Logic of Questions II

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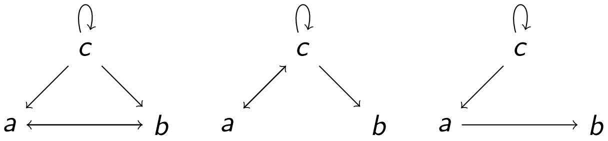
Three aspects of inquisitive logic

1. Questions are types of types (information types)
2. One can define a consequence relation among information types
3. Information types can be combined by logical connectives

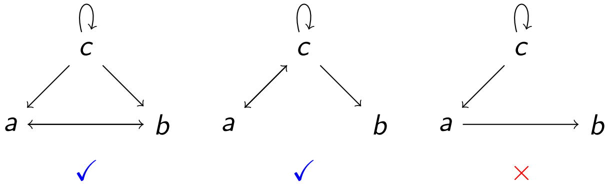
Questions are types of types

- Statements classify structures.
- Questions classify statements.

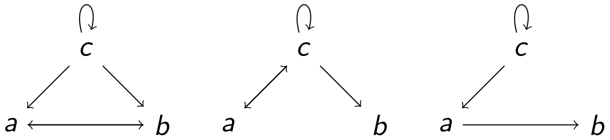
$\forall x Rcx$



$\forall x Rcx$

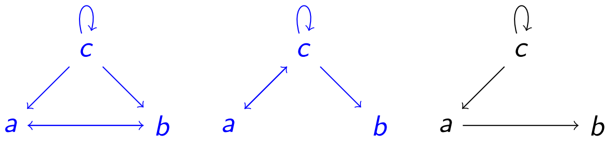


$?\forall x Rcx$



$?\forall x Rcx$

✓ (this state provides the answer YES)



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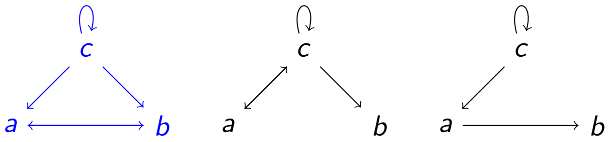
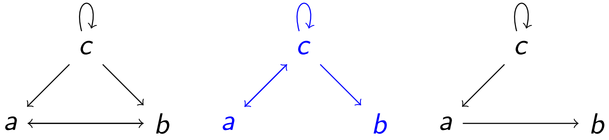
✓ (this state provides the answer YES)



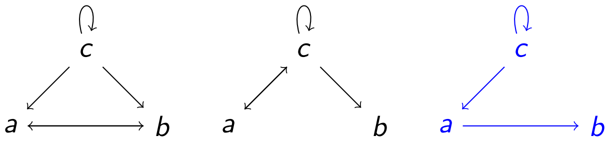
$?\forall x Rcx$

✓ (this state provides the answer YES)



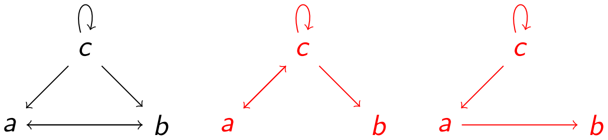
$?\forall x Rcx$

✓ (this state provides the answer NO)



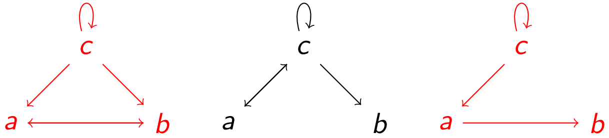
$?\forall x Rcx$

× (this state provides no answer)



$?\forall x Rcx$

$\times$ (this state provides no answer)



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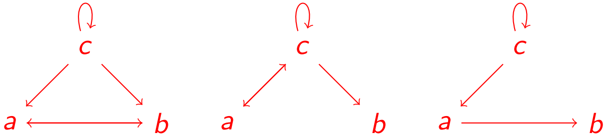
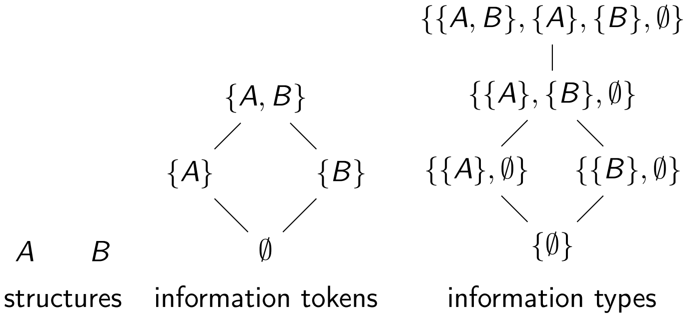


