic idea is to distinguish propositions: those which involve quantifying over individuals, and those which involve quantifying over functions.
- Chwistek and Ramsey (1920s) independently discovered that these levels could be collapsed
- So in the 1930s, Simple Type Theory was developed, and in 1940, Church publishes on his 'simply-typed λ-calculus'.

Church's Type Theory (1940-)

Definition (Church's Type Hierarchy)

- i is a type of **individuals**;
- o is a type of **propositions**;
- If α and β are types, then $\alpha \to \beta$ is a type of **functions** from α to β.

Unlike Russell's type theory, we have the **function type** as a primitive notion. A **function** models a computer program which on an input x computes an output t. Church was interested in mathematical models of computation.

$$\frac{x^\alpha \vdash t^\beta}{(\lambda x.t)^{\alpha\to\beta}} \text{ function abstraction}$$

$$\frac{f^{\alpha\to\beta} \qquad a^\alpha}{fa^\beta} \text{ function application}$$

Logical Connectives in Church's type theory

Church's initial account of logic in his type theory (1940) was concerned about what types of the logical symbols were.

Logical Symbols	Type
$\bot, \top$	o
$\neg$	$o \to o$
$\wedge, \vee, \Rightarrow$	$o \to (o \to o)$
$\forall, \exists$	$(i \to o) \to o$
f	$i \to i$
R	$i \to o$

Curry-Howard Correspondence

Howard (1969) showed that the **types** of Church's type theory corresponded to propositions, and the terms/expressions of those types corresponded to the proofs of the propositions.

$$\frac{x^{\alpha} \vdash t^{\beta}}{(\lambda x.t)^{\alpha \to \beta}} \text{ function abstraction} \qquad \frac{\alpha \vdash \beta}{\alpha \Rightarrow \beta} \Rightarrow I$$

$$\frac{t^{\alpha \to \beta} \qquad u^{\alpha}}{tu^{\beta}} \text{ function application} \qquad \frac{\alpha \Rightarrow \beta \qquad \alpha}{\beta} \Rightarrow E$$

Theorem (Curry-Howard Correspondence (Howard 1969))

Given a derivation of φ in intuitionistic propositional logic, there is a construction of type φ (and the converse).

Curry-Howard Correspondence

LOGIC	Church's Type Theory
proposition P	type P
proof	term t
$A \Rightarrow B$	$A \rightarrow B$
$\perp$	f
$\neg A$	$A \rightarrow f$

Dependent Type Theory

Although Church's λ-calculus was very good as a model of computation, Martin-Löf was not satisfied with its characterisation of the logical quantifiers.

> *[...W]hat cannot be typed within [Church's Type Theory are] the quantifiers, provided you vary the domain of the quantification. (Martin-Löf 1993, 11)*

Dependent Type Theory

Consider the universal quantifier $\forall$ and the expression $\forall x, P(x)$, where P is a unary predicate. When we interpret this set-theoretically/model-theoretically, the universal quantifier applies to the domain D of the model. So P should have the type $D \to o$, since P is a predicate that takes an object from the domain D, and for an object $a \in D$, $P(a)$ is a proposition (i.e. of type o). So P is **dependent** on the domain D. But in Church's type theory, $\forall$ is of type $(i \to o) \to o$, without specifying a domain. Hence, we need dependent types!

Martin-Löf Type Theory

- Martin-Löf first started developing his type theory in 1970s. There are many variations of it, but we focus on his 'intensional type theory'.
- In Martin-Löf type theory, we have **types** and **elements**/**terms**.
- An element is always *of a type*.
  - We cannot talk about 2 by itself. In type theory, 2 is always of type (e.g.) $\mathbb{N}$ (or $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots$)
- We write $a : A$ to express 'a is of type A' or 'a is an element of A'. We call '$a : A$' a **judgement**

Martin-Löf Type Theory

The 'atoms' of Martin-Löf Type Theory are judgements. There are four kinds of judgements:

Definition (Judgements)

Judgements are of the forms:

A Type	$A = B$ Type
$a : A$	$a = b : A$

'A Type' reads 'A is a type'. '$A = B$ type' reads 'A and B are equal types' or 'A and B refer the same type'. (**judgemental equality**)
'$a : A$' reads 'a is of type A' or 'a is A'. And '$a = b : A$' reads 'a and b are equal terms of type A' or 'a and b refer to the same term of type A'.

