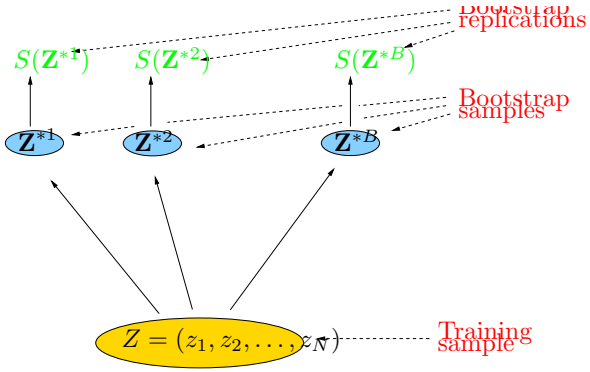
Learning Method Comparison

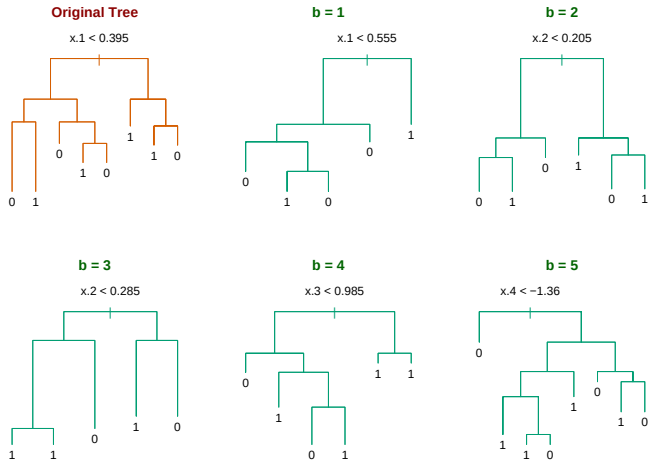
- Neural Networks before deep learning
- SVM - logistic regression; with a non-linear transform. that worstens scalability.

Characteristic	Neural Nets	SVM	Trees	MARS	k-NN, Kernels
Natural handling of data of "mixed" type	▼	▼	▲	▲	▼
Handling of missing values	▼	▼	▲	▲	▲
Robustness to outliers in input space	▼	▼	▲	▼	▲
Insensitive to monotone transformations of inputs	▼	▼	▲	▼	▼
Computational scalability (large N)	▼	▼	▲	▲	▼
Ability to deal with irrelevant inputs	▼	▼	▲	▲	▼
Ability to extract linear combinations of features	▲	▲	▼	▼	◆
Interpretability	▼	▼	◆	▲	▼
Predictive power	▲	▲	▼	◆	▲

Ensemble Methods

- To improve the predictive power of a decision tree, we combine the results from a bag of trees.
- Common methods
  - Random forest (+ Bagging)
  - Boosting
    - Adaboost - classification
    - Gradient boosting - regression and classification
  - Stacking
  - MARS (=earth).





Bootstrap

- Select elements with replacement.
- We have N data samples, we select with replacement N samples – some are selected more than one, some are not selected at all. *The not selected are used for testing.*
- The probability of not-selecting a sample is $\left(1 - \frac{1}{N}\right)^N \approx e^{-1} = 0.368$.
- Selected samples used to learn a model (usually a tree).
- These are used for the **OutOfBag** error computation.
- All today models are implemented in

```
sklearn.ensemble
sklearn.inspection
```

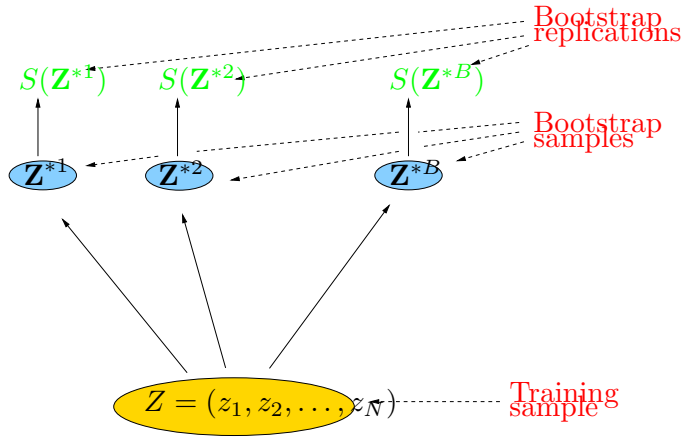


FIGURE 7.12. *Schematic of the bootstrap process. We wish to assess the statistical accuracy of a quantity $S(\mathbf{Z})$ computed from our dataset. B training sets $\mathbf{Z}^{*b}$, $b = 1, \ldots, B$ each of size N are drawn with replacement from the original dataset. The quantity of interest $S(\mathbf{Z})$ is computed from each bootstrap training set, and the values $S(\mathbf{Z}^{*1}), \ldots, S(\mathbf{Z}^{*B})$ are used to assess the statistical accuracy of $S(\mathbf{Z})$.*

Random Forest for Regression or Classification !

procedure RANDOM FOREST:(X, y training data)
  for $b = 1, 2, \ldots, B$ **do**
    Draw a bootstrap sample $\mathbf{Z}^*$ of size N
    repeat ▷ Grow a random forest tree T_b
      Select m variables at random from p variables. ▷ ! crucial
      Pick the best variable/split–point among the m
      Split the node into two children nodes.
    until the minimum node size n_{min} is reached.
  end for
  Output the ensemble of trees $\{T_b\}_1^B$.
end procedure ▷ usually no pruning

To make a prediction at a new point x:

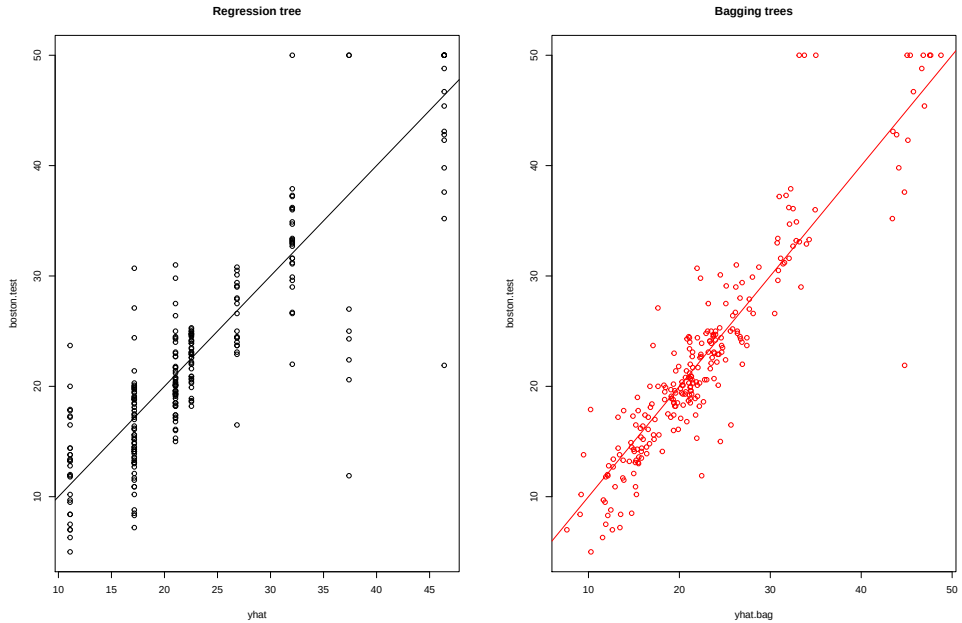
- Regression: $\hat{f}_{rf}^B(x) = \frac{1}{B}\sum_{b=1}^B T_b(x)$.
- Classification: Let $\widehat{C}_b(x)$ be the class prediction of the bth random–forest tree.
  - Predict $\widehat{C}_{rf}^B(x) = \textit{majority vote}\ \{\widehat{C}_b(x)\}_1^B$.