Algebras of information tokens and of their types



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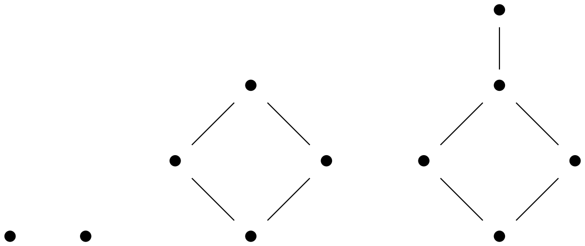
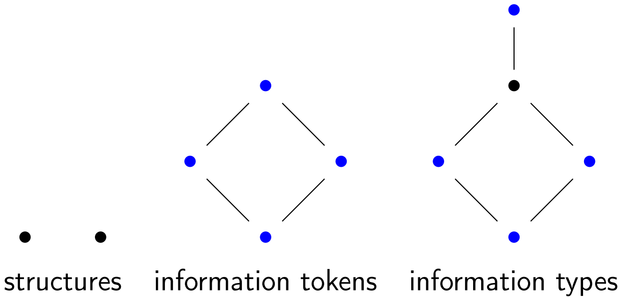
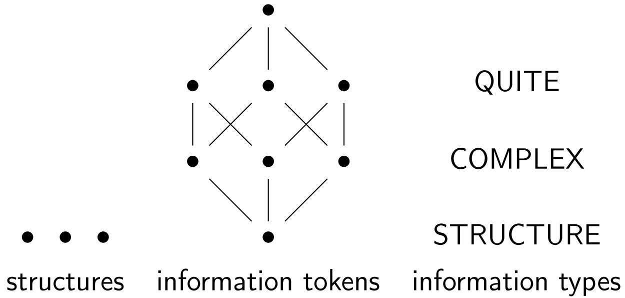


structures information tokens information types

Algebras of information tokens and of their types

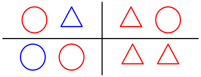


Algebras of information states and their types



Entailment among types of information

The space of possibilities S:



Information tokens:

a is a circle, b is a triangle, a is red, ...

Information types:

shape of a, shape of b, colour of a, colour of b

- a *is a triangle* $\models_S$ *b is red*
- a *is a circle* $\not\models_S$ *b is red*
- *colour of b, shape of a* $\models_S$ *colour of a*
- *colour of b, shape of a* $\not\models_S$ *shape of b*

Entailment among types of information

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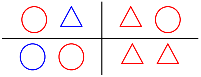

Information tokens:

a is a circle, b is a triangle, a is red, . . .

Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\vDash_S$ *b is red*
- *a is a circle* $\nvDash_S$ *b is red*
- *colour of b, shape of a* $\vDash_S$ *colour of a*
- *colour of b, shape of a* $\nvDash_S$ *shape of b*

Entailment among types of information

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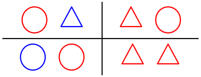

Information tokens:

a is a circle, b is a triangle, a is red, . . .

Information types:

shape of a, shape of b, colour of a, colour of b

- *a is a triangle* $\models_S$ *b is red*
- *a is a circle* $\not\models_S$ *b is red*
- *colour of b, shape of a* $\models_S$ *colour of a*
- *colour of b, shape of a* $\not\models_S$ *shape of b*

Combining information types

- the shape of a and the colour of b (an instance: a is a circle and b is blue)
- the colour of all objects (an instance: a is red and b is blue)
- dependence of the shape of b on the colour of a (an instance: if a is red then b a triangle and if a is blue then b is a circle)

First-order language

Terms are defined in the usual way. Complex formulas are defined as follows:

$$\varphi ::= \bot \mid t_1 = t_2 \mid Pt_1 \ldots t_n \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \exists\!\!\!\exists x \varphi$$

