Standard three-year GAČR grant proposal 2024-2026

The role of set theory in modern mathematics

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Contents

1 Scientific aspects

1.1 Introduction

This projects aims to investigate the role of set theory in modern mathematics.[2] This subject matter can be approached from several perspectives:

One can look at set theory as a convenient universal language for mathematics which is able to express many diverse concepts in one notational system. We believe it is safe to say that in this sense is the role of set theory is generally accepted and viewed favourably. We can say that in this sense set theory appears as a language on which we put minimal requirements with regard to its contents (the "naive set theory").

Or, one can look at set theory more closely also with regard to its contents and propose that by analysing its assumptions, set theory might *actively* help in solving mathematical problems. In this sense, the question of contents – or axioms – gains more importance, and so the notion of the axiomatic set theory (ZFC) replaces the notion of a "naive set theory". From the historical point of view, this second perspective draws its claim for relevance from the remarkable success related to the Axiom of Choice (AC) in the first half of the 20th century. AC was used to gain more understanding of the structure and extent of Lebesgue measure, the characterization of continuity (Hahn theorem), the

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[2] See Remark at the beginning of Section *Aims of the project* for a brief discussion of terminology.

existence of bases of vector spaces, the existence of algebraic closures, and many other. It was not unreasonable to assume that some more potent principles hide in the vague concept of the naive set theory and wait for their discovery.

Arguably, this hope came to an end with the discovery by Cohen in early 1960s of the method of *forcing*, which has been very – some would say perhaps too much – successful in finding non-artificial statements independent over ZFC. If "everything goes", and ZFC cannot even decide the size of the real line, and thus fails to solve the Continuum Hypothesis (CH) – to take the most famous example–, how can be expected that it can help in providing more understanding of mathematical concepts?

In this project, we will investigate the problem from two positions:

($*$) We can attempt to *extend* the axiomatic system ZFC to be able to prove or refute statements otherwise independent over ZFC. The new axioms, or "candidates" for new axioms, should be well motivated, with extensive structural consequences, and if possible easy to use. See for instance [2], [1], [15], [24, 25] or [37] for more extensive discussion. Optimally, we could isolate a few powerful axioms which could impose an additional structure over ZFC, and prove or refute many statements shown independent by forcing.

($**$) We can attempt to actively search for new theorems in ZFC which are relevant for mathematics, for instance by making an independent statement more specific, or redefine notions which tend to behave randomly to have more structure.[3]

Our primary concern for this project is item ($*$), with ($**$) being also important.

What are the candidates for new axioms which we mentioned in ($*$)? Among the most prominent are the axiom of constructibility $V = L$, various axioms of large cardinals, forcing axioms such as Martin's axiom MA_{ω_1} (with ¬CH) or Proper forcing axiom PFA, or the Axiom of Determinacy, AD. We review some of them briefly in Section *Background*.

Let provide some illustration for ($*$) using the Whitehead and Kaplansky problems. *Whitehead problem* related to the characterization of free abelian groups has been formulated by Whitehead in the 1950s: Is every abelian group A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ free (see [8] for more details)? Stein answered the question affirmatively for countable groups, but progress for uncountable groups was slow. It was completely unexpected when Shelah proved in 1970s [33] that the problem is independent over ZFC for groups of size ω_1. Shelah proved that the affirmative answer follows from $V = L$, while the negative one follows from $\mathsf{MA}_{\omega_1} + \neg\mathsf{CH}$. Later on he also showed in [34, 35] that CH alone is not sufficient for the affirmative answer, so the assumption $V = L$ cannot be non-trivially weakened for the affirmative answer. *Kaplansky's conjecture* was formulated in 1948: Kaplansky conjectured that any Banach algebra homomorphism from $C(X)$, for a non-empty compact Hausdorff space X, into any other Banach algebra is necessarily continuous (and thus the notion of continuity – which depends on the norm – is reduced to purely *algebraic* properties of $C(X)$). In the second half of 1970's, in a series of works Dales, Esterle, Solovay and Woodin showed that this problem is independent over ZFC (see for instance [6]): CH implies Kaplansky's conjecture is false, while by forcing, one can show that it is consistent that Kaplansky's conjecture is true. Notice that the consistency of the Kaplansky's conjecture was ascertained only by a forcing construction, which is weaker than having it as a consequence of a (reasonable) forcing axiom.[4] Only

[3] For instance Shelah's pcf theory [36] or the notation of singular compactness [33] related to generalizations of Whitehead's problem which we discuss later on.