Bagging (Bootstrap aggregating)

- It is a Random Forest, where we use all predictors, that is $m = p$.
- both regression and classification.
- Training data $\mathbf{Z} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$

$$\hat{f}_{bag}(x) = \frac{1}{B} \sum_{b=1}^{B} \hat{f}^{*b}(x).$$

- Bagging adds the smoothness in predicted values.
- Left: constant prediction at tree leafs.
- Average over different trees adds smoothness.
- **Random forest** selects a subset of attributes to increase the diversity of trees.



Behind Random Forest

The variance of the random forest estimate $Var(\hat{f}_{rf}^{B}(x)) = \mathbb{E}(\hat{f}(x) - \mathbb{E}\hat{f}(x))^2$ is

- iid data variables, independent features, each with variance σ^2:
  - $\frac{1}{B}\sigma^2$
- id identically distributed data, each with variance σ^2 with positive pairwise correlation ρ:
  - $\rho\sigma^2 + \frac{1-\rho}{B}\sigma^2$.
- The second part is addressed by bagging.
- The idea behind random random forest is to address the first part of the formula.
  - Before each split, select $m \leq p$ variables as candidates for splitting.
  - $m \leftarrow \sqrt{p}$ for regression, even as low as 1. $\frac{p}{3}$ for classification.
- Bagging does not change linear estimates, such as the sample mean
  - The pairwise correlation between bootstrapped means is about 50%.

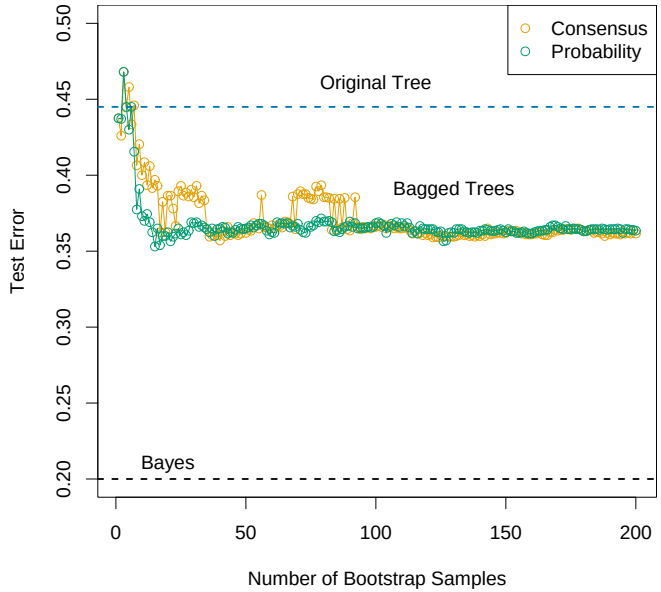
Bagging for Classification

- Training data $\mathbf{Z} = \{(x_1, g_1), (x_2, g_2), \ldots, (x_N, g_N)\}$
- for each bootstrap sample, $b = 1, 2, \ldots, B$, we fit our model, giving prediction $\hat{f}^{*b}(x)$.
- Take either
  - predict probabilities of classes and find the class with the highest predicted probability over the bootstrap samples

$$\hat{G}(x) = \mathit{argmax}_k \sum_{b=1}^{B} \hat{f}^{*b}(x)$$

  - predict class and

$$\hat{G}_{bag}(x) = \mathit{majority\ vote}\ \{\hat{G}^{*b}(x)\}_{b=1}^{B}.$$

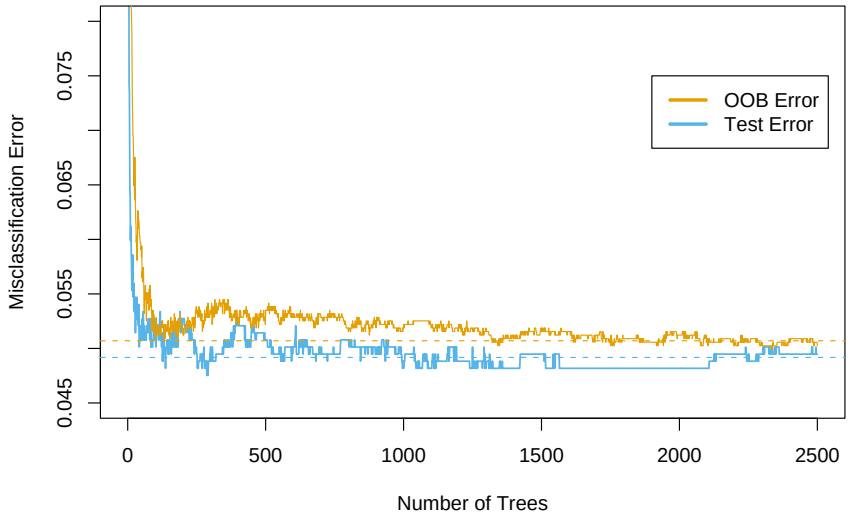


OOB Error

Definition (Out of bag error (OOB))

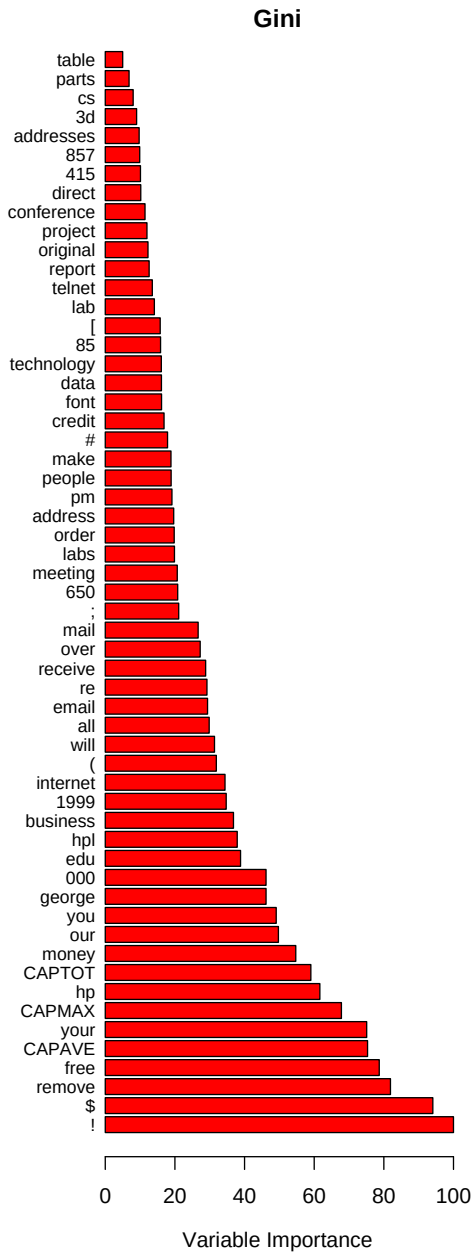
For each observation $z_i = (x_i, y_i)$, construct is random forest predictor by averaging only those trees corresponding to bootstrap samples in which z_i did not appear.

