Department of Logic FFUK

Introduction to Boolean alegbras

Radek Honzík

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Contents

1 Introduction

A Boolean algebra B is a structure over a language L which consists of two constants, denoted 0 and 1, one unary operation of a complement $-$, and two binary operations of meet $\wedge$ and join $\vee$. Using the properties of the binary operations, it is possible to define an ordering $\leq$ on B in which 0 is the least element and 1 the greatest element. The ordering $\leq$ is called a lattice, which means that every two elements of B have the supremum and infimum in $\leq$.

Unlike groups and rings which are motivated by the algebraic properties of the operations $+$ and $\times$ on structures such as $\mathbb{Z}$ or $\mathbb{R}$, Boolean algebras are based on the properties of the classical propositional logic and the set-theoretical operations of intersection, union, and complement.

We will start by introducing the notion of a lattice, distributive lattice, and Boolean algebra. After verifying some basic properties, we review the most familiar examples of Boolean algebras (for instance the clopen algebra of the Cantor space or Lindenbaum-Tarski algebras corresponding to first-order theories). We show how to generalize the binary operations to infinite setting and use these notions to define a complete Boolean algebra. Next we move to representation theorems: we show that up to isomorphism all Boolean algebras are algebras of sets, moreover determined by a simple topology (Stone's duality). We finish by showing that all atomless countable Boolean algebras are isomorphic – a result analogous to the famous theorem of Cantor that all countable dense linear orders without end-points are isomorphic.

Due to the limited scope of the lecture, there are necessarily important and interesting topics which will be omitted. For a more detailed discussion of Boolean algebras, we recommend [1] and [2].

1.1 Notational conventions

We use the same notation for a structure and its domain. For instance if B is a Boolean algebra, we write $B = \langle B, \wedge, \vee, -, 0, 1\rangle$. In rare cases when this convention may be confusing, we explicitly refer to an algebra B, or to the domain of an algebra B. Some special algebras are denoted by descriptions, as in $\mathrm{CO}(X, \tau)$: the clopen (closed and open) algebra of the topological space (X, τ). If we need to distinguish operations between several algebras, we add superscripts, as in $B = \langle B, \wedge^B, \vee^B, -^B, 0^B, 1^B\rangle$ and

$A = \langle A, \wedge^A, \vee^A, -^A, 0^A, 1^A \rangle$.

To avoid mixing up the algebraic join symbol $\vee$ and the disjunction (usually denoted by $\vee$ as well), we use the word "or" instead. For the conjuction, we will use &, so there is no danger of confusing $\wedge$ (meet) and &; however, for reasons of symmetry, we often use "and" instead of &.

The symbol $\Leftrightarrow$ stands for "if and only if". Theories and axioms are denoted by sans serif font, as in AC (Axiom of Choice), CH (Continuum Hypothesis), GCH (Generalized Continuum Hypothesis), T (variable for a theory), PA (Peano Arithmetics), ZF (Zermelo-Fraenkel set theory without AC), ZFC (Zermelo-Fraenkel set theory with AC). We assume throughout that ZFC is a consistent theory.

Natural numbers are denoted by ω, rational by $\mathbb{Q}$, real by $\mathbb{R}$, and irrational by $\mathbb{I}$.

Remark 1.1 Some simpler claims in the text are left without a proof. The idea is that the reader should provide missing proofs as an exercise.

2 Lattices, partial orders, and Boolean algebras

A lattice is a simple algebraic structure which corresponds to a partially ordered set where every two elements have the supremum and the infimum. We will later learn that the canonical partial order of a Boolean algebra is a lattice.

2.1 Lattices

Definition 2.1 *Let M be a nonempty set. We say that $\langle M, \leq \rangle$ is* a non-strict partial order *if $\leq$ is a reflexive, transitive and anti-symmetric relation on M. We say that $\langle M, < \rangle$ is* a strict partial order *if $<$ is an anti-reflexive and transitive relation on M.*

Often we omit the mentioning of "non-strict" and "strict" when we refer to partially ordered sets. We use the typographical convention that $\leq$ (and its variants) denote the non-strict order, while $<$ (and its variants) denote the strict order. It is easy to go from one to another:

Observation 2.2 *(i) Let $\langle M, \leq \rangle$ be a non-strict partial order. If we define for all $x, y \in M$,*

$$x < y \iff x \leq y \text{ and } x \neq y,$$

then $\langle M, < \rangle$ is a strict partial order.

(ii) Let $\langle M, < \rangle$ be a strict partial order. If we define for all $x, y \in M$,

$$x \leq y \iff x < y \text{ or } x = y,$$

then $\langle M, \leq \rangle$ is a non-strict partial order.

Proof. Exercise. □

We now introduce basic terminology concerning partial orders. Let $\langle M, \leq \rangle$ be a partial order and $X \subseteq M$. Let x, y, z range over the elements of M:

(2.1)

- x is the *least* element of $X \iff x \in X$ and for all $y \in X$, $x \leq y$.
- x is a *minimal* element of $X \iff x \in X$ and there is no $y \in X$ such that $y < x$.
- x is a *lower bound* of $X \iff$ for all $y \in X$, $x \leq y$.
- x is the *infimum* of $X \iff x$ is the greatest lower bound of X.
- x is the *greatest* element of $X \iff x \in X$ and for all $y \in X$, $x \geq y$.
- x is a *maximal* element of $X \iff x \in X$ and there is no $y \in X$ such that $y > x$.
- x is a *upper bound* of $X \iff$ for all $y \in X$, $x \geq y$.
- x is the *supremum* of $X \iff x$ is the least upper bound of X.

Note that if a supremum (infimum) exists, then it is unique.

Definition 2.3 (Algebraic definition of a lattice) *Let $\wedge$ and $\vee$ be binary functional symbols. We say that $M = \langle M, \wedge, \vee \rangle$ is a* lattice *if M satisfies the following formulas for all $x, y, z \in M$:*

L1 Associativity. $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
L2 Commutativity. $x \vee y = y \vee x$, $x \wedge y = y \wedge x$
L3 Idempotence. $x \vee x = x, x \wedge x = x$
L4 Absorption. $x \vee (x \wedge y) = x, x \wedge (x \vee y) = x$

If M is a lattice, we can define a partial order $\leq$ on M as follows

$$x \leq y \iff x \wedge y = x \iff x \vee y = y. \qquad (2.2)$$

The relation $\leq$ is called *the canonical ordering* of M, or the *associated ordering* of M.

Observation 2.4 *The relation $\leq$ in (2.2) is a partial order and is correctly defined.*

Proof. We first show that the definition of $\leq$ makes sense, ie. that $x \vee y = y$ is true if and only if $x \wedge y = x$ is true. If $x \vee y = y$, then $x \wedge y = x \wedge (x \vee y) = x$ (by absorption). Conversely, if $x \wedge y = x$, then $x \vee y = y \vee x = y \vee (x \wedge y) = y$ (again by absorption).

We now show that $\leq$ is a partial order (using the definition with $\wedge$). We show first that $\leq$ is reflexive: $x \leq x$ if and only if $x \wedge x = x$, which is true by idempotence. Transitivity: $x \leq y \leq z$ implies $x \leq z$ by the following argument using associativity: if $x \wedge y = x$ and $y \wedge z = y$, then $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$. To show that $\leq$ is anti-symmetric, we need to argue that $x \leq y$ and $y \leq x$ already implies $x = y$: this follows by commutativity $x = x \wedge y = y \wedge x = y$. $\square$

A lattice can be defined equivalently by putting additional requirements on the relation $\leq$.

Definition 2.5 (Set-theoretic definition of a lattice) *We say that $M = \langle M, \leq \rangle$ is a* lattice *if M satisfies the following formulas (axioms) for all $x, y \in M$.*

- L'1 $\leq$ is a partial order on M.
- L'2 Every pair $\{x, y\}$ of elements in M has a supremum and infimum in M. We denote this supremum and infimum as $\sup(x, y)$ and $\inf(x, y)$.

One can show that there is a one-to-one correspondence between the algebraic and set-theoretic definitions of a lattice: if $\langle M, \wedge, \vee \rangle$ is a lattice, then the canonical order $\leq$ yields a lattice $\langle M, \leq \rangle$; conversely, if $\langle M, \leq \rangle$ is a lattice and we define for all x, y in M, $x \wedge y = \inf(x, y)$ and $x \vee y = \sup(x, y)$, then $\langle M, \wedge, \vee \rangle$ is a lattice (the inequivalances (2.4) introduced in the next section are useful for this).

Examples: While all linearly ordered sets are lattices, the notion of a lattice is more interesting with non-linearly ordered sets. An important example of a lattice is the powerset of a non-empty set X with the inclusion as an ordering: $\langle \mathscr{P}(X), \subseteq \rangle$, or equivalently with the intersection and union as the binary operations: $\langle \mathscr{P}(X), \cap, \cup \rangle$. If Y is a subset of $\mathscr{P}(X)$, which is closed under intersection and union (i.e. for all $a, b \in Y, a \cap b \in Y$ and $a \cup b \in Y$), then also $\langle Y, \subseteq \rangle$ is a lattice. Perhaps a less familiar example of a lattice is the natural numbers with the divisibility relation $|$: $\langle \omega, | \rangle$ (for $m, n \in \omega$, $m|n \Leftrightarrow (\exists k \in \omega) mk = n$). It is easy to check that the corresponding binary operations for $\langle \omega, | \rangle$ are the least common multiplier and the greatest common factor.