[4] To make this point more precise: φ being consistent by forcing is in general weaker than having

later in 1980s Todorcevic noticed that Kaplansky's conjecture follows from PFA, see [38].[5] The axioms $V = L$, and PFA respectively, thus settle both conjectures: It will be a part of this proposal to investigate which of these answers is more acceptable, in various aspects, and can therefore be used as an argument for the respective axioms.

Even if there have been examples of usefulness of new axioms, as we reviewed in the previous paragraph, the acceptance of them by wider mathematical community has been slow at best, in contrast to the story of AC several decades earlier. It has often been suggested, see [24, 25] or from a different perspective [4], that the main reason for the acceptance of AC is that it is *self-evident*, and so are its consequences (with the notable exception of non-Lebesgue measurability which, however, has been eventually viewed as intuitively acceptable as well). But this explanation has been questioned, see for instance [29] and [12], with arguments that the self-evidence came only after the fact, through the appreciation and evaluation of the consequence of AC which have been seen as global (i.e. connecting otherwise disparate fields of mathematics), structural (i.e. providing a structure to the objects under consideration) and rich in consequences.

In our project we propose to investigate mathematical consequences of axioms suggested as possible extensions of ZFC and evaluate their global and structural effect in mathematics.[6] We aim to carry out this analysis both from the philosophical/historical and mathematical perspective, and wherever possible identify areas where set theory might be useful in solving open problems, or argue that – at least at the moment – such applications seem unlikely in certain fields, and identify reasons why this is the case.

1.2 Background

Let us briefly review the most widely known principles which may be considered as candidates for new axioms.

(a) **Forcing axioms.** Forcing axioms are formulated to exploit the universality and the power of forcing without doing the forcing construction. In a rough analogy, they are similar to Zorn's lemma which is equivalent to AC, but "keeps under the hood" the details of the transfinite recursion which is typically used to derive consequences from AC. Forcing axioms are not equivalent to forcing in a broad sense (see Footnote 4), but imply many statements which can be shown consistent by forcing. The most widely known are: $\mathsf{MA}_{\omega_1} + \neg\mathsf{CH}$ which claims that whenever $(P, \leq)$ is a partially ordered set which has no uncountable antichains (we say that P satisfies the *countable chain condition, ccc*) and $\mathcal{D}$ is a collection of ω_1-many dense sets in P, then there is a filter $G \subseteq P$ which meets every $D \in \mathcal{D}$.[7] We observed above that $\mathsf{MA}_{\omega_1} + \neg\mathsf{CH}$ implies the negative answer to the Whitehead's problem. If we allow more partial orders $(P, \leq)$, we get stronger forcing axioms. The definition of $(P, \leq)$ being *proper* is outside the scope of this proposal, see for instance [19] for details;

it as a consequence of a forcing axiom. The statement $2^\omega = \aleph_{\alpha+1}$ can be forced to be true for every ordinal α, so clearly a single axiom cannot have the same consequences. Some of these equations are actually consequences of additional axioms: for instance, PFA implies $2^\omega = \aleph_2$. Note also that CH is not really interesting in this regard because it is equivalent to $2^\omega = \aleph_1$, so it does not bring any further insights (while PFA has many other consequences).

[5] This is interesting because PFA has much larger consistency strength; compare with the discussion of large cardinal axioms in Aim (2c).

[6] For instance, the success of $V = L$ and $\mathsf{MA}_{\omega_1} + \neg\mathsf{CH}$ in deciding the Whitehead problem in abelian group theory is unquestionable, but did it lead, or does it have a potential to lead to more interesting consequences (both in terms of methods and the subject matter)?