- An OOB error estimate is almost identical to that obtained by N-fold crossvalidation.
- Unlike many other nonlinear estimators, random forests can be fit in one sequence.



Feature Importance Mean Decrease in Impurity

- **Variable Importance** of a predictor X_ℓ in a single tree T is
  $I_\ell^2(T) = \sum_{t=1}^{J} \hat{i}_t^2 \cdot I(v(t) = \ell)$
  - For each internal node t of the tree, we calculate the *Gini* or *RSS* gain
  - where $\hat{i}_t^2$ is the Gini/RSS improvement of the predictor in the inner node t.
    - Gini $\hat{p}_k(t)(1 - \hat{p}_k(t))$ before and after the split
    - for K goal classes, a separate tree for each class against others
    - weighted by the probability of reaching the node t.
- For a set of trees, we average over M all trees
  $I_\ell^2 = \frac{1}{M} \sum_{i=1}^{M} I_\ell^2(T_m)$.
- Usually scaled to the interval $(0, 100)$.



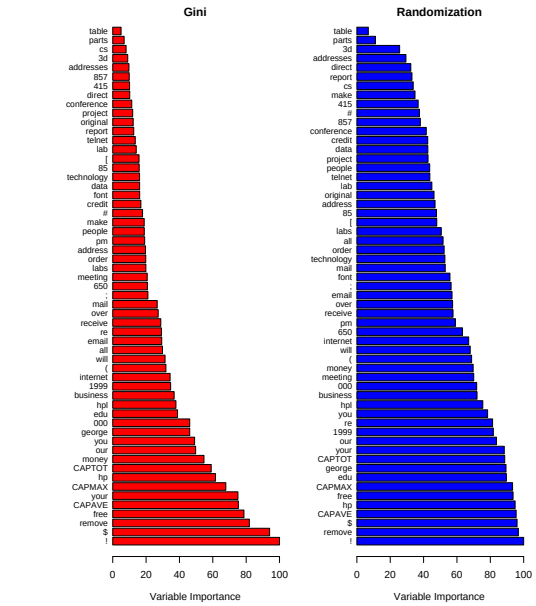
Feature Importance based on Feature Permutation

OOB Variable Importance

1: **procedure** OOBN VarImportance:(data)
2: **for** $b = 1, 2, \ldots, B$ **do**
3: Draw a bootstrap sample $\mathbf{Z}^*$ of size N
4: Grow a random forest tree T_b
5: Calculate accuracy on OOB samples
6: **for** $j = 1, 2, \ldots, p$ **do**
7: permute the values for the jth variable randomly in the OOB samples
8: Calculate the decrease in the accuracy
9: **end for**
10: **end for**
11: Output average accuracy gain for each $j = 1, 2, \ldots, p$.
12: **end procedure**

Alternative Variable Importance

with quite different results



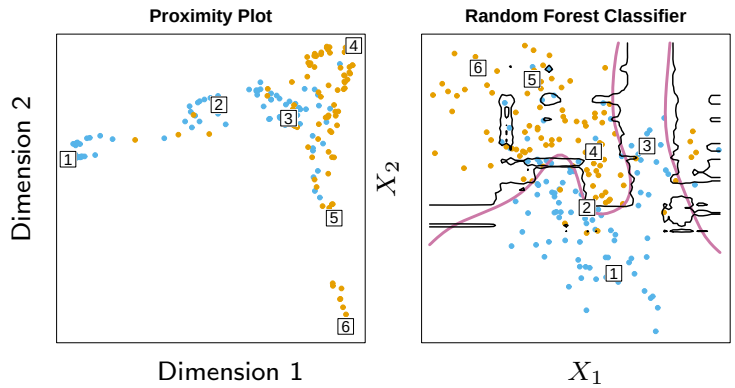
- The randomization voids the effect of a variable.

Proximity plot

Proximity plot

procedure PROXIMITY PLOT(X, y training data)
  for $b = 1, 2, \ldots, B$ **do**
    Draw a bootstrap sample $\mathbf{Z}^*$ of size N
    Grow a random forest tree T_b
    Calculate prediction accuracy on OOB samples
    for any pair of OOB samples sharing the same leaf **do**
      increase the proximity by one.
    end for
  end for
end procedure

- Distinct samples usually come from the pure regions
- Samples in the 'star center' are close to the decision boundary.



Overfitting

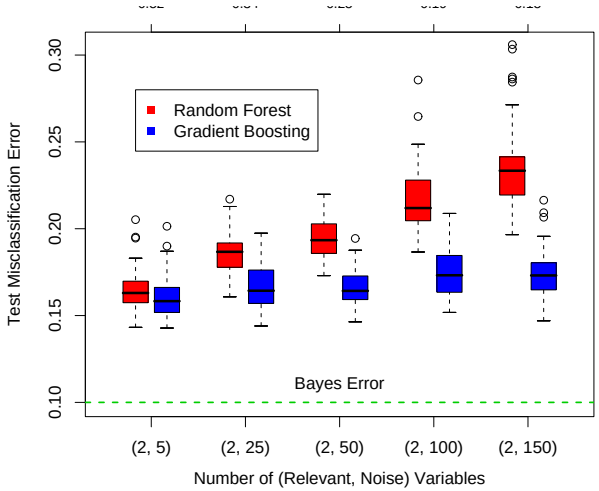
- Though the random forest cannot overfit the limit distribution

$$\hat{f}_{rf}(x) = \mathbb{E}_{\Theta} T(x;\Theta) = \lim_{B\to\infty} \hat{f}_{rf}^{B}(x)$$

- the limit distribution (the average of fully grown trees) may overfit the data.
- Small number of relevant variables with many irrelevant hurts the random forest approach.

$\Rightarrow$ With higher number of relevant variables RF is quite robust.

- 6 relevant and 100 noisy variables, $m = \sqrt{6+100} \sim 10$
- probability of a relevant variable being selected at any split is 0.46.



- Seldom the pruning improves the random forest result
- usually, fully grown trees are used.
  - Two additive vars, 10 noisy,
  - plus additive Gaussian noise.