2.2 Distributive lattices

Sometimes it is useful to add the following axioms to the axioms of a lattice:

(2.3) $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

We call these axioms (2.3) *axioms of distributivity.*

Observation 2.6 *Every lattice satisfies the following formulas, for all x, y, z:*

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \text{ and } (x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

Proof. We first show that each lattice satisfies the following formulas:

(2.4)

- $(\forall x, y, z)(x \leq z \;\&\; y \leq z) \to x \vee y \leq z$,
- $(\forall x, y, z)(x \leq y \;\&\; x \leq z) \to x \leq y \wedge z$.

To prove the first formula in (2.4), we need to show that $x \vee y \vee z = z$; using the assumptions we have: $z = z \vee z = x \vee z \vee y \vee z = x \vee y \vee z$ as required. Note that we have used the definition $x \leq y \Leftrightarrow x \vee y = y$. The proof of the second formula is analogous.

We will prove just the first formula $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ (the proof of the second formula is analogous). As its antecedent is connected by the operation $\vee$, it is enough to show separately that $x \leq (x \vee y) \wedge (x \vee z)$ and $y \wedge z \leq (x \vee y) \wedge (x \vee z)$. However, to prove $x \leq (x \vee y) \wedge (x \vee z)$, it is again by (2.4) sufficient to show $x \leq x \vee y$ and $x \leq x \vee z$, but this is immediate. Similarly for $y \wedge z$: $y \wedge z \leq y \leq x \vee y$, and $y \wedge z \leq z \leq x \vee z$. □

However, the full distributivity in (2.3) does not always hold. For instance, the following lattice violates distributivity:

u

a b c

0

Indeed: $a \vee (b \wedge c) = a \vee 0 = a$, while $(a \vee b) \wedge (a \vee c) = u \wedge u = u$.

We now show that the two formulas in (2.3) are equivalent over lattices, i.e. if we assume distributivity for $\vee$, we can prove it for $\wedge$, and conversely.

Observation 2.7 *In every lattice the following are equivalent:*

(i) $(\forall x, y, z)[x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)]$,
(ii) $(\forall x, y, z)[x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)]$.

Proof. Assume (i) is true. We want to show that (ii) is true as well. By Observation 2.6, it suffices to show that

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z). \tag{2.5}$$

Since we can assume distributivity for $\vee$, we use it to obtain $(x \wedge y) \vee (x \wedge z) = [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z)]$. By (2.4) and another appeal to distributivity for $\vee$ and to absorption, to show (2.5), it suffices to show (a) $x \wedge (y \vee z) \leq (x \wedge y) \vee x = x$ and (b) $x \wedge (y \vee z) \leq (x \wedge y) \vee z = (z \vee x) \wedge (z \vee y)$. The inequality (a) is obvious. As regards the inequality (b), applying (2.4) again, it suffices to show that $x \wedge (y \vee z) \leq z \vee x$ and $x \wedge (y \vee z) \leq z \vee y$. The first inequality holds because $x \wedge (y \vee z) \leq x \leq z \vee x$, and the second is again obvious. The converse is proved analogously. □

2.3 Boolean algebras

Definition 2.8 *Let* $\wedge, \vee$ *be binary functional symbols,* $-$ *a unary functional symbol, and* $0, 1$ *two constants. A Boolean algebra is a structure*

$$B = \langle B, \wedge, \vee, -, 0, 1 \rangle,$$

where $\wedge$ *and* $\vee$ *are binary functions from* B^2 *to* B*,* $-$ *is a unary function from* B *to* B*, and* $0, 1$ *are two elements of* B*.* B *satisfies the following formulas for all* $x, y, z \in B$*:*

B1 Associativity of $\vee, \wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$.

B2 Commutativity of $\vee, \wedge$: $x \wedge y = y \wedge x$, $x \vee y = y \vee x$.
B3 Absorption. $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$.
B4 Distributivity of $\vee, \wedge$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
B5 Complement. $x \vee (-x) = 1$, $x \wedge (-x) = 0$.

Remark 2.9 A one-element structure with $0 = 1$ is called a trivial Boolean algebra. Depending on our choice, we might postulate that such a structure will not be considered: just add a new axiom $0 \neq 1$.

Notice that every Boolean algebra is a lattice (axioms B1–B3). This allows us to define the *canonical ordering* of a Boolean algebra by

$$x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y. \tag{2.6}$$

Also note that for all $x \in B$,

$$0 \leq x \leq 1,$$

i.e. 0 is the least element and 1 the greatest element in the ordering $\leq$. To see this, recall that $0 \leq x$ is equivalent to $0 \wedge x = 0$, and $0 = x \wedge (-x) = x \wedge x \wedge (-x) = x \wedge 0$. The inequality $x \leq 1$ follows similarly, using the axiom $x \vee (-x) = 1$.

Definition 2.10 *A structure* $\langle M, \wedge, \vee, 0, 1\rangle$ *is called* a complemented lattice *if it is a lattice with the least element 0 and greatest element 1 such that for every* $x \in M$ *there exists in* M *a complement* x' *satisfying* $x \vee x' = 1$ *and* $x \wedge x' = 0$.

By the above definition, a Boolean algebra is a distributive complemented lattice. We show that distributivity implies the uniqueness of complement.

Observation 2.11 *If* B *is a Boolean algebra then* $-x$ *is the unique complement of* x.

Proof. Assume that y is a complement of x, i.e. $x \wedge y = 0$ and $x \vee y = 1$. We want to show that $-x = y$.

$$\begin{aligned} -x &= -x \wedge (x \vee y) \\ &= (-x \wedge x) \vee (-x \wedge y) \\ &= 0 \vee (-x \wedge y) \\ &= -x \wedge y \end{aligned}$$

The last line is equivalent to $-x \leq y$. The converse $y \leq -x$ is shown by starting with $-x = -x \vee (x \wedge y)$. □

Now we show some more properties of the operations in Boolean algebras.

Observation 2.12 *The following formulas are true in every Boolean algebra:*

(i) $- - x = x$,
(ii) $-x = -y \rightarrow x = y$,
(iii) *(de Morgan's laws)* $-(x \vee y) = -x \wedge -y, -(x \wedge y) = -x \vee -y$.

Proof. (i) By uniqueness of complement, axiom B5 claims that x is the complement of $-x$.

(ii) If two elements are identical, so are their complements, that is $- - x = - - y$, which by (i) implies $x = y$.

(iii) We argue for the first formula, the second one follows analogously. We show that $-x \wedge -y$ is the complement of $x \vee y$. To this effect it is enough to show that $(-x \wedge -y) \wedge (x \vee y) = 0$ and $(-x \wedge -y) \vee (x \vee y) = 1$. But this follows immediately by distributive laws. □

Now we show how the operations interact with the partial order.

Observation 2.13 *Following formulas hold in all Boolean algebras.*

(i) $x_0 \leq x_1 \;\&\; y_0 \leq y_1 \rightarrow x_0 \vee y_0 \leq x_1 \vee y_1$,
(ii) $x_0 \leq x_1 \;\&\; y_0 \leq y_1 \rightarrow x_0 \wedge y_0 \leq x_1 \wedge y_1$,
(iii) $x_0 \leq x_1 \leftrightarrow -x_1 \leq -x_0$.

Proof. (i)-(ii). Follows by straightforward manipulation. (iii) Follows by de Morgan's laws: assuming $x_0 \vee x_1 = x_1$, we argue that $-x_1 \wedge -x_0 = -(x_1 \vee x_0) = -x_1$. □

2.4 Subalgebras and homomorphisms

Definition 2.14 *Let* $B = \langle B, \wedge^B, \vee^B, -^B, 0^B, 1^B \rangle$ *be a Boolean algebra and let* A *be a subset of* B *which satisfies the following:*

(i) $0^B \in A, 1^B \in A$,
(ii) *For all* $a, b \in A$, $-^B a \in A$, $a \wedge^B b \in A$, *and* $a \vee^B b \in A$.

Then the structure $A = \langle A, \wedge^A, \vee^A, -^A, 0^A, 1^A \rangle$ *is called a* subalgebra *of* B*, where* $0^A = 0^B, 1^A = 1^B$*, and* $\wedge^A$ *is the restriction of* $\wedge^B$ *to* A*, and similarly for* $\vee^A$ *and* $-^A$*.*

Observation 2.15 *Show that if* A *and* B *are as above, then* A *is a Boolean algebra.*

Proof. Exercise. □

Definition 2.16 *Let*

$$A = \langle A, \wedge^A, \vee^A, -^A, 0^A, 1^A \rangle$$

and

$$B = \langle B, \wedge^B, \vee^B, -^B, 0^B, 1^B \rangle$$

be Boolean algebras. Let $f : A \to B$ *be a function.* f *is called a* homomorphism *if the following hold for all* $x, y \in A$*:*

(i) $f(0^A) = 0^B$ *and* $f(1^A) = 1^B$*.*
(ii) $f(x \wedge^A y) = f(x) \wedge^B f(y)$*,* $f(x \vee^A y) = f(x) \vee^B f(y)$*, and* $f(-^A x) = -^B f(x)$*.*

If f *is injective, then* f *is called an* embedding*. If* f *is injective and onto, then* f *is called an* isomorphism*.*

If f is a homomorphism from an algebra A to an algebra B, we write $f : A \to B$. An isomorphism is denoted by $f : A \cong B$.