[7] $A \subseteq P$ is an antichain if for all $a \neq b \in A$, there is no $p \in P$ with $p \leq a, b$. $D \subseteq P$ is dense if for all $p \in P$ there is some $d \leq p$ in D.

PFA is analogous to $\mathsf{MA}_{\omega_1} + \neg\mathsf{CH}$ with *ccc* replaced by *proper*. PFA is very powerful statements which for instance implies the Kaplansky's conjecture, see [38].

(b) **$V = L$, Axiom of Determinacy in L-like models.** $V = L$ claims that the set-theoretic universe V is equal to the iterative union of initial segments L_α in which $L_{\alpha+1}$ is the collection of *definable* subsets of L_α. The definability restriction ensures that the *constructible universe* L has a very complex structure. L was defined by Gödel in the early 1930's in order to show that AC and CH are both consistent with respect to ZF and ZFC, respectively. Jensen [20] analyzed $V = L$ carefully in 1970's and identified many consequences of $V = L$ which are often used instead of $V = L$ to show that statements are independent: for instance $\diamondsuit_{\omega_1}$ (which suffices to show that there exists the so called Suslin line), $\diamondsuit^*_{\omega_1}$ (which is sufficient for the affirmative answer to the Whitehead problem), $\square_{\omega_1}$, and many other. $V = L$ is "practically" complete in the sense that the usual methods for showing independence do not work for $V = L$: every non-trivial forcing construction yields a model of $V \neq L$, and so only autoreferential sentences are known to be independent over $\mathsf{ZFC} + V = L$. However, for many reasons – some mathematical, some more philosophical – $V = L$ is not widely considered to be a good candidate for an axiom. Without going into too many details, $V = L$ it is often viewed as being too restrictive (it for instance disproves the existence of measurable cardinals, and is seen as giving too many ill-behaved subsets of the reals), so extensions of $V = L$ have been considered which contain more sets (see for instance $L[U]$ and $L[E]$ in [21]). Sometimes these extension are considered in the context of the *axiom of determinacy*, AD,[8] which claims that certain infinite games on the reals have winning strategies. Having $V = L(\mathbb{R})$ (least inner model with all the reals) with the additional assumption of AD, has often be proposed to be a good candidate for a new axiom (see [39, 40] for a popular discussion).

(c) **Large cardinals.** Large cardinals are regular uncountable cardinals for which we postulate some properties which hold on ω (if we assume AC). Typical examples are the following: (a) strong limitness and regularity, which leads to the notion of an *inaccessible cardinal*, (b) the compactness theorem for the classical logic $L_{\omega,\omega}$ generalized to an uncountable $\kappa > \omega$ and infinitary logic $L_{\kappa,\kappa}$, leading to the notions of *weakly compact* and *strongly compact cardinals* (depending on the size of the underlying alphabet), with *supercompact cardinals* being a version of this property, (c) Ramsey theorem on partitions leading again to weakly compact cardinals and also to *Ramsey cardinals*, (d) the possibility to extend κ-complete filters to κ-complete ultrafilters which leads to *measurable cardinals* and also to *strongly compact cardinals*, etc.

The underlying assumption is that by having these principles, set theory can decide more interesting mathematical statements. While these hopes failed for CH, in a more structured way Gödel's hope from [11] about solving CH did materialize: for instance the existence of a supercompact cardinal implies that all projective subsets of the reals are Lebesgue measurable, while $V = L$ implies that there is a Σ^1_2-set which is not Lebesgue measurable. Observe that while a supercompact implies projective measurability as a consequence, from the consistency point of view only one inaccessible cardinal is sufficient (see [32]).

[8] AD can also be considered by itself, but since it contradicts AC, it is often viewed in the context of the model $L(\mathbb{R})$ which violates AC under AD, but AC can still hold in V. There are weakenings of AD, such as the *Projective Determinacy*, which are consequences of large cardinals and consistent with AC.

