Autotopisms of a group

Lemma. *Let Q be a loop (i.e., a quasigroup with a unit, say 1) and let (α, β, γ) be an autotopism of Q.*

(i) *If $\alpha(1) = 1$ and $\beta(1) = b$, then $\beta = \gamma = R_b\alpha$.*
(ii) *If $\beta(1) = 1$ and $\alpha(1) = a$, then $\alpha = \gamma = L_a\beta$.*

Proof. Since (i) and (ii) are mirror symmetric, only (i) will be proved. If $x \in Q$, then $\beta(x) = 1 \cdot \beta(x) = \alpha(1)\beta(x) = \gamma(1 \cdot x) = \gamma(x)$. Thus $\beta(x) = \gamma(x) = \gamma(x \cdot 1) = \alpha(x)\beta(1) = R_b\alpha(x)$. □

Proposition. *Let G be a group. Then* $\mathrm{Atp}(G) = \{(L_a\,\varphi, R_b\,\varphi, L_aR_b\,\varphi);\ a, b \in G$ *and* $\varphi \in \mathrm{Aut}(G)\}$.

Proof. If $a, b \in G$, then $(L_a, \mathrm{id}_G, L_a) \in \mathrm{Atp}(G)$ and $(\mathrm{id}_G, R_b, R_b) \in \mathrm{Atp}(G)$. Consider $(\alpha, \beta, \gamma) \in \mathrm{Atp}(G)$. Put $a = \alpha(1)$, $b = \beta(1)$. Then $L_a^{-1}\alpha(1) = R_b^{-1}\beta(1) = 1$. Applying the lemma to the autotopism $(L_a^{-1}\alpha, R_b^{-1}\beta, L_a^{-1}R_b^{-1}\gamma)$ shows that the latter autotopism is equal to $(\varphi, \varphi, \varphi)$ for a permutation φ of G. Thus $\varphi \in \mathrm{Aut}(G)$. □