Observation 2.17 *Assume* $f : A \to B$ *is a homomorphism. Then the following hold:*

(i) $f[A] = \{b \in B \mid (\exists a \in A)(f(a) = b)\}$ *is closed under the constants and operations in* B *and therefore determines a subalgebra of* B*, which we denote* $f[A]$*.*
(ii) If $f : A \to B$ *is an embedding,* $f[A]$ *is isomorphic to* A *via* f*.*

Proof. Exercise. □

We finish this section with a useful observation which says that a question concerning the existence of an isomorphism between two Boolean algebras can be reduced to a question concerning the existence of an isomorphism between the two canonical partial orders (the upside being that an isomorphism between two partial orders is easier to verify).

Observation 2.18 *Two Boolean algebras A and B are isomorphic as algebraic structures if and only if their canonical partial orders are isomorphic (with 0 being the least and 1 the greatest element in the order).*

Proof. We give only a sketch of proof. Observation 2.11 implies that it is enough to find an isomorphism between A and B with respect to $\wedge, \vee, 0, 1$ because the operation of complement is definable in the language $\wedge, \vee, 0, 1$. Using ideas appearing in the paragraph after Definition 2.5, one can conclude that the existence of an isomorphism with respect to $\wedge, \vee, 0, 1$ is equivalent to the existence of an isomorphism with respect to the canonical ordering $\leq$ with the condition that we map 0^A to 0^B and 1^A to 1^B. □

3 Examples of Boolean algebras

In this section, we give examples of Boolean algebras and discuss their properties. Let us note that if B is a Boolean algebra, then by the size of B we mean the size of the domain of B.

3.1 The two-element Boolean algebra

By Observation 2.18, all two-element Boolean algebras A and B are isomorphic via a function which sends 0^A to 0^B and 1^A to 1^B. In particular up to isomorphism the truth table algebra for propositional logic $\langle \{0,1\}, \wedge, \vee, \neg, 0, 1\rangle$ is the unique two-element Boolean algebra. Another representation of the same algebra is the powerset algebra of a one-element set (see the next section 3.2). We denote this unique two-element Boolean algebra by **2**, and its domain by $\{0,1\}$.

3.2 Powerset algebras

Let X be a set and $\mathscr{P}(X)$ its powerset. Then the algebra

$$\mathscr{P}(X) = \langle \mathscr{P}(X), \cup, \cap, -, \emptyset, X\rangle$$

is a Boolean algebra, with 0 being interpreted by $\emptyset$ and 1 by X.

The size of a powerset algebra is determined by the size of X: if the size of X is κ (κ can be a finite or infinite cardinal number), then

$$|\mathscr{P}(X)| = 2^\kappa.$$

We will later show that every finite Boolean algebra is isomorphic to a powerset algebra for some finite X; in particular, a finite Boolean algebra has size 2^n for some $n \in \omega$. However, it is not true that every infinite Boolean algebra is isomorphic to a powerset algebra: for instance in Section 3.3.1 we define a countable Boolean algebra which cannot be a powerset algebra because there is no cardinal number κ such that $2^\kappa = \omega$.

3.3 Algebras of sets

Let X be a non-empty set and let $Y \subseteq \mathscr{P}(X)$ be closed under the operations of the powerset algebra $\mathscr{P}(X)$. Then

$$\langle Y, \cup, \cap, -, \emptyset, X\rangle$$

is a subalgebra of $\mathscr{P}(X)$ and it is a Boolean algebra (see Definition 2.14).

Definition 3.1 *Let X, Y be as above. Then we call $\langle Y, \cup, \cap, -, \emptyset, X\rangle$ an* algebra of sets (on X).

It is worth noting that the canonical ordering on an algebra of sets is the inclusion relation $\subseteq$. Also note that a powerset algebra is a special case of an algebra of sets. We shall prove later in Theorem 5.14 that up to isomorphism, all Boolean algebras are algebras of sets.

We give some examples of algebras of sets below in Sections 3.3.1, 3.4.1, 3.4.2, and 3.4.3.

3.3.1 Finite-cofinite algebras

Let X be a non-empty set. We call $a \subseteq X$ *cofinite* in X if $X \setminus a$ is finite. Let

$$B = \{a \subseteq X \mid a \text{ is finite or cofinite}\}. \tag{3.7}$$

Then $\langle B, \cup, \cap, -, \emptyset, X \rangle$ is an algebra of sets on X because B is closed under the operations of the powerset algebra $\mathscr{P}(X)$. (Exercise.)

Lemma 3.2 *Assume X is a set of size $\kappa \geq \omega$. Then the finite-cofinite algebra B defined in (3.7) has size κ.*

Proof. The set of all finite subsets of an infinite κ has size κ; similarly, the set of all complements of finite subsets has size κ. Since the union of two infinite sets of size κ has size κ, we are done. □

3.4 Topological constructions

Let X be a set. We call a subsystem $\tau \subseteq \mathscr{P}(X)$ a *topology* on X if τ satisfies the following properties:

(i) $X, \emptyset \in \tau$,
(ii) If $A, B \in \tau$ then also $A \cap B \in \tau$,
(iii) If $\mathscr{A}$ be a subset of τ, then $\bigcup \mathscr{A}$ is in τ.

The pair (X, τ) is called a *topological space*. The sets in τ are called *open*; if A is open, then $X \setminus A$ is called *closed*. A system $\mathscr{B} \subseteq \mathscr{P}(X)$ is called a *basis* of τ if $\mathscr{B} \subseteq \tau$ and every $A \in \tau$ can be expressed as a union of elements in $\mathscr{B}$.

The system of open sets together with operations $\cup, \cap, -$ does not necessarily form an algebra of sets because open sets are generally not closed under the complement operation. However, we can use a topological space to define a Boolean algebra if we restrict our attention to open sets which are both open and closed:

Definition 3.3 *If (X, τ) is a topological space, we call $A \subseteq X$ a* clopen *set if both A and $X \setminus A$ are open. Let $\mathrm{CO}(X, \tau)$ denote the system of clopen subsets of (X, τ).*

Observation 3.4 *If (X, τ) is a topological space, then the system of clopen subsets of X forms a Boolean algebra of sets which we denote $\mathrm{CO}(X, \tau) = \langle \mathrm{CO}(X, \tau), \cup, \cap, -, \emptyset, X \rangle$.*

Proof. By the definition of a topological space, $\emptyset$ and X are clopen. It remains to verify that $\mathrm{CO}(X, \tau)$ is closed under operations $-, \cap, \cup$. Let A, B

be clopen. By definition of being clopen, $-A$ is clopen. To show that $A \cup B$ is clopen, we need to show that it is both open and closed. $A \cup B$ is open because A and B are open and hence their union is open. $A \cup B$ is also closed because $-(A \cup B) = -A \cap -B$ is open since A, B are closed. Closure under intersection follows by de Morgan's rules. □

While the definition of the clopen algebra seems somewhat special, we will learn that every Boolean algebra is up to isomorphism a clopen algebra of some topological space (see Theorem 5.16).

We get some more familiarity with this notion in the following sections where we introduce the standard topologies on the real numbers, irrational numbers and the Cantor space.

3.4.1 Real numbers

Now we define the standard topology of the real line $\mathbb{R}$.

Definition 3.5 (Topology of the real line $\tau_{\mathbb{R}}$) *A set $A \subseteq \mathbb{R}$ is in $\tau_{\mathbb{R}}$ if and only if for every $x \in A$ there is an interval (a, b) of real numbers $a < b$ such that $x \in (a, b)$ and $(a, b) \subseteq A$.*

Observation 3.6 *The system $\tau_{\mathbb{R}}$ is a topology on $\mathbb{R}$.*

Proof. Exercise. □

Observation 3.7 *Assume $\mathscr{B}_{\mathbb{R}}$ contains as elements all intervals of the form (r_0, r_1) where $r_0 < r_1$ are* rational *numbers (we call (r_0, r_1) an open interval with rational endpoints). Then $\mathscr{B}_{\mathbb{R}}$ is a basis of $\tau_{\mathbb{R}}$.*

Proof. We know that rational numbers are dense in $\mathbb{R}$: for every two real numbers $r_0 < r_1$ there is a rational number q such that $r_0 < q < r_1$. If A is open then for each $x \in A$ we can find by denseness an open interval $B_x \subseteq A$ with rational endpoint (that is $B_x \in \mathscr{B}_{\mathbb{R}}$) such that $x \in B_x$; thus $A = \bigcup_{x \in A} B_x$. □

It follows that the topology of real numbers has a *countable* basis. Moreover, since rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$ in the topological sense (every open set contains a rational number), we say that the topology of real numbers is *separable.*

Corollary 3.8 *There are $2^{\aleph_0}$ open subsets of $\mathbb{R}$, i.e. $|\tau_{\mathbb{R}}| = |\mathbb{R}|$.*

Proof. Clearly there are at least $2^{\aleph_0}$ many open sets because $(-r, r)$ is an open set for every $r \in \mathbb{R}$, $r \neq 0$.