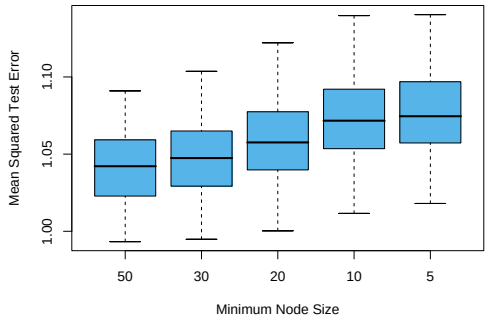


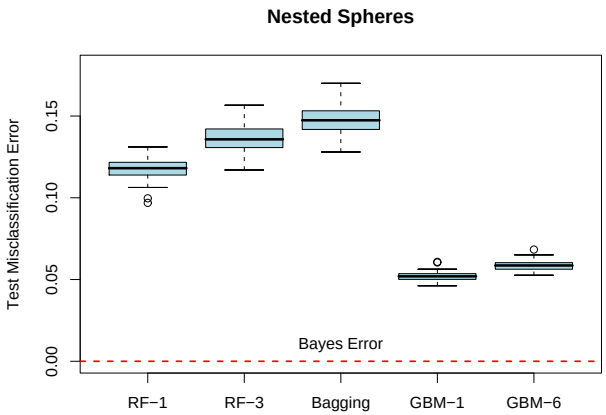
FIGURE 15.8. *The effect of* tree size *on the erro*

Random Forest Experiments

Spam example misclassification error

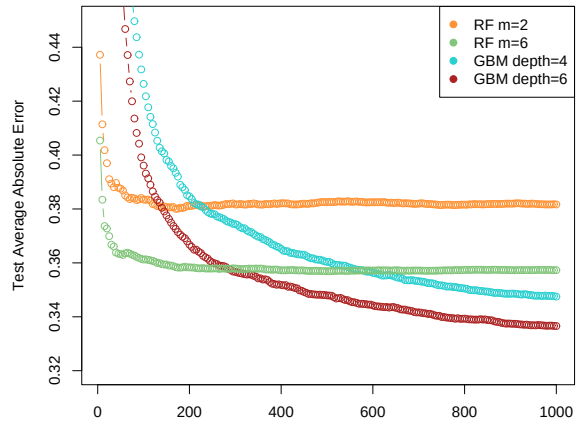
- bagging 5.4%
- random forest 4.88%
- gradient boosting 4.5%.

Nested spheres in $\mathbb{R}^{10}$, 2500 trees, the number selected by 10–fold crossvalidation



California housing data

- Random forests stabilize at about 200 trees, while at 1000 trees boosting continues to improve.
  - Boosting is slowed down by the shrinkage
  - the trees are much smaller (decision stumps, interaction depth=1 or 2).
- Boosting outperforms random forests here.



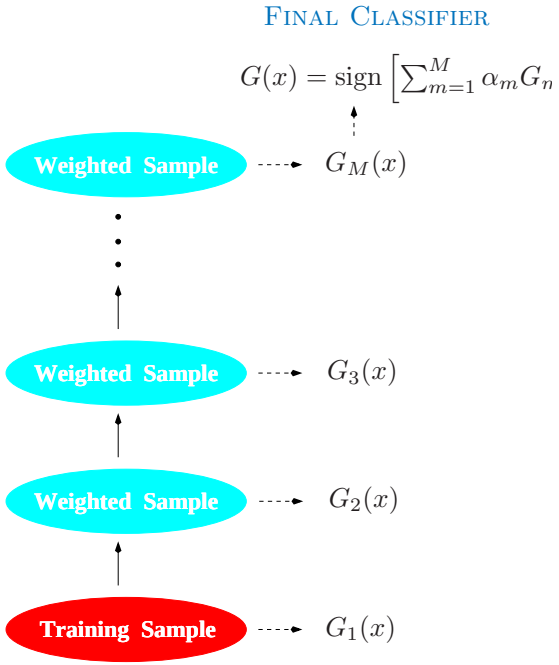
Boosting

! Use a week classifier as a decision stump (a decision tree with the depth= 1).

AdaBoost.M1

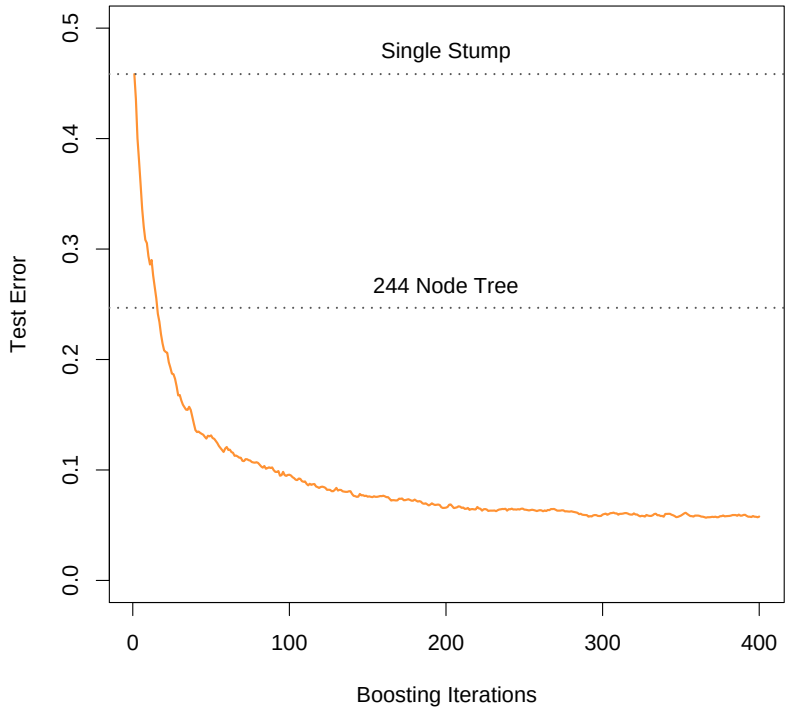
1: **procedure** ADABOOST CLASSIFIER(X, G)
2: Initialize the observation weights $w_i \leftarrow \frac{1}{N}$.
3: **for** $m = 1, 2, \ldots, M$ **do**
4: Fit a classifier $G_m(x)$ to the training data using weights w_i
5: compute $err_m \leftarrow \frac{\sum_{i=1}^{N} w_i I(y_i \neq G_m(x_i))}{\sum_{i=1}^{N} w_i}$
6: compute $\alpha_m \leftarrow log\frac{(1-err_m)}{err_m}$
7: Set $w_i \leftarrow w_i \cdot e^{I(y_i \neq G_m(x_i)) \cdot \alpha_m}$
8: (normalize weights)
9: **end for**
10: Output $G(x) = sign[\sum_{m=1}^{M} \alpha_m G_m(x)]$.
11: **end procedure**

- Two class problem with encoding $Y \in \{-1, 1\}$
- $\overline{err} = \frac{1}{N} \sum_{i=1}^{N} I(y_i \neq G(x_i))$.



Nested Spheres Example

- The features $X_1, \ldots, X_{10}$ are standard independent Gaussian
- deterministic target
  - $Y = 1$ iff $\sum_{j=1}^{10} X_j^2 > \chi^2_{10}(0.5) = 9.34$,
  - $Y = -1$ otherwise.
- 2000 training cases
- 10000 test observations.
- Decision stumps.