1.3 Aims of the project

The project will evaluate the role and prospects of set theory in modern mathematics, both from the philosophical/ historical and mathematical perspectives, and will (i) identify areas where set theory has potential to resolve open problems or potential to open new approaches to solving them, (ii) identify reasons why such applications seem unlikely at the moment at specific areas and explore the potential of alternative foundational frameworks such as category theory or type theory.

Remark. For the purposes of this project we mean by "modern mathematics" results and methods which emerged after the stabilization of the axiomatic system ZFC, when the role of AC has been clarified, and also after the discovery and development of forcing. As such, modern mathematics is more aware of the boundaries and restrictions set by the independence results in set theory (and also in logic).

Summary. The project will last 3 years.

The following participants will work on the project:

Radek Honzik (PI), PhD student 1 (focused on history and philosophy of mathematics), PhD student 2 (focused on mathematics and set theory). See Section *Human resources and institutional support* for more details regarding the proposed members of the grant.

Aims

(1) Summarize and analyze applications of more advanced set-theoretical techniques and results in modern mathematics. The typical areas which are liable to independence phenomenon due to their sensitivity to underlying set theory are: (a) abelian group theory and its generalizations (for instance to modules) stemming from Shelah's work on Whitehead conjecture [8] and singular compactness [9]; (b) functional analysis, with problems such as Kaplansky's conjecture [6], or existence of the outer automorphisms of the Calkin algebra [10] or [31]; (c) abstract algebra and homology, see for instance [30]; (d) measure theory, see for instance [14] or more contextual [13]; (e) general topology and analysis, in particular regularity properties of subsets of reals, see for instance [22], and [21] or [19] for global reference.

The survey may be rather extensive, and we plan to publish in at least two parts, divided by topics. In this survey we will focus both the technical details of the applications and the set-theoretical techniques used, and on the historical aspect, identifying the relative value of the problem to the community. The survey will also focus on subsequent development of the topics and whether the answer to the given problem led to further development and progress in the area. As we already observed, many results showed to be *independent* over ZFC are actually *consequences* of axioms such as PFA or $V = L$; we expect that many independence results can actually be subsumed in this survey and identified as consequence some known new axioms.

The research reasons for such a survey are as follows:

(a) The results in this area are scattered and no uniform survey of these applications exists. By providing a careful survey, we expect to see patterns emerge which would indicate which set-theoretical methods and axioms have more extensive consequences. This will allows us to focus on these in further work, and put some structure in their use.

(b) By identifying relevant patterns, we can isolate areas and questions which are solvable by set-theoretical methods and prove more general theorems, or apply the techniques in related subject areas.

(2) We will investigate and catalogue the main differences in use and reception of these assumptions. The focus will be a combination of mathematical and historical and philosophical perspective. For the purposes of this group we can divide the principles into several groups, which we used already in Section *Background*:

(a) **Forcing axioms.** Forcing axioms tend to decide globally many independent statements, in the opposite way than $V = L$. They can be regarded as a non-trivial strengthening of $V \neq L$ or $\neg\mathsf{CH}$, by having a rich structural content. It is relevant for our project that forcing axioms come in different forms, and have different consistency strength: while $\mathsf{MA}_{\omega_1} + \neg\mathsf{CH}$ has the consistency strength of ZFC, PFA has a very high consistency strength on the level of supercompact cardinals. In this way, forcing axioms are naturally connected to large cardinal axioms in item (2c).

(b) $V = L$**, Axiom of determinacy in** L**-like models.** The analysis of principles which are consequences of $V = L$, or more general models of the form $V = L[E]$, or $V = L(\mathbb{R})$ with AD, tends to be rather technical. These assumptions are often considered more as the source of counterexamples, than genuine candidates for new axioms (see [12], [24, 25], [37], and [39, 40]). We will analyse these objections carefully, with focus on the (supposedly) counterintuitive results in mathematics (for instance for the affirmative solution to the Whitehead problem).