- $\neg\varphi =_{def} \varphi \to \bot$
- $\varphi \vee \psi =_{def} \neg(\neg\varphi \wedge \neg\psi)$
- $\exists x \varphi =_{def} \neg\forall x \neg\varphi$
- $?\varphi =_{def} \varphi \mathbin{\mathbb{V}} \neg\varphi$

- $Pa \mathbin{\mathbb{V}} Qa$ represents the question *whether a has the property P or the property Q*
- $\exists\!\!\!\exists x Px$ represents the question that asks *what is an object that has the property P*

Some examples

- Is Alice married to Bob? $?Mab$
- Is Alice married to Bob$^{\uparrow}$ or to Charlie$^{\downarrow}$? $Mab \veebar Mac$
- Is Allice married to Bob or to Charlie$^{\uparrow}$ $?(Mab \vee Mac)$
- Who did Alice invite to her wedding? $\forall x?Iax$
- What is Bob's favorite dish? $\exists xFbx$

Some examples

$$\exists! x \varphi(x) =_{def} \exists x(\varphi(x) \wedge \forall y(\varphi(y) \rightarrow y = x))$$

- What is the largest city in the world?
- Who is the current president of France?
- Who was the best man at your wedding?

Inquisitive model

An inquisitive model (for a given signature) is a pair $\mathcal{M} = \langle D, W \rangle$, where

- D is a nonempty set,
- W is a set of first-order structures on the domain D.

We can assume that the interpretations of names and function symbols are rigid. Given an evaluation of variables e every term t has a fixed value $t^{\mathcal{M},e}$.

An information state in $\mathcal{M}$ is a subset of W.

Inquisitive semantics

Given an inquisitive model $\mathcal{M} = \langle D, W \rangle$, and an evaluation of variables e in $\mathcal{M}$, we define a support relation between information states in $\mathcal{M}$ and formulas.

- $s \Vdash_e \bot$ iff $s = \emptyset$,
- $s \Vdash_e t_1 = t_2$ iff $t_1^{\mathcal{M},e}$ is identical with $t_2^{\mathcal{M},e}$,
- $s \Vdash_e Pt_1 \dots t_n$ iff $M \vDash_e Pt_1 \dots t_n$, for every $M \in W$,
- $s \Vdash_e \varphi \wedge \psi$ iff $s \Vdash_e \varphi$ and $s \Vdash_e \psi$,
- $s \Vdash_e \varphi \to \psi$ iff for every $t \subseteq s$, if $t \Vdash_e \varphi$, then $t \Vdash_e \psi$,
- $s \Vdash_e \forall x \varphi$ iff for every $o \in D$, $s \Vdash_{e(o/x)} \varphi$,
- $s \Vdash_e \varphi \vee\!\!\!\vee \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

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- $s \Vdash_e \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x \varphi$ iff for some $o \in D$, $s \Vdash_{e(o/x)} \varphi$.

Key properties

Proposition

The following two properties hold generally for every formula φ:

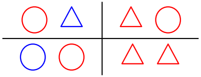
1. *Empty-set property:* $\emptyset \Vdash_e \varphi$,
2. *Persistence:* $s \Vdash_e \varphi$ *and* $t \subseteq s$ *implies* $t \Vdash_e \varphi$.

The following property holds for every $\{\exists, \vee\!\!\!\vee\}$*-free formula* α*:*

3. *Truth-support bridge:* $s \Vdash_e \alpha$ *iff for all* $M \in s$, $M \vDash_e \alpha$.

Inquisitive vs. declarative existential quantifier

- $s \Vdash_e \exists x Px$ means: in every structure from s there is some object that has the property P.
- $s \Vdash_e \exists\!\!\!\exists x Px$ means: there is some object that in every structure from s has the property P.

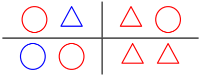


In this state s we have

- $s \Vdash_e \exists x Red(x)$,
- but $s \nVdash_e \exists\!\!\!\exists x Red(x)$.