Additive Model

- We encode the binary goal by $Y \in \{-1, +1\}$.
- Boosting fits an additive model:

$$f(x) = \sum_{m=1}^{M} \beta_m b(x; \gamma_m)$$

- where β_m for $m = 1, \ldots, M$ are the expansion coefficients
- $b(x; \gamma) \in \mathbb{R}$ are usually simple functions of the multivariate argument x
  - characterized by a set of parameters γ.
- For trees, γ parametrizes the split variables and split points at the internal nodes, and the predictions at the terminal nodes.
- Forward stagewise Additive Modeling sequentially adds one new basis function without adjusting the parameters and coefficients of the previously fitted.
- For squared–error loss

$$L(y, f(x)) = (y - f(x))^2,$$

we have

$$\begin{aligned} L(y_i, f_{m-1}(x) + \beta_m b(x_i; \gamma_m)) &= (y_i - f_{m-1}(x) - \beta_m b(x_i; \gamma_m))^2 \\ &= (r_{im} - \beta_m b(x_i; \gamma_m))^2 \end{aligned}$$

Exponential Loss and AdaBoost

- Let us use the $Y \in \{-1, 1\}$ encoding and the **exponential loss**

$$L(y, f(x)) = e^{-yf(x)}.$$

- We have to solve

$$\begin{aligned}
(\beta_m, G_m) &= \arg\min_{\beta, G} \sum_{i=1}^{N} e^{[-y_i(f_{m-1}(x_i)+\beta G(x_i)]} \\
&= \arg\min_{\beta, G} \sum_{i=1}^{N} e^{[-y_i(f_{m-1}(x_i)]} e^{[-y_i \beta G(x_i)]} \\
&= \arg\min_{\beta, G} \sum_{i=1}^{N} w_i^{(m)} e^{[-y_i \beta G(x_i)]}
\end{aligned}$$

- where $w_i^{(m)} = e^{[-y_i f_{m-1}(x_i)]}$ does not depend on β nor $G(x)$.
- this weight depends on $f_{m-1}(x_i)$ and change with each iteration m.

Exponential Loss and AdaBoost

- From

$$
\begin{aligned}
(\beta_m, G_m) &= \arg\min_{\beta, G} \sum_{i=1}^{N} w_i^{(m)} e^{[-y_i \beta G(x_i)]} \\
&= \arg\min_{\beta, G} \left[e^{\beta} \cdot \sum_{y_i \neq G(x_i)} w_i^{(m)} + e^{-\beta} \cdot \sum_{y_i = G(x_i)} w_i^{(m)} \right] \\
&= \arg\min_{\beta, G} \left[(e^{\beta} - e^{-\beta}) \cdot \sum_{i=1}^{N} w_i^{(m)} I(y_i \neq G(x_i)) + e^{-\beta} \cdot \sum_{i=1}^{N} w_i^{(m)} \right]
\end{aligned}
$$

- For any $\beta > 0$ the solution for $G_m(x; \gamma)$ is

$$
G_m = \arg\min_{\gamma} \sum_{i=1}^{N} w_i^{(m)} I(y_i \neq G(x_i; \gamma)),
$$

- Recall the error definition:

$$
err_m = \frac{\sum_{i=1}^{N} w_i^{(m)} I(y_i \neq G_m(x_i))}{\sum_{i=1}^{N} w_i^{(m)}}
$$

Adaboost Update

$$\arg\min_{\beta,G}\left[(e^{\beta}-e^{-\beta})\cdot\sum_{i=1}^{N}w_i^{(m)}I(y_i\neq G(x_i))+e^{-\beta}\cdot\sum_{i=1}^{N}w_i^{(m)}\right]$$

- The minimum w.r.t. β_m is:

$$\beta_m=\frac{1}{2}log\frac{1-err_m}{err_m}$$

- The approximation is updated

$$f_m(x)=f_{m-1}(x)+\beta_m G_m(x)$$

- which causes the weights for the next iteration to be:

$$w_i^{m+1}=w_i^m\cdot e^{-\beta_m y_i G_m(x_i)}.$$

- using the fact $-y_i G_m(x_i)=2\cdot I(y_i\neq G_m(x_i))-1$ we get

$$w_i^{m+1}=w_i^m\cdot e^{\alpha I(y_i\neq G_m(x_i))}\cdot e^{-\beta_m}.$$

Why exponential loss?

- The population minimizer is

$$f^*(x) = \arg\min_{f(x)} \mathbb{E}_{Y|x}(e^{-Yf(x)}) = \frac{1}{2} log \frac{P(Y=1|x)}{P(Y=-1|x)}.$$

- therefore

$$P(Y=1|x) = \frac{1}{1+e^{-2f^*(x)}}.$$

- The same function $f^*(x)$ minimizes also deviance (cross–entropy, binomial negative log–likelihood)
  - interpreting f^* as the logit transform. Let:

$$p(x) = P(Y=1|x) = \frac{e^{f^*(x)}}{e^{-f^*(x)}+e^{f^*(x)}} = \frac{1}{1+e^{-2f^*(x)}}.$$

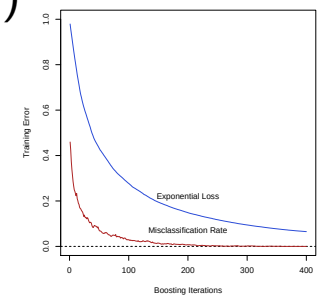
  - and define $Y^| = (Y+1)/2 \in \{0,1\}$. Log–likelihood is

$$\ell(Y, p(x)) = Y^| \log p(x) + (1-Y^|) \log(1-p(x))$$

  - or equivalently the deviance:

$$-\ell(Y, f(x)) = log\left(1+e^{-2Yf(x)}\right).$$

- Exponential loss decreases long after misclassification loss is stable at zero.



Forward Stagewise Additive Modeling

- A general iterative fitting approach.
- In each step, we select the best function from the dictionary $b(x_i; \gamma)$, fit its parameters γ and the weight of this basis function β_m.
- Stagewise approximation is often faster then iterative fitting of the full model.

Forward Stagewise Additive Modeling

procedure FORWARD STAGEWISE ADDITIVE MODELING(L, X, Y, b)
    Initialize $f_0 \leftarrow 0$.
    for $m = 1, 2, \ldots, M$ **do**
        Compute $(\beta_m, \gamma_m) \leftarrow \arg\min_{\beta,\gamma} \sum_{i=1}^{N} L(y_i, f_{m-1}(x_i) + \beta b(x_i; \gamma))$.
        Set $f_m(x) \leftarrow f_{m-1}(x) + \beta_m b(x_i; \gamma_m)$
    end for
end procedure

- For example, our basis functions are decision trees, γ represents the splits and fitted values $T(*; \gamma)$).
- For square error loss, any new tree $T(*; \gamma)$ is the best tree fitting residuals $r_i = y_i - f_{m-1}(x_i)$.