(c) **Large cardinal axioms.** A special case in point is the role of large cardinals. They are easy to formulate, but they are looked at by some with scepticism for *philosophical reasons* as not being sufficiently well-motivated, while some other accept them for the same reasons (see [17]). Also based on the survey in Aim (1), the role of large cardinals in mathematics deserves a detailed treatment, with evaluations both as regards the technical contents involved and historical and philosophical considerations. For instance, the use of inaccessible cardinals in the proof of Fermat's last theorem, see for instance [28], has often been dismissed as a non-essential feature of the proof. While this may the case, the use of them may be interpreted as providing an easier path to results provable without them, which is reminiscent of Hilbert's discussion of "ideal mathematics" (see [18], [41]).

As a sub-aim in this area we will investigate and provide an up-to-date account of the original hopes of Gödel, see [11], that large cardinal axioms can decide many (interesting) independent question, such as CH. This naturally leads to a technical inquiry as to the nature of "mathematically relevant questions"; for instance Shelah argues [37] that CH is *not* an interesting question – information we learned only after extensive set theoretical research (compare also with Hamkin's position in [16]). Other, more rigid principles, has been proposed instead of CH. We aim to analyze them and put into context of the original Gödel's suggestion.

Note that from the philosophical perspective it is instructive to compare the reception of AC and of large cardinals: as we observed, large cardinals can be often seen as a generalized version of AC in which we postulate some extra properties (such as "extending a filter to an ultrafilter" is generalized to extending a "κ-complete filter into a κ-complete ultrafilter"); in both cases the non-constructive nature of the assumption has been seen as a problem from the beginning, even for AC: from philosophical as in Brouwer's [5] to more mathematical from Baire, Borel and Lebesgue (see for instance [29] for more details).

(3) We will critically compare alternative foundational frameworks for mathematics in the sense of their potential for identifying and solving open problems in mathematics. We will focus primarily on *category theory* and *type theory*. While these alternative frameworks are not the primary concern of this project, we feel that without a thorough comparison of philosophical foundations and mathematical consequences, the role of set theory could not be ascertained properly.

   (a) Category theory has its foundations in 1950s in works of Eilenberg and Mac Lane in algebraic topology and instead of sets and elementship, it uses the concepts of homomorphisms and their composition. It offers an easier way of representing certain theorems and generalizing them in algebra-oriented fields, but it is often less suitable in more abstract settings. We will start by considering surveys as in [27, 26] or the extensive treatment in [23] and move to discussing the potential for discovering and solving open problems in mathematics.

   (b) Type theory, and especially homotopy type theory and univalent foundations of mathematics due to Voevodsky, has been proposed by some as an alternative framework for foundations of mathematics, and has been discussed extensively recently in certain areas. Originating from a rather narrow field, it has laid claims for relevance both in terms of mathematics and philosophy. We will critically evaluate this framework, in terms of its actual potential for solving open problems in mathematics. See [7] or [3] for some examples.

(4) Based on the survey and analysis in Aim (1) and the research in Aim (2), as refined with comparison with the results of Aim (3), we will identify set-theoretical principles and axioms which has proved to be most efficient in current mathematics and discuss them both in terms of their technical contents and consequences, and as regards their intuitive acceptability.

   We plan to do the following:

   (a) We will further develop and refine these set-theoretical axioms whenever possible and look for new areas of application. These areas are expected to be identified as a result of the analysis in Aim (1).

   (b) We will put the axioms identified in Aims (1) and (2) into the proper historical and philosophical context, with the aim of providing common background for enhancing the transfer of methods and results between set theory and general mathematics (in both directions). Based on results in Aim (3), we will also include alternative frameworks, such as category theory and type theory, in this discussion and indicate whether and where they could have additional benefits.

2 Procedural aspects

2.1 Methods and techniques

We will carefully analyze results and set-theoretical methods and assumptions identified in results in Aim (1). The analysis will be carried out both from the point of set theory and mathematics, and from the point of philosophical considerations (see Aim 2). We will identify the areas where set theory, and possibly other foundational frameworks, have potential to provide new insights, based on results in Aim (4).