Inquisitive vs. declarative disjunction

- $s \Vdash_e Pa \vee Qa$ means: in every structure from s, the object a either has the property P or the property Q.
- $s \Vdash_e Pa \mathbin{\mathbb{V}} Qa$ means: either the object a has the property P in all structures from s, or the object a has the property Q in all structures from s.



In this state s we have

- $s \Vdash_e Circle(a) \vee Red(a)$,
- but $s \nVdash_e Circle(a) \mathbin{\mathbb{V}} Red(a)$.

Inquisitive consequence relation

We define the consequence relation $\vDash$ as preservation of support.

Proposition

For the $\{\exists, \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\}$-free fragment of the language, the logic corresponds to classical first-order logic.

Disjunction and existence property

Theorem (Grilletti 2018)

Let Γ *be a set of* $\{\exists, \mathbb{V}\}$*-free formulas and* φ, ψ *arbitrary formulas. Then*

(a) *if* $\Gamma \vDash \varphi \mathbb{V} \psi$ *then* $\Gamma \vDash \varphi$ *or* $\Gamma \vDash \psi$*,*

(b) *if* $\Gamma \vDash \exists x \varphi$ *then for some term* t*,* $\Gamma \vDash \varphi[t/x]$*.*

Compactness

Theorem

If every finite subset of Δ is satisfiable then Δ is satisfiable.

Compactness for entailment is an open problem:

- if $\Delta \vDash \varphi$ then for some finite $\Delta' \subseteq \Delta$, $\Delta' \vDash \varphi$.

More open problems

- Is the set of valid formulas recursively enumerable? (axiomatization)
- If φ is not valid, is there a counterexample $\langle D, W \rangle$ with countable D and W? (Löwenheim-Skolem)

A fragment of the language $\mathcal{L}^{-}_{inq}$

Only declarative antecedents are allowed:

$$\varphi ::= \bot \mid t_1 = t_2 \mid Pt_1 \dots t_n \mid \varphi \wedge \varphi \mid \alpha \to \varphi \mid \forall x \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \exists x \varphi$$

where α is $\{\exists, \vee\!\!\!\vee\}$-free

Inquisitive logic in the language $\mathcal{L}_{inq}^{-}$

Intuitionistic logic plus (where α is declarative)

DN $\neg\neg\alpha \to \alpha$,

CD $\forall x(\varphi \vee\!\!\!\vee \psi) \to (\varphi \vee\!\!\!\vee \forall x\psi)$, if x is not free in φ,

$\vee\!\!\!\vee$-split $(\alpha \to (\varphi \vee\!\!\!\vee \psi)) \to ((\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi))$,

$\exists\!\!\!\exists$-split $(\alpha \to \exists\!\!\!\exists x\varphi) \to \exists\!\!\!\exists x(\alpha \to \varphi)$, if x is not free in φ.

The derivability relation is denoted by $\vdash$.

Theorem (Grilletti 2020)

Let $\Phi \cup \{\varphi\}$ be a set of $\mathcal{L}_{inq}^{-}$-sentences. Then,

$$\Phi \models \varphi \textit{ iff } \Phi \vdash \varphi.$$

Mention-some fragment

$$\chi ::= \alpha \mid \chi \mathbin{\mathbb{V}} \chi \mid \exists x \chi \mid \chi \wedge \chi \mid \alpha \to \chi$$

where α is $\{\exists, \mathbb{V}\}$-free

Theorem (Ciardelli 2016)

For every χ from the mention-some fragment there are declarative $\alpha_1, \ldots, \alpha_n$ and tuples of variables $\overline{x}_1, \ldots, \overline{x}_n$ such that:

$$\vdash \chi \leftrightarrow \exists \overline{x}_1 \alpha_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} \exists \overline{x}_n \alpha_n.$$