Gradient Tree Boosting Algorithm

Gradient Tree Boosting Algorithm

procedure GRADIENT TREE BOOSTING ALGORITHM(X, Y, L)
  Initialize $f_0(x) \leftarrow \arg\min_\gamma \sum_{i=1}^{N} L(y_i, \gamma)$.
  for $m = 1, 2, \ldots, M$ **do**
    for $i = 1, 2, \ldots, N$ **do**
      compute $r_{im} = -\left[\frac{\partial L(y_i, f(x_i))}{\partial f(x_i)}\right]_{f(x_i)=f_{m-1}(x_i)} =^{[*]} y_i - f_{m-1}(x_i)$
    end for
    Fit reg. tree to the target r_{im} giving regions $\{R_{jm}\}_{j=1,\ldots,J_m}$.
    for $j = 1, 2, \ldots, J_m$ **do**
      Compute $\gamma_{jm} \leftarrow \arg\min_\gamma \sum_{i \in R_{jm}} L(y_i, f_{m-1}(x_i) + \gamma)$.
    end for
    Set $f_m(x) \leftarrow f_{m-1}(x) + \sum_{j=1}^{J_m} \gamma_{jm} I(x \in R_{jm})$.
  end for
  Output $\hat{f}(x) = f_M(x)$.
end procedure

[*] for square error loss.

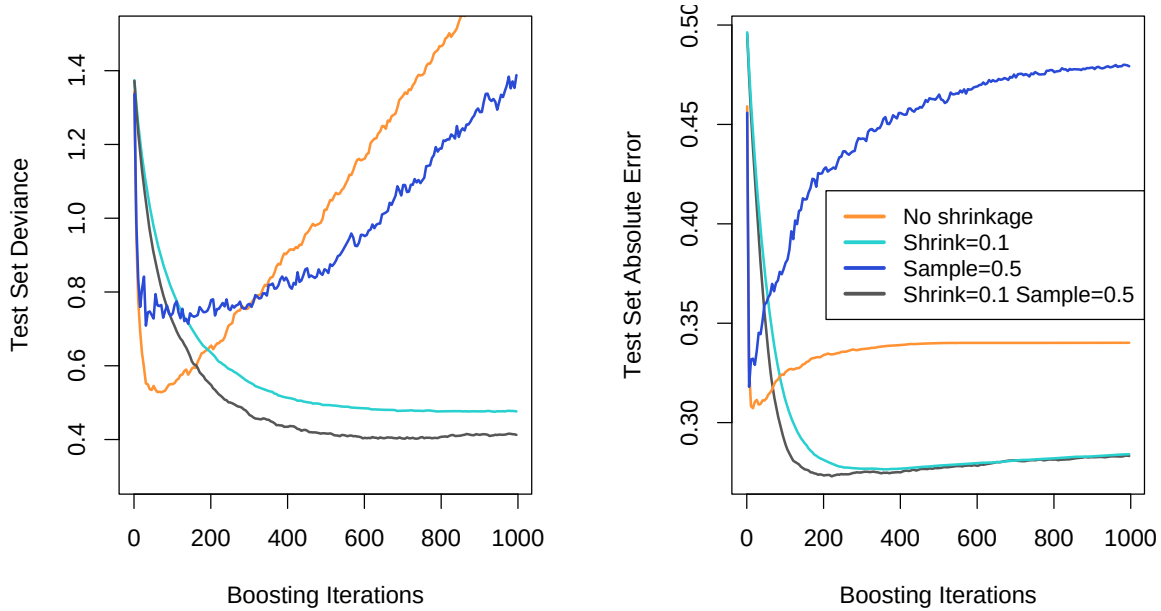
Regularization: Shrinkage, Subsampling

- **Shrinkage** adds the shrinkage parameter $0 < \nu < 1$ to the model construction at line 11:

$$f_m(x) \leftarrow f_{m-1}(x) + \nu \sum_{j=1}^{J_m} \gamma_{jm} I(x \in R_{jm})$$

- This slows down the the learning; this may be both an advantage and an disadvantage.
- **Subsampling** select without replacement only $\eta = \frac{1}{2}$ of data samples in each step.

- Figure: Nested sphere example
- Left: binomial deviance fit
- Right: square error.
- Shrinkage avoids overfitting.



Stacking

- Over a set of models (possibly different types) learn a simple model (like a linear regression)
- Assume predictions $\hat{f}_1(x), \hat{f}_2(x), \ldots, \hat{f}_M(x)$ under square error loss
- Predictors trained without ith example are denoted
  - $\hat{f}_1^{-i}(x), \hat{f}_2^{-i}(x), \ldots, \hat{f}_M^{-i}(x)$
- we can seek weights $w = (w_1, \ldots, w_m)$ such that

$$\hat{w}^{st} = \arg\min_w \sum_{i=1}^{N} \left[y_i - \sum_{m=1}^{M} w_m \hat{f}_m^{-i}(x) \right]^2 .$$

- The final prediction is

$$\hat{f}^{st}(x) = \sum_{m=1}^{M} w_m^{st} \hat{f}_m(x).$$

  - Using cross–validated predictions $\hat{f}_m^{-i}(x)$ stacking avoids giving unfairly high weight to models with higher complexity
  - Better results can be obtained by restricting the weights to be nonnegative and to sum to 1.

Decision Rules from Decision Trees

- We can represent a tree as a set of rules
  - one rule for each leaf.
- These rules may be improved by testing each attribute in each rule
  - Has the rule without this test a better precision than with the test?
  - Use validation data
  - May be time consuming.
- These rules are sorted by (decreasing) precision.

Patient Rule Induction Method PRIM = Bump Hunting

- Rule induction method
- We iteratively search regions with the high Y values
  - for each region, a rule is created.
- CART runs of data after aproximately $\log_2(N) - 1$ cuts.
- PRIM can affort $-\frac{\log(N)}{\log(1-\alpha)}$.
  For $N = 128$ data samples and $\alpha = 0.1$ it is 6 and 46 respectively 29, since the number of observations must be a whole number.

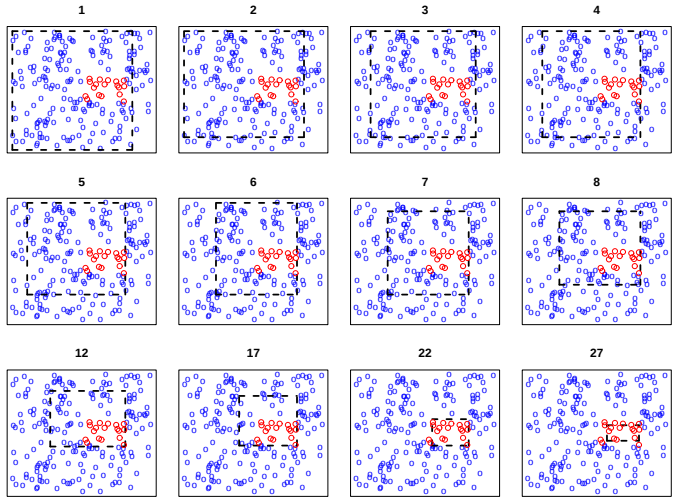
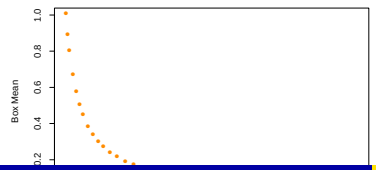


FIGURE 9.7. *Illustration of PRIM algorithm. There are two classes, indicated by the blue (class 0) and red (class 1) points. The procedure starts with a rectangle (broken black lines) surrounding all of the data, and then peels away points along one edge by a prespecified amount in order to maximize the mean of the points remaining in the box. Starting at the top left panel, the sequence of peelings is shown, until a pure red region is isolated in the bottom right panel. The iteration number is indicated at the top of each panel.*

PRIM Patient Rule induction Algorithm

PRIM

- Consider the whole space and all data. Set $\alpha = 0.05$ or 0.10.
- Find X_j and its upper or lower boundary such that the cut of $\alpha \cdot 100\%$ observations leads to the maximal mean of the remaining data.
- Repeat until less then 10 observations left.
- Enlarge the region in any direction that increases the mean value.
- Select the number of regions by the crossvalidation. All regions generated 1-4 are considered.
- Denote the best region B_1.
- Create a rule that describes B_1.
- Remove all data in B_1 from the dataset.
- Repeat 2-5, create B_2 continue until final condition met.