Conversely, every open set $O \in \tau_{\mathbb{R}}$ is expressible as a union of some sets from the basis. It follows that the number of all open sets is at most the number of all subsets of the basis: if O is open, then for some $X_O \subseteq \mathscr{B}_{\mathbb{R}}$, $O = \bigcup\{B \mid B \in X_O\}$. Since the basis is countable, the number of all subsets of the basis is $2^{\aleph_0}$. It follows that there at most $2^{\aleph_0}$ open sets. □

The system $\langle \tau_{\mathbb{R}}, \cup, \cap, -, \emptyset, \mathbb{R} \rangle$ is not an algebra of sets since it is not closed under complements (for instance the complement of an open interval (a, b) is not open because there is no open interval which contains a (or b) and is included in $\mathbb{R} \setminus (a, b)$). However, we know that $\mathrm{CO}(\mathbb{R}, \tau_{\mathbb{R}})$ is a Boolean algebra. We show that it is in fact isomorphic to $\mathbf{2}$:

Observation 3.9 *The topological space $(\mathbb{R}, \tau_{\mathbb{R}})$ has only two clopen sets: $\emptyset$ and $\mathbb{R}$. The algebra $\mathrm{CO}(\mathbb{R}, \tau_{\mathbb{R}})$ is therefore isomorphic to $\mathbf{2}$.*

Proof. Assume A is open but not equal to $\emptyset$ or $\mathbb{R}$. We will show that the complement of A (denoted as $-A$) is not open. Choose some real number x which is not in A. Then either $A \cap \{r \in \mathbb{R} \mid r < x\}$ or $A \cap \{r \in \mathbb{R} \mid r > x\}$ must be non-empty. Without loss of generality assume that $A_x = A \cap \{r \in \mathbb{R} \mid r < x\}$ is non-empty. A_x is bounded from above and so has a supremum; let x_0 be this supremum. It is obvious that $x_0 \leq x$.

We argue that x_0 cannot be in A, and hence is in $-A$, but no open interval containing x_0 can be a subset of $-A$. This will imply that $-A$ is not open.

If $x = x_0$, then $x_0 \in -A$ and no open interval (r_0, r_1) can contain x and be included in $-A$: assume for contradiction that (r_0, r_1) contains x and is included in $-A$; then $A_x \cap (r_0, x) = \emptyset$, and because x_0 is the supremum of A_x, it must be that $x_0 \leq r_0$; but because $r_0 < x = x_0$ this is a contradiction.

Assume $x_0 < x$. If x_0 were in A (and so also in A_x), then there must be by openness of A_x an open interval (r_0, r_1) included in A_x containing x_0 such that $r_0 < x_0 < r_1$. However as x_0 is the supremum, this would imply that $x_0 \geq r_1$, which is a contradiction. It follows that x_0 is not in A. Now we can argue as in the case $x = x_0$ and conclude there can be no open interval included in $-A$ containing x_0. □

3.4.2 Irrational numbers

Let $\mathbb{I}$ denote the set of irrational numbers and let $(\mathbb{I}, \tau_{\mathbb{I}})$ be the topology of the real line restricted to $\mathbb{I}$, that is

$$A \subseteq \mathbb{I} \text{ is in } \tau_{\mathbb{I}} \;\Leftrightarrow\; \text{there is } A' \text{ in } \tau_{\mathbb{R}} \text{ such that } A = A' \cap \mathbb{I}.$$

Observation 3.10 *$\mathscr{B}_{\mathbb{I}} = \{(r_0, r_1) \cap \mathbb{I} \,|\, r_0 < r_1$ rational numbers$\}$ is a clopen base of $\tau_{\mathbb{I}}$.*

Proof. Let $\langle B_i \,|\, i \in \omega\rangle$ be some enumeration of the base $\mathscr{B}_{\mathbb{R}}$ of $\tau_{\mathbb{R}}$ where each B_i is of the form (r_0, r_1) for some rational numbers $r_0 < r_1$. We will show that $\mathscr{B}_{\mathbb{I}} = \{B_i \cap \mathbb{I} \,|\, i \in \omega\}$ is a clopen base of $\tau_{\mathbb{I}}$.

Each $B_i \cap \mathbb{I}$ is easily seen to be closed since the end points r_0 and r_1 are not elements of $\mathbb{I}$, and so $\mathbb{I} \setminus (B_i \cap \mathbb{I})$ is open. Thus each $B_i \cap \mathbb{I}$ is clopen.

To show that $\mathscr{B}_{\mathbb{I}}$ is a base, let A be in $\tau_{\mathbb{I}}$. By definition of $\tau_{\mathbb{I}}$, there is $A' \in \tau_{\mathbb{R}}$ and $A = A' \cap \mathbb{I}$. We can write $A' = \bigcup_{i \in J} B_i$ for some J. But clearly, $(\bigcup_{i \in J} B_i) \cap \mathbb{I} = \bigcup_{i \in J}(B_i \cap \mathbb{I})$ as required. □

Since $\mathscr{B}_{\mathbb{I}}$ is infinite, $\mathrm{CO}(\mathbb{I}, \tau_{\mathbb{I}})$ is an infinite Boolean algebra. In fact, the following holds:

Corollary 3.11 *The size of $\mathrm{CO}(\mathbb{I}, \tau_{\mathbb{I}})$ is $2^{\aleph_0}$.*

Proof. There are at most $2^{\aleph_0}$ clopen sets, since there are at most $2^{\aleph_0}$ open sets in $\tau_{\mathbb{R}}$, and so in $\tau_{\mathbb{I}}$.

Conversely, for every $n \in \omega$, consider the clopen set $I_n = (n, n+1) \cap \mathbb{I}$. We will use the sets I_n to express $2^{\aleph_0}$ many clopen sets. For every $a \subseteq \omega$ consider the union $X_a = \bigcup\{I_n \,|\, n \in a\}$; it is easy to check that X_a is a clopen set (X_a is a closed set because the complement $\mathbb{I} \setminus X_a$ is open: given any $r \in \mathbb{I} \setminus X_a$, there are $s_1 < r < s_2$ rational number such that $(s_1, s_2) \cap \mathbb{I}$ is included in $\mathbb{I} \setminus X_a$). Now notice that if $a \neq a'$ are two subsets of ω, then $X_a \neq X_{a'}$. It follows that there are at least as many clopen sets as the number of subsets of ω, i.e. $2^{\aleph_0}$. □

3.4.3 Cantor space

An important property of topological spaces is *compactness*. We will now introduce a compact space which has many similarities with $(\mathbb{R}, \tau_{\mathbb{R}})$ and $(\mathbb{I}, \tau_{\mathbb{I}})$, but also some differences (one of them is compactness because neither $(\mathbb{R}, \tau_{\mathbb{R}})$ nor $(\mathbb{I}, \tau_{\mathbb{I}})$ is compact). Let (X, τ) be a topological space. We say that $\mathscr{C} \subseteq \mathscr{P}(X)$ is an *open cover* if $\bigcup \mathscr{C} = X$ and all elements in $\mathscr{C}$ are open.

Definition 3.12 *We say that* (X, τ) *is* compact *if every open cover of* X *has a finite subcover, i.e. if* $\mathscr{C}$ *is an open cover of* X*, then there exists finite* $\mathscr{C}^* \subseteq \mathscr{C}$ *such that* $\bigcup \mathscr{C}^* = X$.

Notice that $(\mathbb{R}, \tau_{\mathbb{R}})$ is not compact: for instance no cover formed by open intervals of length 1 has a finite subcover.

If (X, τ) is a topological space, there is a natural construction for a topology on the *product space* $X \times X$: the basis of the topology $\tau_{X \times X}$ is the set of all "rectangles" $A \times B$, where $A, B \in \tau$. A basis uniquely determines the topology $\tau_{X \times X}$: a set C is in $\tau_{X \times X}$ if $C = \bigcup \mathscr{A}$ for some collection $\mathscr{A}$ of basic sets.

In general, we can define a product of any length, but we will limit ourselves to the following special case.

Let us consider the two-element space $2 = \{0, 1\}$ with the *discrete* topology τ, i.e. $\tau = \mathscr{P}(2)$ (a topology is discrete if all subsets of X are open). Let us consider the product space which takes as its domain ω-many copies of 2, that is the set 2^ω, with the following base of open sets: a set $O \subseteq 2^\omega$ is a basic open set whenever there exists $n = \{0, \ldots, n-1\}$ and $\sigma : n \to 2$ such that

$$O = O_\sigma = \{f \in 2^\omega \mid \sigma \subseteq f\}, \tag{3.8}$$

i.e. O contains all sequences of $0, 1$ which on the first n arguments agree with σ. The product topology $\tau_{2^\omega} = \tau$ on 2^ω is determined by basic open sets:

$$\tau = \{X \subseteq 2^\omega \mid X \text{ is a union of a collection of basic open sets}\}.$$

It is easy to check that the collection of basic open sets is closed under intersection (if we add $\emptyset$ to it), and is therefore a base of the topology τ. The topology τ is called the *product topology* on 2^ω.

Definition 3.13 *The topological space* $(2^\omega, \tau)$ *is called the* Cantor space.

The space $(2^\omega, \tau)$ is similar to the space $(\mathbb{R}, \tau_\mathbb{R})$ – it has the same size as $\mathbb{R}$, the respective topologies have the same size, it has a countable base and it is separable – but also some important differences, one of them being:

Lemma 3.14 *Every basic open set in 2^ω is* clopen*.*

Proof. Let O be a basic open set determined by some $\sigma : n \to 2$. We need to show that it is also closed, or equivalently that $2^\omega - O$ is open. Clearly,

$$2^\omega - O = \bigcup \{O_{\sigma'} \mid \sigma' : n \to 2, \sigma \neq \sigma'\}.$$

Hence $2^\omega - O$ is a union of open sets, and is therefore open. It follows that O is closed. □

Since every base set is clopen, the family of clopen sets $\mathrm{CO}(2^\omega, \tau)$ is infinite. In order to determine the size of $\mathrm{CO}(2^\omega, \tau)$, we will use the notion of compactness. The discrete topological space on 2 is trivially compact because it is finite. A natural question is whether the product space $(2^\omega, \tau)$ is also compact. By Tychonoff product theorem, which we take as a fact (for more details, see for instance [3, Theorem 7.4]), the answer is positive: the product space $(2^\omega, \tau)$ is compact.