Antecedents from the mention-some fragment

$$\chi ::= \alpha \mid \chi \mathbin{\mathbb{V}} \chi \mid \exists x \chi \mid \chi \wedge \chi \mid \alpha \to \chi$$

$$\varphi ::= \bot \mid t_1 = t_2 \mid Pt_1 \ldots t_n \mid \varphi \wedge \varphi \mid \chi \to \varphi \mid \forall x \varphi \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \exists x \varphi$$

where α is $\{\exists, \mathbb{V}\}$-free

What creates the problem

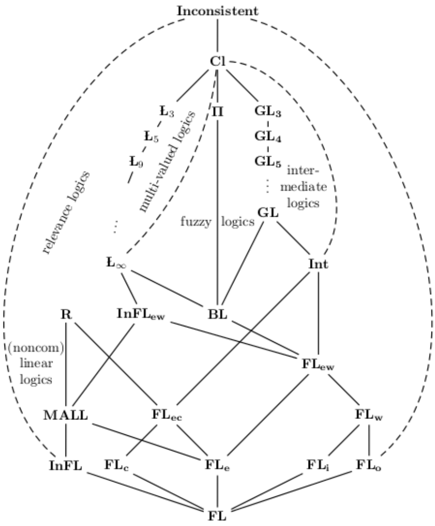
Formulas like this:

- $\forall x ? Px \rightarrow \exists x Sx$

Is inquisitive logic a non-classical logic?

Two alternative approaches:

- inquisitive logic as a superintuitionistic logic in the standard propositional language
- inquisitive logic as a conservative extension of classical logic in an enriched language



Picture taken from Galatos, N. Jipsen, P. Kowalski, T., Ono, H. (2007) *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier Science.

Punčochář, V. (2023). Fuzzy Truth, Fuzzy Support and Fuzzy Information States for Inquisitive Semantics. *Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning*. Pages 572–581.

Vagueness-based false dilemma fallacy

- *Is the enemy weak or strong?*
- *Is Ann a cat person or a dog person?*
- *Are you with us or against us?*
- *Are you an early bird or a night owl?*
- *Do you like beer or wine?*

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Language

Declarative language:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha$$

Defined symbols:

- $\neg\alpha =_{def} \alpha \to \bot$
- $\alpha \vee \beta =_{def} ((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)$

To this declarative language we will add inquisitive disjunction: $\mathbin{\vee\!\!\!\vee}$.

α, β declarative formulas
φ, ψ arbitrary formulas

Crisp models

Definition

A c-model (a shorthand for "crisp model") is a pair $\mathcal{M} = \langle W, V \rangle$, where

- W is a non-empty set (of possible worlds);
- V is a c-valuation (crisp valuation), i.e. a function assigning to each atomic formula an information c-state

An information c-state is any subset of W.

Truth conditions

$$w \not\models \bot,$$

$$w \models p \text{ iff } w \in V(p), \text{ for each atomic formula } p,$$

$$w \models \alpha \wedge \beta \text{ iff } w \models \alpha \text{ and } w \models \beta,$$

$$w \models \alpha \to \beta \text{ iff } w \not\models \alpha \text{ or } w \models \beta.$$

We define tc-consequence relation as the preservation of truth.

Support conditions

$s \Vdash \bot$ iff $s = \emptyset$,

$s \Vdash p$ iff $s \subseteq V(p)$, for each atomic formula p,

$s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$,

$s \Vdash \varphi \to \psi$ iff $\forall t \subseteq W$, if $t \Vdash \varphi$, then $s \cap t \Vdash \psi$,

$s \Vdash \varphi \bigvee \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$.

We define sc-consequence relation as the preservation of support.

Basic results

Proposition (Truth-Support Bridge-I)

For any information c-state s of any c-model, and for any declarative formula α:

$$s \Vdash \alpha \text{ iff } w \vDash \alpha, \text{ for all } w \in s.$$

Proposition

For any set of declarative formulas $\Delta \cup \{\alpha\}$:

$$\Delta \vDash_{sc} \alpha \text{ iff } \Delta \vDash_{tc} \alpha$$

Proposition

In any c-model and for any formula φ:

(a) $\emptyset \Vdash \varphi$ *(empty-state property),*

(b) *if $s \Vdash \varphi$ and $t \subseteq s$ then $t \Vdash \varphi$ (persistence property).*

Continuous t-norms

Definition

A (continuous) t-norm is a continuous, commutative, associative and monotone binary function $*$ on the interval $[0, 1]$ such that $1 * x = x$ and $0 * x = 0$, for each x from $[0, 1]$.