CART Weaknesses

- the high variance
  - the tree may be very different for very similar datasets
  - ensemble learning addresses this issue
- the cuts are perpendicular to the axis
- the result is not smooth but stepwise.
  - MARS (Multivariate Adaptive Regression Splines) addresses this issue.
- it is difficult to capture an additive structure

$$Y = c_1 I(X_1 < t_1) + c_2 I(X_2 < t_2) + \ldots + c_k I(X_k < t_k) + \epsilon$$

  - MARS (Multivariate Adaptive Regression Splines) addresses this issue.

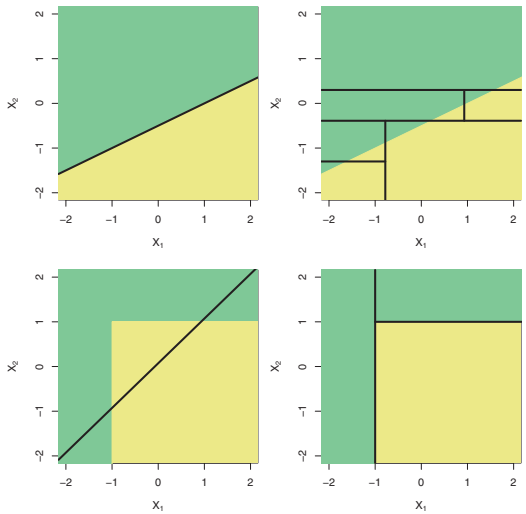


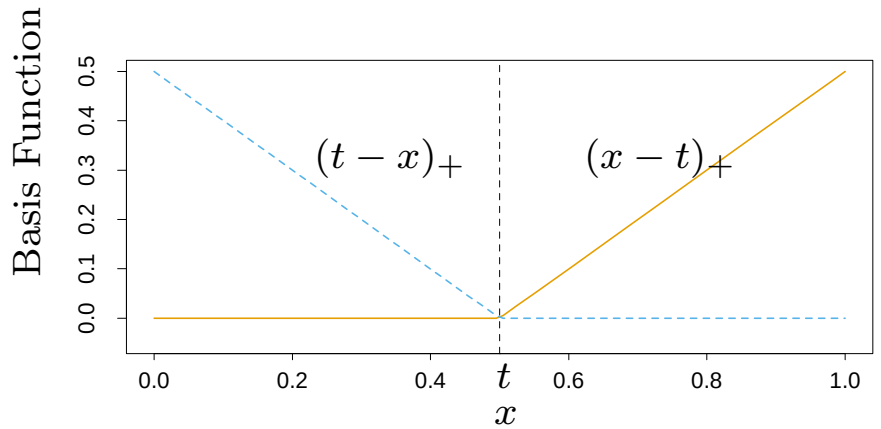
FIGURE 8.7. Top Row: *A two-dimensional classification example in which the true decision boundary is linear, and is indicated by the shaded regions. A classical approach that assumes a linear boundary (left) will outperform a de-*

MARS Multivariate Adaptive Regression Splines

- generalization of linear regression and decision trees CART
- for each feature and each data point we create a **reflected pair** of basis functions
- $(x - t)_+$ and $(t - x)_+$ where $+$ denotes non–negative part, minimum is zero.
- we have the set of functions

$$\mathcal{C} = \{(X_j - t)_+, (t - X_j)_+\}_{t \in \{x_{1,j}, x_{2,j}, \ldots, x_{N,j}\}, j=1,2,\ldots,p}$$

- that is $2Np$ functions for non–duplicated data points.

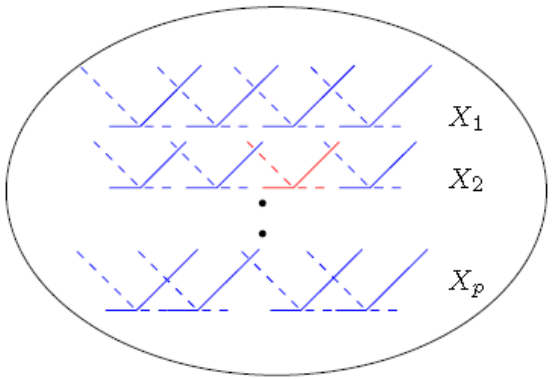


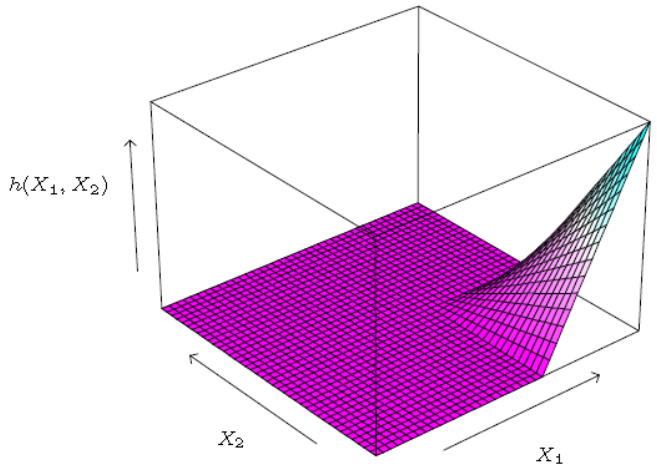
MARS – continuation

- our model is in the form

$$f(X) = \beta_0 + \sum_{m=1}^{M} \beta_m h_m(X)$$

  where $h_m(X)$ is a function from $\mathcal{C}$ or a product of any amount of functions from $\mathcal{C}$
- for a fixed set of h_m's we calculate coefficients β_m by usual linear regression (minimizing RSS)
- the set of functions h_m is selected iteratively.