We use compactness of $(2^\omega, \tau)$ to argue that any clopen set $A \subseteq 2^\omega$ is in fact a finite union of basic clopen sets. Since the number of finite subsets of a countable set is again countable, it follows that the number of clopen sets is just countable. Before we do this properly (in Theorem 3.16), let us define another important property of Boolean algebra, which will be useful in discussing the properties of $\mathrm{CO}(2^\omega, \tau)$.

Definition 3.15 *Let B be a Boolean algebra. An element a in B is an atom if $0 < a$ and there is no $b \in B$ such that $0 < b < a$; in other words a is a minimal element in the set $B - \{0\} = B^+$ (positive elements of B). If B has no atoms, we call B* atomless*.*

Theorem 3.16 *The algebra $\mathrm{CO}(2^\omega, \tau)$ is an atomless countable Boolean algebra.*

Proof. We first show that every clopen set is a union of basic sets. So let $A \subseteq 2^\omega$ be clopen. Since it is open, there is a family X of basic open sets such that $A = \bigcup X$. Since A is also closed, the collection of sets $\mathscr{C} = X \cup \{2^\omega \setminus A\}$ is an open cover of 2^ω. Since the Cantor space is compact, there exists a finite subcover $\{C_0, \ldots, C_{n-1}\} \subseteq \mathscr{C}$ such that $C_0 \cup \ldots \cup C_{n-1} = 2^\omega$. It follows that for some $i \in \{0, \ldots, n-1\}$, $2^\omega \setminus A = C_i$, and $A = \bigcup\{C_j \mid j \in \{0, \ldots, n-1\}, j \neq i\}$. Thus A is a finite union of basic sets.

To show that the algebra is atomless, we need to show that if A is a clopen set, then there is a non-empty clopen set A' such that $A' \subsetneq A$. Clearly, it suffices to consider the case when A is a base set. So let σ be a finite sequence of 0 and 1 such that $A = O_\sigma$. Pick any σ' which properly extends σ: $\sigma \subsetneq \sigma'$, then $A' = O_{\sigma'} \subsetneq O_\sigma$ is as required. □

In Theorem 6.5, we will learn that up to isomorphism there is exactly one atomless countable Boolean algebra (and $\mathrm{CO}(2^\omega, \tau)$ is its canonical representation).

Remark 3.17 The Cantor space can also be described as follows. Consider the closed interval $[0, 1]$ on $\mathbb{R}$. Define a sequence $\langle C_i \mid i \in \omega\rangle$ of closed subsets of $[0, 1]$ as follows: $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. (in the next step divide each segment into three equal pieces and remove the open middle interval). Denote

$$\mathbb{K} = \bigcap_{n \in \omega} C_n.$$

It can be shown that $\mathbb{K}$ has size 2^ω and it is a compact subset of $[0, 1]$ in the topology $\tau_{\mathbb{R}}$ restricted to $\mathbb{K}$. It can also be shown that $\mathbb{K}$ (with the topology $\tau_{\mathbb{R}}$) is *homeomorphic* to the Cantor space $(2^\omega, \tau)$, where a homeomorphism is a bijection between the spaces which preserves open sets in both directions (the existence of a homeomorphism means that the two topological spaces are the same as regards topology).

3.5 Lindenbaum-Tarski algebras

Let T be a first-order theory over a language L. For φ, ψ formulas in L, define the following relation:

$$\varphi \equiv \psi \;\Leftrightarrow\; \mathsf{T} \vdash \varphi \leftrightarrow \psi. \tag{3.9}$$

The relation $\equiv$ is an equivalence relation. We denote by $[\varphi]$ the equivalence class of φ.

Let $B(\mathsf{T})$ the set of all equivalence classes:

$$B(\mathsf{T}) = \{[\varphi] \mid \varphi \text{ a formula of } L\} \tag{3.10}$$

The operations on $B(\mathsf{T})$ are defined as follows:

(i) $[\varphi] \vee [\psi] = [\varphi \vee \psi]$
(ii) $[\varphi] \wedge [\psi] = [\varphi \mathbin{\&} \psi]$
(iii) $-[\varphi] = [\neg\varphi]$
(iv) $1 = [\varphi \to \varphi]$
(v) $0 = [\varphi \wedge \neg\varphi]$

We show that the operations are correctly defined and

$$B(\mathsf{T}) = \langle B(\mathsf{T}), \wedge, \vee, -, 0, 1\rangle$$

is a Boolean algebra.

By correctness of definition we mean that the definition of the operations does not depend on the particular formula we pick from the equivalence class of $[\varphi]$ or $[\psi]$ (we call this property *congruence*): if $\alpha \equiv \varphi$ and $\beta \equiv \psi$, then $[\varphi \vee \psi] = [\alpha \vee \beta]$, and similarly for other operations. The congruence follows easily by the axioms of the predicate calculus:

$$\vdash (\alpha \leftrightarrow \varphi \mathbin{\&} \beta \leftrightarrow \psi) \to (\alpha \vee \beta \leftrightarrow \varphi \vee \psi), \tag{3.11}$$

and similarly for all operations. [Hint. To argue for (3.11), notice that it suffices to show that (3.11) is a *propositional* tautology. It means that we can disregard the issue of free variables in $\alpha, \beta, \varphi, \psi$ which complicates some arguments for *predicate* tautologies.][1]

To argue that $B(\mathsf{T})$ is a Boolean algebra, we need to show that $B(\mathsf{T})$ satisfies all axioms of Boolean algebras. This follows easily by the fact that propositional connectives over $\{0, 1\}$ behave as a Boolean algebra: for instance to show that $[\alpha] \wedge ([\beta] \vee [\gamma]) = ([\alpha] \wedge [\beta]) \vee ([\alpha] \wedge [\gamma])$, it suffices to notice that $[\alpha \mathbin{\&} (\beta \vee \gamma)] = [(\alpha \mathbin{\&} \beta) \vee (\alpha \mathbin{\&} \gamma)]$ is trivially true since

$$\vdash \alpha \mathbin{\&} (\beta \vee \gamma) \leftrightarrow (\alpha \mathbin{\&} \beta) \vee (\alpha \mathbin{\&} \gamma). \tag{3.12}$$

The other axioms are similar.

[1] The so called Equivalences Theorem in predicate logic states that $\alpha \leftrightarrow \alpha' \vdash \varphi \leftrightarrow \varphi'$, where φ' is created from φ by replacing occurrences of α in φ by α'; however it is not generally true that $\vdash (\alpha \leftrightarrow \alpha') \to (\varphi \leftrightarrow \varphi')$. The problem is in free variables possibly occurring in the formulas.

An important subalgebra of $B(\mathsf{T})$ is called the *Lindenbaum-Tarski algebra of* T, which we will denote $LT(\mathsf{T})$. The domain of $LT(\mathsf{T})$ is defined as follows:

$$LT(\mathsf{T}) = \{[\sigma] \mid \sigma \text{ is a sentence in } L\}. \tag{3.13}$$

Note that $LT(\mathsf{T})$ is generally a proper subalgebra of $B(\mathsf{T})$ because it is not true that every formula is provably equivalent to a sentence.[2]

Theorem 3.18 *Every Boolean algebra is isomorphic to a Lindenbaum-Tarski algebra* $LT(\mathsf{T})$ *for some* T*.*

Proof. We will not prove this theorem. If you are interested, see [2, Theorem 9.10, and Exercise 5 in Section 9]. □

We will now study how the properties of T influence the properties of $LT(\mathsf{T})$.

Observation 3.19 *Let* T *be a theory.*

(i) T *is consistent* $\Leftrightarrow$ $LT(\mathsf{T})$ *is a non-trivial algebra (i.e. contains at least two elements).*
(ii) T *is complete and consistent* $\Leftrightarrow$ $LT(\mathsf{T})$ *has exactly two elements.*

Proof. Exercise. [Hint to (ii): If $\mathsf{T} \vdash \sigma$ then $\sigma \in 1$, and if $\mathsf{T} \nvdash \sigma$ then $\sigma \in 0$.] □

The above observation shows that from the point of Boolean algebras, complete theories T are uninteresting because their associated algebras $LT(\mathsf{T})$ are the unique two-element algebra.