Proposition

For every a t-norm $*$ *there is a unique binary residual operation* $\Rightarrow_*$ *on* $[0, 1]$ *satisfying:*

$$x * y \leq z \textit{ iff } x \leq y \Rightarrow_* z.$$

The residual of minimum

the residual of *min* is the function $\Rightarrow_{min}$ defined as follows:

$$x \Rightarrow_{min} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Łukasiewicz t-norm

Łukasiewicz t-norm defined as follows:

$$x *_L y = max\{0, x + y - 1\}.$$

The residual of the Łukasiewicz t-norm $*_L$ is the function $\Rightarrow_L$ defined in this way:

$$x \Rightarrow_L y = \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{otherwise} \end{cases}$$

Truth conditions

$$
\begin{aligned}
&w(\bot) = 0,\\
&w(p) = V(p)(w), \text{ for each atomic formula } p,\\
&w(\alpha \wedge \beta) = min\{w(\alpha), w(\beta)\},\\
&w(\alpha \to \beta) = w(\alpha) \Rightarrow_* w(\beta).
\end{aligned}
$$

We define tc_*-consequence as preservation of the value 1.

Fuzzy sets

Definition

A fuzzy subset of W is a function from W to $[0, 1]$.

- $\emptyset_f$ and W_f the fuzzy empty set and the fuzzy full set
- fuzzy subset relation: $s \sqsubseteq t$ iff $s(w) \leq t(w)$, for all $w \in W$,
- fuzzy intersection: $(s \sqcap t)(w) = min\{s(w), t(w)\}$
- fuzzy union: $(s \sqcup t)(w) = max\{s(w), t(w)\}$

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f-models and f-states

Definition

An f-model (fuzzy model) is a pair $\mathcal{M} = \langle W, V \rangle$, where

- W is a non-empty set (of possible worlds)
- V is an f-valuation (fuzzy valuation), i.e. a function assigning to each atomic formula an information f-state

An information f-state (fuzzy information state) is a fuzzy subset of W.

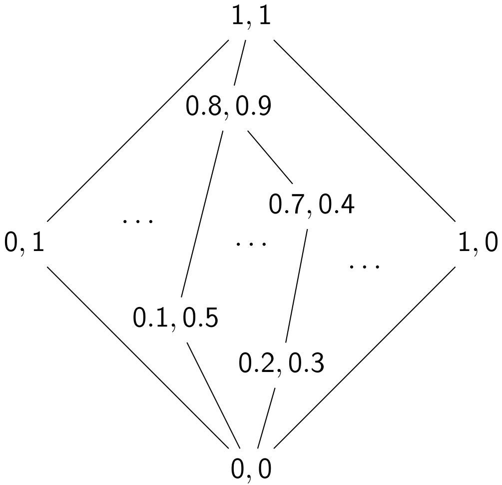
	w_1	w_2	w_3
the culprit is a man	1	1	0
the culprit is tall	0.7	0.8	0.5
the culprit is smart	0.5	0.9	0.8
the culprit knew well the victim	0.8	0.2	0.7
	0.5	0.2	0

Table: An example illustrating fuzzy states

Algebra of crisp information states for two worlds

$$\begin{tikzcd} & 1,1 \arrow[dl, dash] \arrow[dr, dash] & \\ 0,1 \arrow[dr, dash] & & 1,0 \arrow[dl, dash] \\ & 0,0 & \end{tikzcd}$$

Algebra of fuzzy information states for two worlds



Two dimensions of fuzzyfication

Four options:

1. crisp states, crisp support
2. crisp states, fuzzy support
3. fuzzy states, crisp support
4. fuzzy states, fuzzy support

Support conditions: fuzzy states, crisp support

We define the operation $*$ also on the level of f-states:

$$(s * t)(w) = s(w) * t(w),$$

The generalized support conditions are defined as follows:

$s \Vdash \bot$ iff $s = \emptyset_f$,

$s \Vdash p$ iff $s \sqsubseteq V(p)$, for each atomic formula p,

$s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$,

$s \Vdash \varphi \to \psi$ iff $\forall t \sqsubseteq W_f$, if $t \Vdash \varphi$, then $s * t \Vdash \psi$,

$s \Vdash \varphi \mathbin{\rotatebox[origin=c]{180}{$\bigwedge$}} \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$.