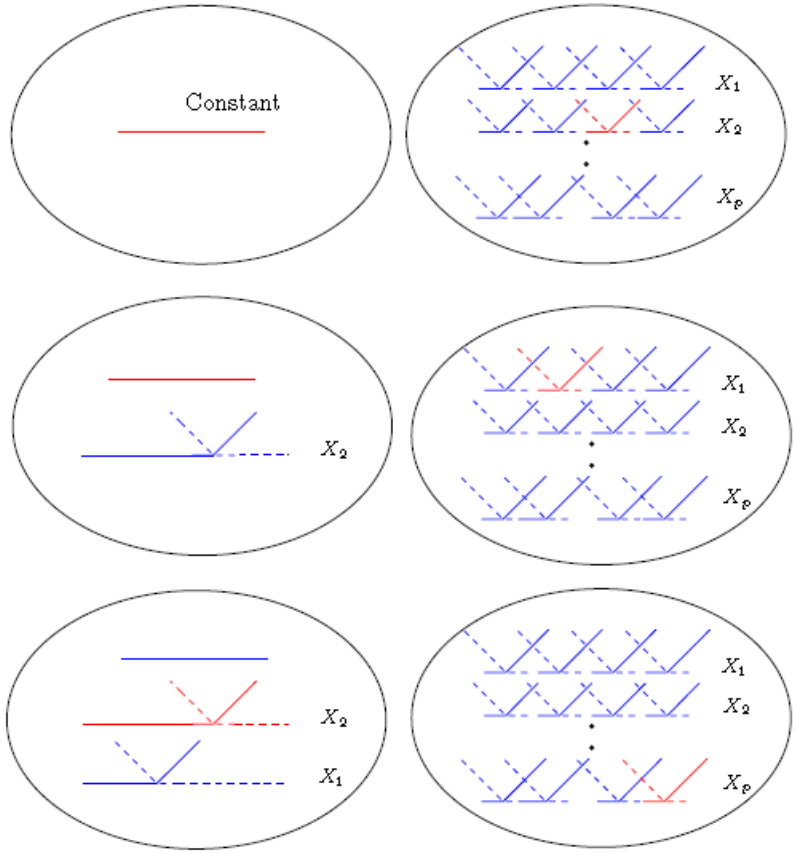


MARS – basis selections

- We start with $h_0 = 1$, we put this function into the model $\mathcal{M} = \{h_0\}$.
- We consider the product of any member $h_\ell \in \mathcal{M}$ with any pair from $\mathcal{C}$,

  $$\hat{\beta}_{M+1} h_\ell(X) \cdot (X_j - t)_+ + \hat{\beta}_{M+2} h_\ell(X) \cdot (t - X_j)_+$$

  we select the one minimizing training error RSS (for any product candidate, we estimate $\hat{\beta}$).
- Repeat until predefined number of functions in $\mathcal{M}$



MARS – model pruning

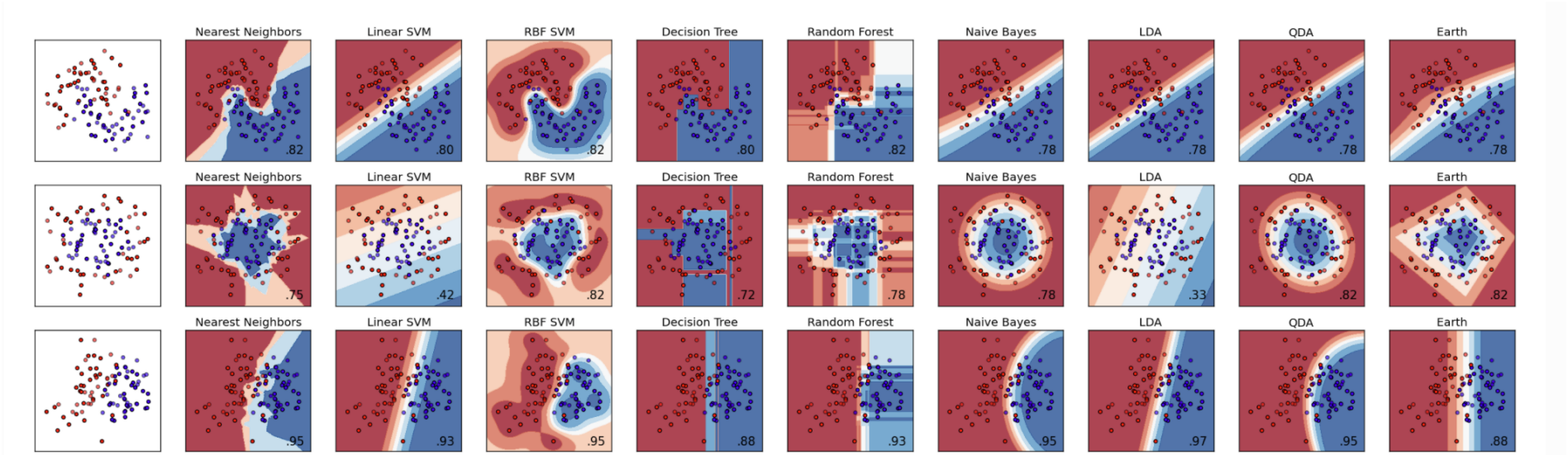
- The model is usually overfitted. We select (remove) iteratively the one minimizing the increase of training RSS. We have a sequence of models $\hat{f}_\lambda$ for different numbers of parameters λ.
- (we want to speed–up cross-validation for computational reasons)
- we select λ (and the model) minimizing **generalized cross-validation**

$$GCV(\lambda) = \frac{\sum_{i=1}^{N}(y_i - \hat{f}_\lambda(x_i))^2}{(1 - M(\lambda)/N)^2}.$$

- where $M(\lambda)$ is the number of effective parameters, the number of function h_m (denoted r) plus the number of knots K, the authors suggest to multiply K by 3: $M(\lambda) = r + 3K$.

MARS is a generalization of CART

- We select piecewise constant functions $I(x - t > 0)$ and $I(x - t \leq 0)$
- If h_m uses multiplication we remove this function from the candidate list. It cannot be used any more.
  - This guarantees binary split.
- Its CART.



https://contrib.scikit-learn.org/py-earth/auto_examples/plot_classifier_comp.html

https://contrib.scikit-learn.org/py-earth/auto_examples/index.html

List of topics

1. Linear, ridge, lasso regression, k-neares neighbours,(formulas) overfitting, curse of dimensionality, (LARS)
2. Splines - the base, natural splines, smoothing splines; kernel smoothing: kernel average, Epanechnikov kernel.
3. Logistic regression, Linear discriminant analysis, generalized additive models
4. Train/test error and data split, square error, 0-1, crossentropy, AIC, BIC,(formulas) crossvalidation, one-leave-out CV, wrong estimate example
5. decision trees, information gain/entropy/gini, CART prunning,(formulas)
6. random forest (+bagging), OOB error, Variable importance, boosting (Adaboost(formulas) and gradient boosting), stacking, MARS,
7. Bayesian learning: MAP, ML hypothesis (formulas), Bayesian optimal prediction, EM algorithm
8. Clustering: k-means, Silhouette, k-medoids, hierarchical
9. Apriori algorithm, Association rules, support, confidence, lift
10. Inductive logic programming basic: hypothesis space search, background knowledge, necessity, sufficiency and consistency of a hypothesis, Aleph
11. Undirected graphical models, Graphical Lasso procedure, deviance, MRF
12. Gaussian processes: estimation of the function and its variance (figures, ideas).

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Table of Contens