If T is incomplete, the algebra $LT(\mathsf{T})$ may become more complex. In fact, if T is consistent, recursively axiomatizable and extends Peano Arithmetics PA, its algebra $LT(\mathsf{T})$ is isomorphic to the algebra $\mathrm{CO}(2^\omega, \tau)$ introduced in Section 3.4.3. In order to show this, we will use the following Fact:

Fact 3.20 (2nd Gödel Incompleteness Theorem) *Let* T *be a consistent first-order theory which is recursively axiomatizable and such that* T *contains* PA*. Then for every sentence* σ *such that* $\mathsf{T} \cup \{\sigma\}$ *is consistent, there exists a formal* Σ_1 *definition* $\ulcorner \mathsf{T} \urcorner$ *of* T *such that*

$$\mathsf{T} \cup \{\sigma\} \nvdash \mathrm{Con}(\ulcorner \mathsf{T} \urcorner \cup \{\ulcorner \sigma \urcorner\}),$$

[2] Recall that the following is true: $\mathsf{T} \vdash \varphi(x) \Leftrightarrow \mathsf{T} \vdash (\forall x)\varphi$. This however does *not* imply $\mathsf{T} \vdash \varphi(x) \leftrightarrow (\forall x)\varphi$.

where $\mathrm{Con}(\ulcorner \mathsf{T} \urcorner \cup \{\ulcorner \sigma \urcorner\})$ *is a sentence in the language of* T *which claims that the formal theory* $\ulcorner \mathsf{T} \urcorner \cup \{\ulcorner \sigma \urcorner\}$ *is consistent.*

Theorem 3.21 *If* PA *is consistent, then* $LT(\mathsf{PA})$ *is a countable atomless Boolean algebra. In fact if* T *is consistent, recursively axiomatizable, has a finite language, and contains* PA*, then* $LT(\mathsf{T})$ *is a countable atomless Boolean algebra (for instance* ZFC *is such a theory).*

Proof. Since in PA has a finite language, the number of all formulas is countable. It follows that $LT(\mathsf{PA})$ is at most countable because it is a partition of the set of all sentences. We will show that if σ is a sentence such that $[\sigma] \neq 0$, then we can find a sentence σ' such that $[\sigma'] < [\sigma]$. This will show that $LT(\mathsf{PA})$ is atomless, and also infinite (and therefore countable).

Let a sentence σ be given. If $\mathsf{PA} \cup \{\sigma\}$ is inconsistent, then $\mathsf{PA} \vdash \neg\sigma$, and hence $[\sigma] = 0$ and we are done.

If $\mathsf{PA} \cup \{\sigma\}$ is consistent, set

$$\sigma' = \sigma \,\&\, \mathrm{Con}(\ulcorner \mathsf{PA} \urcorner \cup \{\ulcorner \sigma \urcorner\}).$$

Notice that $[\sigma'] \leq [\sigma]$ because $\mathsf{PA} \vdash \sigma' \to \sigma$. It remains to show $\mathsf{PA} \nvdash \sigma \to \sigma'$, and so $[\sigma'] \neq [\sigma]$. Assume for contradiction that $\mathsf{PA} \vdash \sigma \to \sigma'$. This implies

$$\mathsf{PA} \cup \{\sigma\} \vdash \mathrm{Con}(\ulcorner \mathsf{PA} \urcorner \cup \{\ulcorner \sigma \urcorner\}).$$

This contradicts the 2nd Gödel Incompleteness Theorem stated above, with PA substituted for T. □

In Theorem 6.5, we show that up to isomorphism there is exactly one atomless countable Boolean algebra. It follows in particular that $\mathrm{CO}(2^\omega, \tau)$ and $LT(\mathsf{PA})$ are isomorphic.

4 Properties of Boolean algebras

We know that the operations $\wedge$ and $\vee$ on a Boolean algebra B correspond to the infima and suprema of two-element subsets of B. In order to obtain more information about B, we will focus our attention on the canonical ordering of B. This will allow us to extend the notions of supremum and infimum to infinite subsets of B. These considerations will lead to important notions of infinite operations, complete algebras, and regular subalgebras.

4.1 Infinite operations

Let B be a Boolean algebra and $M = \{m_i \mid i \in I\}$ a subset of B. If M has a supremum, or infimum, in the canonical ordering of a Boolean algebra, then we denote it as $\bigvee M = \bigvee_{i \in I} m_i$, or $\bigwedge M = \bigwedge_{i \in I} m_i$. To denote in which algebra we are working at the given moment, we may write $\bigvee^B M$, or $\bigwedge^B M$.

Example. It often happens that some subsets of B do not have a supremum or infimum. Let B be the finite/cofinite algebra on ω, i.e. $B = \{A \subseteq \omega \mid A$ is finite or cofinite$\}$. For $n \in \omega$, let us denote by A_n the set of all even numbers smaller or equal to n; note that each A_n is in B. Then $A = \{A_n \mid n \in \omega\}$ does not have a supremum in B.

We show some simple arithmetics concerning infinite operations.

Lemma 4.1 *Let B be a Boolean algebra and let $M = \{m_i \mid i \in I\}$ be a subset of B and assume that the supremum $\bigvee_{i \in I} m_i$ of $\{m_i \mid i \in I\}$ and infimum $\bigwedge_{i \in I} m_i$ of $\{m_i \mid i \in I\}$ exist. Then the right-hand sides in (i) and (ii) also exists and:*

(i) (de Morgan's law) $-\bigvee_{i \in I} m_i = \bigwedge_{i \in I} -m_i$, $-\bigwedge_{i \in I} m_i = \bigvee_{i \in I} -m_i$
(ii) (distributivity) $b \wedge \bigvee_{i \in I} m_i = \bigvee_{i \in I}(b \wedge m_i)$, $b \vee \bigwedge_{i \in I} m_i = \bigwedge_{i \in I}(b \vee m_i)$

Proof. We show just $-\bigvee_{i \in I} m_i = \bigwedge_{i \in I} -m_i$ and $b \wedge \bigvee_{i \in I} m_i = \bigvee_{i \in I}(b \wedge m_i)$; the dual versions are proved analogously.

(i). Denote $a = \bigvee_{i \in I} m_i$. We want to show that $-a$ is the infimum of $-M = \{-m_i \mid i \in I\}$. For every m_i, $a \geq m_i$ implies $-a \leq -m_i$ and so $-a$ is a lower bound of $-M$. Let b another lower bound of $-M$: it means that $-b$ is an upper bound of M; this implies that $a \leq -b$ which is equivalent to $-a \geq b$ as required.

(ii). Note that this proof will use the existence of a complement, not just the (finite) distributivity of a Boolean algebra guaranteed by the axioms. First note that the relationships in (2.4) are in fact equivalences; each Boolean algebra satisfies:

(4.14)

– $(\forall x, y, z)(x \leq z \wedge y \leq z) \leftrightarrow x \vee y \leq z$,
– $(\forall x, y, z)(x \leq y \wedge x \leq z) \leftrightarrow x \leq y \wedge z$.

We will show that $b \wedge \bigvee_{i\in I} m_i$ is the supremum of $\{b \wedge m_i \mid i \in I\}$. It is clearly an upper bound. To show that it is the least such we argue as follows: let a be an upper bound of $\{b \wedge m_i \mid i \in I\}$. By monotonicity of $\vee$, this implies $-b \vee a \geq -b \vee (b \wedge m_i)$, which implies $-b \vee a \geq -b \vee m_i$ by distributivity. By (4.14), it also in particular holds that $-b \vee a \geq m_i$. As i was arbitrary, it means that $-b \vee a$ is an upper bound of $\{m_i \mid i \in I\}$. This implies that $\bigvee_{i\in I} m_i \leq -b \vee a$. By monotonicity of $\wedge$, we have that $b \wedge \bigvee_{i\in I} m_i \leq b \wedge (-b \vee a) = b \wedge a$. Again by (4.14), we in particular have that $b \wedge \bigvee_{i\in I} m_i \leq a$, so that $b \wedge \bigvee_{i\in I} m_i$ is really the supremum as desired. □

4.2 Complete algebras, the σ-algebra of Borel sets

Definition 4.2 *Let κ be an infinite regular cardinal. We call a Boolean algebra B κ-complete if supremum and infimum exists for every M such that $|M| < \kappa$. If κ is ω_1-complete, we say that B is σ-complete. If B is κ-complete for all κ, then we say that B is complete (equivalently, B is complete if every subset of B has the supremum and the infimum).*

Every Boolean algebra is ω-complete, because if $X = \{x_1, \ldots, x_n\} \subseteq B$ is a finite subset of B, then $x_1 \wedge \cdots \wedge x_n$ is the infimum of X in B, and $x_1 \vee \cdots \vee x_n$ the supremum of X in B. This implies that every finite Boolean algebra is complete.

Observation 4.3 *For every non-empty X, the powerset algebra $\mathscr{P}(X)$ is complete.*

Proof. Hint. For every $Y \subseteq \mathscr{P}(X)$, $\bigcup Y$ is the supremum and $\bigcap Y$ is the infimum of Y in $\mathscr{P}(X)$. □

The following lemma says that with Boolean algebras, it suffices to verify completeness just for either of the two infinite operation: infimum or supremum.

Lemma 4.4 *Let B be a Boolean algebra and let κ be a regular infinite cardinal. Then the following are equivalent:*

(i) Every $X \subseteq B$ of size $< \kappa$ has the supremum.
(ii) Every $X \subseteq B$ of size $< \kappa$ has the infimum.

Proof. (i)→(ii). This is a simple consequence of Lemma 4.1(i): if X has size $< \kappa$, then $-X = \{-x \,|\, x \in X\}$ has size the same size as X, and by our assumption $-X$ has the supremum and so X has the infimum:

$$-\bigvee -X = \bigwedge - - X = \bigwedge X.$$

The proof of the converse direction is analogous. □

Corollary 4.5 *The following are equivalent for a Boolean algebra B:*

(i) B is complete.
(ii) Every subset of B has the supremum.
(iii) Every subset of B has the infimum.

Proof. As in Lemma 4.4. □

Definition 4.6 *We say that an algebra of sets $B = \langle B, \cap, \cup, -, 0, X\rangle$ on some non-empty set X is a* σ-algebra (of sets) *if B is closed under countable unions and intersections of elements in B. So in particular B is σ-complete.*

Remark 4.7 Note that the notion of a σ-algebra is stronger than the notion of a σ-complete algebra of sets: with the σ-algebra we require that the supremum and infimum of a countable X are of a specific form, i.e. equal to $\bigcup X$ or $\bigcap X$, respectively; σ-completeness just requires that there is *some* element of B which is the supremum, or infimum, of X in B.