Instead of Church's account of typing logical symbols, Martin-Löf type theory extends Church's type theory via Curry-Howard Correspondence.

- $a : A$ means a is a proof of the proposition A
- $f : A \rightarrow B$ means f is a proof of the conditional $A \Rightarrow B$.

So if we want to say '$A \Rightarrow B$ is true', we express it as 'there is a term of type $A \rightarrow B$.

Dependent Types

Definition (Type Family)

Let A be a type. We say that the type $P(x)$ is a type family over A, where $x : A$.

With the notion of type family, we can interpret 'predicates' in logic as 'type families' in type theory.

Martin-Löf Type Theory

Logic	Type Theory
proposition P	type P
proof p of proposition P	term p of type P
	$p : P$
$\top, \bot$	$\mathbf{1}, \mathbf{0}$
$A \Rightarrow B$	$A \rightarrow B$
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$
$\neg A$	$A \rightarrow \mathbf{0}$
predicate $P(x)$	type family $P(x)$
$\forall x \; P(x)$	$\Pi_{x:A} P(x)$
$\exists x \; P(x)$	$\Sigma_{x:A} P(x)$

In what sense is Martin-Löf type theory a foundation?

Answer 1a Martin-Löf's Motivation: a foundation for constructive mathematics

Answer 1b Types as Cantor's Sets (manifolds)

Answer 2 Definition of mathematical concepts in type theory

Answer 3 Foundational goals/aims of type theory

A Foundation for Constructive Foundation

Answer 1a Martin-Löf's Motivation: a foundation for constructive mathematics

The theory of types with which we shall be concerned is intended to be a full scale system for formalizing [constructive] mathematics as developed, for example, in the book by Bishop. (Martin-Löf 1975)

Constructive Mathematics

Very roughly...

- Mathematical objects are shown to exist when they are constructed: i.e. showing that there is a contradiction if such object doesn't exist is not allowed
- Only constructive proofs are allowed: e.g. we cannot assume the law of excluded middle $P \vee \neg P$. So showing that $\neg P$ infers a contradiction is not enough to show that P is true.

Martin-Löf Type Theory is a logical system that tracks (constructive) *proofs*! The conjunction $A \wedge B$ is true iff there is a term (i.e. a proof) (a, b) for the type $A \times B$, where $a : A$ and $b : B$.

Types as Domains

Answer 1b Types as Domains

In his lectures, *Philosophical Aspects of Intuitionistic Type Theory* (1993), Martin-Löf suggests that his notion of type is the same as Cantor's original notion of set:

> *What is a type? The simplest answer seems to me to be that a type is defined by what it means to be an object of the type as well as what it means for two objects of the type to be the same. (Martin-Löf 1993, Lecture 3, p.18)*

Definition (Judgements)

Judgements are of the forms:

A Type	$A = B$ Type
$a : A$	$a = b : A$

Types as Domains

> *[In Cantor's lecture in 1880,] he says that a set has to be defined, first, by explaining what an element of a set is [...], and, secondly, we have to explain what it means for two elements of the set to be equal. (Martin-Löf 1993, Lecture 9, p.126)*

Indeed, we can find Cantor's own description of a 'well-defined set'...

Types as Domains

> *A manifold (a sum, a set) of elements [...] is called well-defined if [the manifold is] regarded as internally determined, [by whether or not something is an element], and also whether two objects belonging to the set [...] are equal to each other or not (Cantor 1882, pp.114-115; quoted from Klev 2019, 273)*

In the same lecture, Martin-Löf claims that types are then the '**domains** of individuals', just as sets were for Cantor.

Types as Domains

- Recall that Cantor's notion of set was inspired by Riemann's notion of manifold, as a **domain**
- Martin-Löf's notion of type conceptually is the notion of domain
- Martin-Löf continues on Riemann's foundational project by looking at domains (i.e. types) as fundamental to mathematics

Mathematical Concepts and Type Theory

Answer 2 Definition of mathematical concepts in type theory

In set theory, mathematical concepts were defined as sets and proven to exist according to the set theoretic axiom.
In type theory, each concept is defined by its introduction rule, and its elimination rule.