We define sc_*-consequence as preservation of support in W_f.

Basic results

Proposition (Truth-Support Bridge-II)

For any information f-state s of any f-model, and for any declarative formula α:

$$s \Vdash \alpha \text{ iff } s(w) \leq w(\alpha), \text{ for all } w \in W.$$

Proposition

For any set of declarative formulas $\Delta \cup \{\alpha\}$:

$$\Delta \vDash_{sc_*} \alpha \text{ iff } \Delta \vDash_{tc_*} \alpha$$

Proposition

In any f-model and for any formula φ:

(a) $\emptyset_f \Vdash \varphi$ *(empty-state property),*

(b) *if $s \Vdash \varphi$ and $t \sqsubseteq s$ then $t \vDash \varphi$ (persistence property).*

Graded support

	w_1	w_2	w_3
s	0.3	0.4	0.9
p	0.6	0.7	0.89
q	0.2	0.3	0.1

Table: An example illustrating graded support

Graded support

For any f-state s and any formula α

$$s \rightsquigarrow_* \alpha \text{ is a shorthand for } \bigwedge_{w \in W} (s(w) \Rightarrow_* w(\alpha)).$$

for any f-state s and any formula α:

$$s \Vdash \alpha \text{ iff } s \rightsquigarrow \alpha = 1.$$

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Table: An example illustrating graded support

If $*$ is the Łukasiewicz t-norm we obtain:

- $s \rightsquigarrow_* p = 0.99$
- $s \rightsquigarrow_* q = 0.2$

- Our aim now is to capture the notion of graded or fuzzy support that would determine to what degree s supports φ. We denote this value as $s_*[\varphi]$.
- For declarative α we want to obtain $s_*[\alpha] = s \leadsto_* \alpha$.

The main idea

Crisp semantic clause for implication:

- $s \Vdash \varphi \to \psi$ iff $\forall t \sqsubseteq W_f$, if $t \Vdash \varphi$, then $s * t \Vdash \psi$.

Graded semantic clause for implication:

- $s[\varphi \to \psi] = \bigwedge_{t \sqsubseteq W_f} (t[\varphi] \Rightarrow_* s * t[\psi])$.

Support conditions: fuzzy states, fuzzy support

$$
\begin{aligned}
&s[\bot] = s \rightsquigarrow \bot, \\
&s[p] = s \rightsquigarrow p, \text{ for every atomic formula } p, \\
&s[\varphi \wedge \psi] = min\{s[\varphi], s[\psi]\}, \\
&s[\varphi \to \psi] = \textstyle\bigwedge_{t \sqsubseteq W_f}(t[\varphi] \Rightarrow s * t[\psi]), \\
&s[\varphi \vee\!\!\!\vee \psi] = max\{s[\varphi], s[\psi]\}.
\end{aligned}
$$

We define fsc$_*$-consequence as preservation of 1 in W_f.

Basic results

Proposition (Truth-Support Bridge-III)

For any information f-state $s \sqsubseteq W_f$ of any f-model and any declarative formula α:

$$s_*[\alpha] = s \rightsquigarrow_* \alpha.$$

Proposition

For any set of declarative formulas $\Delta \cup \{\alpha\}$:

$$\Delta \vDash_{fsc_*} \alpha \textit{ iff } \Delta \vDash_{tc_*} \alpha$$

Proposition

For every formula φ:

(a) $\emptyset_f[\varphi] = 1$ *(empty-set property),*

(b) *if $s \sqsubseteq t$ then $t[\varphi] \leq s[\varphi]$ (persistence property).*

Declarative vs. inquisitive propositions

Proposition

For any declarative α:

$$\bigwedge_i (s_i[\alpha]) = (\bigvee_i s_i)[\alpha]$$

A counterexample to $inf\{s[p \vee q], t[p \vee q]\} = (s \sqcup t)[p \vee q]$:

	v	w
s	1	0
t	0	1
p	1	0
q	0	1

Setting	Truth-Support Bridge
crisp states, crisp support	$s \Vdash \alpha$ iff $\forall w \in s, w \vDash \alpha$
fuzzy states, crisp support	$s \Vdash \alpha$ iff $\forall w \in W, s(w) \leq w(\alpha)$
fuzzy states, fuzzy support	$s[\alpha] = \bigwedge_{w \in W}(s(w) \Rightarrow w(\alpha))$