An important example of a σ-algebra of sets is the collection of *Borel sets* on the real line $\mathbb{R}$. Let us denote Borel sets on $\mathbb{R}$ by Borel($\mathbb{R}$). Borel($\mathbb{R}$) is defined as the least σ-algebra of sets containing all open sets in the topology $(\mathbb{R}, \tau_{\mathbb{R}})$:
(4.15)

$$\text{Borel}(\mathbb{R}) = \bigcap\{B \,|\, B \text{ is a } \sigma\text{-algebra on } \mathbb{R} \text{ containing all open sets in } \tau_{\mathbb{R}}\}.$$

Lemma 4.8 $\langle \text{Borel}(\mathbb{R}), \cup, \cap, -, \emptyset, \mathbb{R}\rangle$ *is a σ-algebra of sets.*

Proof. First note that the system

$$Y = \{B \,|\, B \text{ is a } \sigma\text{-algebra on } \mathbb{R} \text{ containing all open sets in } \tau_{\mathbb{R}}\}$$

is non-empty because the power set algebra $\mathscr{P}(\mathbb{R})$ is closed under countable unions and intersections.

We show that Borel($\mathbb{R}$) is closed under complement, and countable intersections and unions. Since $X \in \text{Borel}(\mathbb{R})$ is equivalent to X being in every $B \in Y$, it follows immediately that $-X$ is in every $B \in Y$, and so $-X \in \text{Borel}(\mathbb{R})$. Similarly, if Z is a countable subset of Borel($\mathbb{R}$), then it is a countable subset of every σ-algebra in Y, and hence its intersection (and union) is present in every σ-algebra in Y, and so in Borel($\mathbb{R}$). □

Remark 4.9 It can be shown that CH (The Continuum Hypothesis) holds for the Borel sets in the following sense: every Borel set A is either at most countable, or has the size $|\mathbb{R}|$. Hence if there is a counterexample to CH, it must be more complicated than Borel sets. Intuitively, Borel sets are the "well-behaved subsets" of the real line.

4.3 Regular subalgebras and completions

Definition 4.10 *A subalgebra A of a Boolean algebra B is called a* regular subalgebra *if for every $M \subseteq A$ the following hold:*

(i) If $\bigvee^A M$ exists, then $\bigvee^B M$ exists, and they are equal
(ii) If $\bigwedge^A M$ exists, then $\bigwedge^B M$ exists, and they are equal.

Definition 4.11 *We say that A is a* dense subalgebra *of B if for every $0 < b \in B$ there is $0 < a \in A$ such that $a \leq_B b$.*

Notice that the definition of denseness refers to elements in B which are greater than 0 (such elements are called *positive*).

Before proceeding, we show a useful equivalent formulation of the canonical ordering $x \leq y$ on a Boolean algebra.

Observation 4.12 *Let B be a Boolean algebra, and $\leq$ its canonical ordering. The for all $x, y \in B$,*

$$x \leq y \iff x \wedge -y = 0. \tag{4.16}$$

Proof. From left to right: Since $x \wedge y = x$, $x \wedge -y$ is equal to $x \wedge y \wedge -y = 0$. From right to left: $x = x \wedge (y \vee -y) = (x \wedge y) \vee (x \wedge -y) = x \wedge y$ because the second expression is equal to 0 by our assumption. □

Note that $x \not\leq y$ is thus equivalent to $x \wedge -y$ being a positive element.

Denseness is an important concept because it (among other things) implies regularity.

Lemma 4.13 *Every dense subalgebra of a Boolean algebra B is regular in B.*

Proof. Let A be a dense subalgebra of B.

Let $M \subseteq A$ be a subset of A and let $a = \bigvee^A M$ exist. We want to show that a is the supremum of M in B. a is clearly an upper bound of M in B because the canonical orderings of A and B coincide on elements of A. So let $b \in B$ be an upper bound of M, we want to show that $a \leq b$. Assume for contradiction that $a \not\leq b$. By (4.16), $a \wedge -b$ is a positive element in B. By denseness of A in B, there exists a' in A such that $0 < a' \leq a \wedge -b$. We show that $a \wedge -a'$ is an upper bound of M in A and that $a \wedge -a' < a$; this leads to contradiction because all upper bounds must be greater or equal to a. Let $m \in M$ be given: $m \leq a$ is true because a is an upper bound; but since b is also an upper bound, we obtain $m \leq b \leq -a'$ because $a' \leq -b$. It follows that $m \leq a \wedge -a'$ and so $a \wedge -a'$ is indeed an upper bound. It holds $a \wedge -a' \leq a$; assume now that $a \wedge -a' = a$: this means that $a \leq -a'$; however we also know that $a' \leq a$, which together implies $a' \leq -a'$ (which is equivalent to $a' \wedge -a' = a'$). The last inequality can only be true if a' is equal to 0, but $a' > 0$. Contradiction.

For the case of $\bigwedge^A M$ we argue as follows. Denote by $-M$ the set $\{-m \mid m \in M\}$). The following holds:

$$\bigwedge\nolimits^A M = -\bigvee\nolimits^A -M = -\bigvee\nolimits^B -M = \bigwedge\nolimits^B M,$$

where the first and the third identity holds because of the de Morgan laws in Lemma 4.1, and the middle one follows from the proof for $\bigvee^A M$ in the first part of this proof. □

The following theorem claims that there is a canonical procedure which will construct for every Boolean algebra B a complete algebra B' such that B is a dense (and hence regular) subalgebra of B'. Moreover, this B' is unique and is called the completion of B. Complete Boolean algebras can be used to define a generalized truth-evaluation of first-order formulas, which takes its values in a complete Boolean algebra. It can be shown that the usual Completion theorem for first order logic still holds with respect to this generalized satisfaction.

Fact 4.14 *Every Boolean algebra has a unique completion where the original algebra is dense (and hence regular).*

We will not prove this theorem; for a proof, see [1] or [2, Section 4.3].

5 Representations of Boolean algebras

We will show in this section that every Boolean algebra is isomorphic to an algebra of sets. In Section 5.1 we show this for a special case of algebras, the so called *atomic* Boolean algebras. In Section 5.2 we show this for all algebras. Finally, in Section 5.3, we give a topological version of the representation theorem (Stone's duality).

5.1 Atomic and complete Boolean algebras

Recall that if B is a Boolean algebra, we denote by B^+ the set of all non-zero elements of B. We call an element of B^+ a positive element of B. A positive element $a \in B^+$ is called an atom if there is no $b \in B^+$ such that $b < a$. The set of all atoms of B will be denoted by $\mathrm{At}(B)$. B is called *atomless* if it has no atoms. B is called *atomic* if there is an atom below every element of B^+.

Examples. Every finite algebra is atomic. For every X, the powerset algebra $\mathscr{P}(X)$ is atomic. The clopen algebra $\mathrm{CO}(2^\omega, \tau)$ is atomless.

We will now give several equivalent definitions of an atom.

Lemma 5.1 *Let B be a Boolean algebra. The following are equivalent for every $a \in B$:*

(i) a is an atom.
(ii) For every $x \in B$, $a \leq x$ or $a \leq -x$, but not both.
(iii) $a > 0$ and for all $x, y \in B$, $a \leq x \vee y \Leftrightarrow (a \leq x$ or $a \leq y)$.

Proof. (i) $\to$ (ii). If $a \leq x$ and $a \leq -x$, then $a \leq x \wedge -x = 0$ which cannot be true because a is an atom. Assume now that $a \not\leq x$; by (4.16) this is equivalent to $a \wedge -x$ being a positive element: $0 < a \wedge -x \leq a$; since a is an atom, it must hold $a \wedge -x = a$ which is equivalent to $a \leq -x$.

(ii) $\to$ (iii). a is clearly positive because $a = 0$ would imply $a \leq x \wedge -x$. One side of the equivalence in (iii) is obvious: if $a \leq x$ or $a \leq y$ then also $a \leq x \vee y$. We show the converse. So assume $a \leq x \vee y$ and $a \not\leq x$. By (ii) this means that $a \leq -x$, and hence also $a \leq (x \vee y) \wedge -x = y \wedge -x$, and so $a \leq y$ as required.

(iii) $\to$ (i). Assume $0 < b \leq a$; we wish to show $b = a$. $a = a \wedge (b \vee -b) = (a \wedge b) \vee (a \wedge -b) = b \vee (a \wedge -b)$ since by our assumption $b \leq a$, $a \wedge b = b$. By (iii), either $a \leq b$ or $a \leq (a \wedge -b)$; in the first case, we have $a \leq b$ and $b \leq a$, and therefore $a = b$. In the second case, we have $a \leq -b$, which together with $b \leq a$ implies $b = 0$, and this contradicts our assumption. □

For the following theorem, recall the notion from Section 2.4.

Theorem 5.2 *For every Boolean algebra B, the map f from B to the powerset algebra $\mathscr{P}(\mathrm{At}(B))$ defined by*

$$f(x) = \{a \in \mathrm{At}(B) \mid a \leq x\} \tag{5.17}$$

is a homomorphism.[3] *Moreover:*

(i) If B is atomic, f is an embedding.
(ii) If B is a complete Boolean algebra, f is onto.

Thus if (i) and (ii) hold together, f is an isomorphism.

Proof. We first verify that f is a homomorphism. Let $B = \langle B, \wedge, \vee, -, 0, 1\rangle$ be a Boolean algebra and x, y elements of B.