Mathematical Concepts and Type Theory

Definition (Natural Number Type)

The natural number type is defined by the introduction rules and the elimination rule.

Introduction Rules

$$\frac{}{0 : \mathbb{N}} \qquad\qquad \frac{n : \mathbb{N}}{\mathrm{succ}(n) : \mathbb{N}}$$

Elimination Rule. For any type family $P(x)$ that depends on $x : \mathbb{N}$

$$\frac{p_0 : P(0) \qquad p_s : \Pi_{n:\mathbb{N}} P(n) \to P(\mathrm{succ}(n)) \qquad n : \mathbb{N}}{\mathrm{Elim}_{\mathbb{N}}(p_0, p_s, n) : P(n)}$$

Mathematical Concepts and Type Theory

- So there isn't a strict difference between 'logic' and 'mathematics'.
- (e.g.) The type of natural numbers is defined by introduction and elimination rules, just as the logical connectives are.

Foundational Goals of Type Theory

Answer 3 What are the foundational goals/aims of type theory?

- Computational guidance for mathematical proofs

  The formal system of type theory is suited to computer systems and has been implemented in existing proof assistants. A proof assistant is a computer program which guides the user in construction of a fully formal proof, only allowing valid steps of reasoning. It also provides some degree of automation, can search libraries for existing theorems, and can even extract numerical algorithms for the resulting (constructive) proofs. (Univalent Foundations Program, 2013, 10)

Computation Guidance

- A proof assistant checks whether a judgement (e.g. $(a, b) : A \times B$) is expressed properly by checking that the term (a, b) is appropriately typed as $A \times B$ following the introduction and elimination rules.
- Logical reasoning is interpreted in type theory following Curry-Howard Correspondence.
- So a computer assistant can check whether the proof (a, b) is indeed a proof of the conjunction $A \times B$, and guide the user through any errors.

Final Remarks

Summary

In what sense is set theory a foundation?
Is set theory *the* foundation of mathematics?

- Set theory is an *axiomatic* foundation which defines all mathematical concepts as sets according to the axioms (Generous Arena, Shared Standard)
- Set theory offers classical first-order logic with the axioms as the guidance for what kinds of proofs are allowed in mathematics (Meta-mathematical Corral)
- Set theory offers a way to rank different axioms (extending ZFC) to compare the consistency strengths of different mathematical results. (Risk Assessment)

Summary

- Type theory is a *inferential* foundation which defines all mathematical concepts by introduction and elimination rules.
- Type theory offers constructive (intuitionistic) logic and their extensions, including classical logic, as the guidance for proofs
- Type theory offers a practical utility for mathematicians to verify their proof steps by running them on computer programs.

Conclusion

- Set theory is not *the* foundation of mathematics, but rather *a* foundational theory that has its strengths
- Type theory is another foundational theory with different strengths.
- Set theory and type theory offer conceptually and practically different foundational theories for mathematics.
- Mathematics can have a variety of foundations, each of which can have different foundational roles.

Suggested Reading

History of Set Theory

- **Hallett, Michael. 2024.** "Zermelo's Axiomatization of Set Theory", The Stanford Encyclopedia of Philosophy (Spring 2024 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL =https://plato.stanford.edu/archives/spr2024/entries/zermelo-set-theory/.
- **Ferreirós, José. 2023.** "The Early Development of Set Theory", The Stanford Encyclopedia of Philosophy (Summer 2023 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL = https://plato.stanford.edu/archives/sum2023/entries/settheory-early/.

Suggested Reading

History of Type Theory

- **Coquand, Thierry. 2022** "Type Theory", The Stanford Encyclopedia of Philosophy (Fall 2022 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL = https://plato.stanford.edu/archives/fall2022/entries/type-theory/.

Suggested Reading

Comparing set theory and type theory

- **Klev, Ansten. 2019.** "A Comparison of Type Theory with Set Theory." In Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts, edited by Stefania Centrone, Deborah Kant, and Deniz Sarikaya, 271–92.
- **Maddy, Penelope. 2019.** "What Do We Want a Foundation to Do?" In Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts, edited by Stefania Centrone, Deborah Kant, and Deniz Sarikaya, 293–311.