Beyond that, we will apply the standard methods of pure mathematics and philosophy – reading new results in papers, and applying them to questions under consideration, discussion with fellow colleagues, raising new hypotheses, refuting them, modifying them, etc.

2.2 Human resources and institutional support

The project proposes to hire the following researchers for the duration of the project (3 years) (capacity 1.0 means 100% contract, i.e. 40 working hours per week):

- Radek Honzik, the leader of the project (PI), capacity 0.7. The capacity is set to 0.7 because I also have some duties at the university where I teach.
- PhD student 1, capacity 0.5. PhD student with the primarily focus on philosophy and history of modern mathematics. The capacity is 0.5 because I expect that the student will have some other duties as well (teaching, exams, etc.).
- PhD student 2, capacity 0.5. PhD student with the primary focus on mathematics and set theory. The capacity is 0.5 because I expect that the student will have some other duties as well (teaching, exams, etc.).

Radek Honzik works as a regular professor at the Faculty of Arts; his eduction is both mathematical and philosophical, his primary research focus being mostly set-theoretical. It is expected he will be able to lead and coordinate the research in both areas. The PhD students will be carefully chosen from the pool of students available when the grant starts. One student will be chosen with primary focus on philosophy and history to provide more focus and expertise in this area, the other student with focus on mathematics and set theory. It is expected that joint cooperation of these three members of the team across fields will provide solid foundation for success of the project.

Apart from people directly hired by the project, we plan to have research visits and invite researchers to Prague to actively engage internationally: among the good candidates for visits are: Andrey Brooke Taylor, Joan Bagaria, Neil Barton, J. D. Hamkins, Philipp Lucke, Manachem Magidor.

The institutional support will be provided by Faculty of Arts and in particular the Department of Logic, the home department of Radek Honzik. It contains all necessary equipment and space to host the members of the project team.

2.3 Work plan, strategies for dissemination of results

We propose the following tentative schedule for the results:

Year 1. We will primarily work on Aims (1) and (2).

Year 2. We will finish our work on Aim (1). We will continue on Aims (2) and will start Aim (3).

Year 3. We will consolidate and finalize results obtained in Years 1-2, and will finalize the project by working on Aim (4).

With regard to a contingency plan, we have formulated the aims with flexibility in mind: while we expect that there is potential for set theory in resolving or clarifying problems in mathematics, we are aware that there are mathematical and philosophical considerations which must be addressed: that is why we also propose to discuss the philosophical and historical background and also evaluate the potential of other foundational frameworks.

Strategies for dissemination of results will include research visits in universities and research centers specialised in philosophy of set theory and set theory, talks given in seminars, workshops and conferences (such as: Logica, European Summer School in logic, language and information, Logic Colloquium, and other international events), and the preparation of several papers which will collect the results obtained in the project and will be sent for publication in recognised international journals, like The Review of Symbolic Logic, The Bulletin of Symbolic Logic, Synthese, The Journal of Philosophical Logic, Philosophia Mathematica, Annals of Pure and Applied Logic, The Journal of Symbolic Logic. We will use the fact that the proposer works at a university (home institution Logic Department at the Faculty of Arts, but he also regularly teaches set theory at the Faculty of Mathematics and Physics) and have an active and rich contact with young students and researchers: we plan to popularize our results within communities of students and early-career researchers and attract their attention to the topic.

2.4 HR Award certificate

The Department of Logic of the Charles university, the working place of Radek Honzik and the seat of the grant, will provide full institutional and administrative support for the grant, including necessary infrastructure such as computers, journal subscriptions and etc. In particular, the department will provide expertise and help in managing grant projects, to make sure that the project runs smoothly with regard to administrative processes.

With regard to equal opportunities, Charles university has obtained an HR Award in 2019 for the development of human potential, and the department's employment policy for the grant is therefore guaranteed to be in full compliance with established standards for the support of Equal Opportunities in the academic and research environment.[9]

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