Table: Truth-Support Bridge in different versions of inquisitive semantics

Setting	Propositions expressed by formulas
crisp states, crisp support	downward closed crisp sets of crisp information states
fuzzy states, crisp support	downward closed crisp sets of fuzzy information states
fuzzy states, fuzzy support	antitone fuzzy sets of fuzzy information states

Table: Propositions in different versions of inquisitive semantics

The language $\mathcal{L}_{IEL}$

$$\varphi := \ p \mid \bot \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid E_a\varphi$$

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- $\neg\varphi =_{def} \varphi \to \bot$
- $\varphi \vee \psi =_{def} \neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \leftrightarrow \psi =_{def} (\varphi \to \psi) \wedge (\psi \to \varphi)$
- $?\varphi =_{def} \varphi \vee\!\!\!\vee \neg\varphi$
- $W_a\varphi =_{def} E_a\varphi \wedge \neg K_a\varphi$

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Epistemic modalities

the formula	represents
$K_a p$	The agent a knows that p.
$K_a?p$	The agent a knows whether p.
$E_a?p$	The agent a entertains whether p.
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- $K_a(\text{question}) = \text{statement}$
- $E_a(\text{question}) = \text{statement}$
- $W_a(\text{question}) = \text{statement}$

Declarative formulas

Definition

The set of declarative $\mathcal{L}_{IEL}$-formulas is the least set that contains all atomic formulas, $\perp$, $K_a\varphi$ and $E_a\varphi$, for any $\mathcal{L}_{IEL}$-formula φ, and is closed under $\wedge$ and $\rightarrow$.

Models

Definition

A concrete inquisitive epistemic model (CIE-model) is a triple $\langle W, \Sigma_{\mathcal{A}}, V \rangle$, where

- W is a nonempty set of possible worlds
- $\Sigma_{\mathcal{A}} = \{\Sigma_a \mid a \in \mathcal{A}\}$ is a set of inquisitive state maps
- V is a valuation assigning subsets of W to atomic formulas

Inquisitive state maps

- Σ_a assigns to every world w the issue of the agent a in the world w
- every issue is represented by a set of information states (those states that resolve the issue)
- every information state is represented by a set of possible worlds (those worlds that are compatible with the information, i.e. that are not excluded by the information)
- the information state of the agent in a world determines the boundaries for the issue of the agent in the world

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Inquisitive state maps

$\Sigma_a : W \to \mathcal{P}(\mathcal{P}(W))$, $\sigma_a :\to \mathcal{P}(W)$ satisfying:

- $\Sigma_a(w)$ is nonempty downward closed,
- $\sigma_a(w) = \bigcup \Sigma_a(w)$,
- for any $w \in W$, $w \in \sigma_a(w)$ (factivity),
- for any $w, v \in W$, if $v \in \sigma_a(w)$, then $\Sigma_a(v) = \Sigma_a(w)$ (introspection).

Support conditions

- $s \vDash K_a\varphi$ iff $\forall w \in s$: $\sigma_a(w) \vDash \varphi$,
- $s \vDash E_a\varphi$ iff $\forall w \in s\ \forall t \in \Sigma_a(w)$: $t \vDash \varphi$.

Theorem

In every inquisitive epistemic model:

(a) *every formula is supported by the empty state,*

(b) *support is downward persistent for all formulas,*

(c) *support of declarative formulas is closed under arbitrary unions,*

(d) *every formula is equivalent to the inquisitive disjunction of a finite set of declarative formulas.*

Axiomatization of IEL

INT	Axioms of intuitionistic logic and modus ponens
split	$(\alpha \to (\varphi \vee\!\!\!\vee \psi)) \to ((\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi))$
rdn	$\neg\neg\alpha \to \alpha$
S5	S5-axioms and necessitation for K_a and E_a
K2	$K_a(\varphi \vee\!\!\!\vee \psi) \leftrightarrow (K_a\varphi \vee K_a\psi)$
KE	$E_a\alpha \leftrightarrow K_a\alpha$

(α ranges over declarative formulas)