Maximally nonassociative quasigroups

Lemma 1. *Let Q be a maximally nonassociative quasigroup. Then Q is idempotent.*

Proof. Consider $a \in Q$. The triple (e_a, a, f_a) is associative. Hence $e_a = a = f_a$. Since $a = e_a a$, there has to be $a = aa$. $\square$

Lemma 2. *Let x, y and z be elements of a quasigroup Q. Any of the following conditions makes (x, y, z) an associative triple.*

(i) $z = f_y = f_{xy}$,
(ii) $x = e_y = e_{yz}$, *and*
(iii) $y = f_x = e_z$.

The value of $x \cdot yz = xy \cdot z$ is equal to xy in case (i), to yz in case (ii) and to xz in case (iii).

Proof. (i) $x \cdot yz = xy = xy \cdot z$, (ii) $x \cdot yz = yz = xy \cdot z$ and (iii) $x \cdot yz = xz = xy \cdot z$. $\square$

Proposition. *Let Q be a quasigroup of finite order n. Let k be the number of associative triples. Then $k \geq n$. The equality holds if and only if Q is maximally nonassociative.*

Proof. There has to be $k \geq n$ since each (e_a, a, f_a) is an associative triple. If $k = n$, then there are no other associative triples and $x \cdot yz = y$ whenever (x, y, z) is an associative triple. Suppose that there exist $a, y \in Q$ such that $z = f_a = f_y$ and $a \neq y$. Then $a = xy$ for some $x \in Q$. By point (i) of Lemma 2, (x, y, z) is associative and $x \cdot yz = xy = a \neq y$. This contradicts $x \cdot yz = y$. Therefore $x \mapsto f_x$ is a permutation of Q. By mirrory symmetry $x \mapsto e_x$ is a permutation too.

Choose $y \in Q$. Since the left and right units yields permutations, there exist $x, z \in Q$ such that $y = f_x = e_z$. The triple (x, y, z) is associative, by point (iii) of Lemma 2. Hence $x = e_y$ and $z = f_y$. Therefore $y = xy = xf_x = x$ and $y = yz = e_z z = z$. $\square$