Clearly, $f(0) = \emptyset$ and $f(1) = \mathrm{At}(B)$. The complement: $f(-x) = \{a \in \mathrm{At}(B) \mid a \leq -x\}$; by Lemma 5.1(ii), $f(-x) = \mathrm{At}(B) \setminus f(x)$. The join operation: $f(x \vee y) = \{a \in \mathrm{At}(B) \mid a \leq x \vee y\}$; by Lemma 5.1(iii), $f(x \vee y)$ is equal to the union of $f(x) = \{a \in \mathrm{At}(B) \mid a \leq x\}$ and $f(y) = \{a \in \mathrm{At}(B) \mid a \leq y\}$ and so $f(x \vee y) = f(x) \cup f(y)$. The meet operation: $f(x \wedge y) = f(x) \cap f(y)$ because $a \leq x \wedge y \;\Leftrightarrow\; a \leq x \;\&\; a \leq y$.

Assume now that B is atomic. We want to show that f is an embedding. Let x, y be elements in B such that $x \neq y$. Without loss of generality let $x \not\leq y$.

[3] If B has no atoms, then $\mathscr{P}(\mathrm{At}(B))$ is a degenerate Boolean algebra with $0 = 1$.

But then $x \wedge -y$ is positive and there is an atom $a \leq x \wedge -y$. This implies that $a \leq x$ and so $a \in f(x)$, while $a \leq -y$ implies that $a \in f(-y)$, and so $a \notin f(y)$. Hence $f(x) \neq f(y)$ as required.

Finally, let B a complete algebra. Let Y be an arbitrary non-empty element of $\mathscr{P}(\mathrm{At}(B))$: $\emptyset \neq Y = \{y_i \mid i \in I\} \subseteq \mathrm{At}(B)$, for some index set I. We will show that $Y = f(s)$ where $s = \bigvee Y$, proving that f is onto $\mathscr{P}(\mathrm{At}(B))$. First we show $Y \subseteq f(s)$. Let a in Y be arbitrary. Then $a \leq s$ since s is the supremum of Y, and hence $a \in f(s)$. For the converse direction $f(s) \subseteq Y$, first realize that if $a \neq a'$ are two distinct atoms, then

$$a \wedge a' = 0. \tag{5.18}$$

To see that (5.18) is true, realise that if we assume $0 < a \wedge a'$, then $0 < a \wedge a' \leq a$ and $0 < a \wedge a' \leq a'$, and since a, a' are atoms, this implies $a = a \wedge a' = a'$, which contradicts our assumption $a \neq a'$. If a is in $f(s)$, then $a \leq s = \bigvee Y$, and so $a \wedge \bigvee_{i \in I} y_i = a > 0$. By the generalized distributivity, this is equal to $\bigvee_{i \in I}(a \wedge y_i)$. For contradiction assume that $a \notin Y$; then $\bigvee_{i \in I}(a \wedge y_i) = \bigvee_{i \in I} 0 = 0$ by (5.18), which is a contradiction. It follows that $a \in Y$, concluding $f(s) = Y$. □

Corollary 5.3 *(i) Every atomic Boolean algebra is isomorphic to an algebra of sets. Every complete and atomic Boolean algebra is isomorphic to a powerset algebra.*

(ii) The finite Boolean algebras are, up to isomorphism, exactly the powerset algebra of finite sets. In particular, the size of a finite algebra B is 2^n for some $n \in \omega$.

(iii) Two finite Boolean algebras are isomorphic if and only if they have the same cardinality.

Proof. (i). If a Boolean algebra B is atomic, then the homomorphism f from Theorem 5.2 is an embedding and therefore B is isomorphic to $f[B]$, which is a subalgebra of $\mathscr{P}(\mathrm{At}(B))$. If B is also complete, then f is an isomorphism.

(ii). Every finite algebra B is both complete and atomic; by (i), B is isomorphic to $\mathscr{P}(\mathrm{At}(B))$. It follows that the size of B is 2^n, where n is the number of atoms of B.

(iii). The direction from left to right follows because an isomorphism is a bijection. To prove the converse direction, let A, B be finite Boolean algebras of the same cardinality. By (ii), $A \cong \mathscr{P}(\mathrm{At}(A))$ and $B \cong \mathscr{P}(\mathrm{At}(B))$. Since A

and B have the same cardinality, the cardinality of $\mathrm{At}(A)$ and $\mathrm{At}(B)$ must also be the same. It follows that any bijection between $\mathrm{At}(A)$ and $\mathrm{At}(B)$ generates an isomorphism between $\mathscr{P}(\mathrm{At}(A))$ and $\mathscr{P}(\mathrm{At}(B))$, and hence between A and B. □

5.2 Stone's representation theorem

In order to generalize Theorem 5.2 to a non-atomic Boolean algebra B, we need an object which will play the role of atoms in Theorem 5.2. Instead of atoms, we will consider certain subsets of B, called *ultrafilters*. Ultrafilters are more complex than atoms, and to obtain non-trivial ultrafilters, it is necessary to use some form of AC.

Definition 5.4 *A* filter *F on a Boolean algebra B is a subset F of B such that:*

(i) $1 \in F$,
(ii) If $x \in F$ and $x \leq y$, then $y \in F$,
(iii) If $x, y \in F$, then $x \wedge y \in F$.

We say that F is *a proper filter* if $0 \notin F$, or equivalently, if $F \neq B$.

Lemma 5.5 *The property of being a filter F on B can be equivalently defined by these two conditions.*

(i) $1 \in F$,
(ii) $x \wedge y \in F \;\Leftrightarrow\; x \in F$ *and* $y \in F$

Proof. Let F be a filter on B according to Definition 5.4. We need to show that $x \wedge y \in F$ implies $x \in F$ and $y \in F$. This is true because $x \wedge y \leq x$ and $x \wedge y \leq y$.

Conversely, let F satisfy the conditions in the lemma. We need to show that $x \in F$ and $x \leq y$ implies $y \in F$. Since $x \leq y$ is equivalent to $x \wedge y = x$, it follows $x \wedge y \in F$ and hence $y \in F$ by (ii). □

Notice that if F is a filter, the condition (ii) in Lemma 5.5 cannot be generalized for disjunction: it is not true that every filter satisfies

$$x \vee y \in F \;\Leftrightarrow\; x \in F \text{ or } y \in F. \tag{5.19}$$

Consider the Frechet filter on ω which we denote by Frechet(ω), where

$$\text{Frechet}(\omega) = \{x \subseteq \omega \,|\, \omega \setminus x \text{ is finite}\}.$$

Frechet(ω) is a proper filter on the Boolean algebra $\mathscr{P}(\omega)$. If x is the set of all even numbers including 0 and y is the set of all odd numbers, $x \cup y \in \text{Frechet}(\omega)$, but $x \notin \text{Frechet}(\omega)$ and $y \notin \text{Frechet}(\omega)$. The generalization for disjunction will hold for a suitable strengthening of the notion of a filter, called an ultrafiter. Ultrafilters will be introduced below in Definition 5.7.

Let B be a Boolean algebra. We say that a subset $X \subseteq B$ has the *finite intersection property (FIP)* if for every $n \in \omega$ and every sequence $x_0, \ldots, x_n$ of elements of X, $x_0 \wedge \ldots \wedge x_n \neq 0$.

Lemma 5.6 *Let B be a Boolean algebra. Every $X \subseteq B$ with FIP can be extended into a proper filter F.*

Proof. Define

(5.20) $F = \{y \in B \,|\, \exists n \in \omega \text{ and } x_0, \ldots, x_n \in X \text{ such that } x_0 \wedge \ldots \wedge x_n \leq y\}.$

As F has FIP, no element in F is equal to 0, and hence F is proper. If x, y are in F, then $x \geq x_0 \wedge \ldots \wedge x_n$ and $y \geq y_0 \wedge \ldots \wedge y_m$ for some n, m and elements $x_0, \ldots, x_n, y_0, \ldots, y_m$ in X. Clearly $x \wedge y$ is in F because it is greater than $x_0 \wedge \ldots \wedge x_n \wedge y_0 \wedge \ldots \wedge y_m$. Lastly, if $y \geq x$ for some $x \in F$, then $y \geq x \geq x_0 \wedge \ldots \wedge x_n$, for some $x_0, \ldots, x_n$ in X. ☐

Definition 5.7 *A proper filter F on a Boolean algebra B is called*

(i) *maximal* if there is no proper filter F' which strictly includes F, i.e. there is no F' such that $F \subseteq F'$ but $F \neq F'$.
(ii) an *ultrafilter* if for all $x \in B$, either $x \in F$ or $-x \in F$.
(iii) *prime* if (5.19) holds for F.

Observation 5.8 *Let B be a Boolean algebra and F a proper filter on B. The F is maximal if and only if for every $x \notin F$, there is some $x' \in F$ such that $x' \wedge x = 0$.*

Proof. To prove the equivalence from left to right, assume for contradiction that there exists $x \notin F$ such that $x' \wedge x$ is non-zero for every $x' \in F$. Then

$F \cup \{x\}$ has FIP because for every finite sequence $x_1, \ldots, x_n$ from F, $x' = x_1 \wedge \ldots \wedge x_n$ is also in F, and so $x_1 \wedge \ldots x_n \wedge x = x' \wedge x$. By Lemma 5.6, there is a proper filter extending $F \cup \{x\}$, which is impossible if F is maximal.

Conversely, assume F is not maximal and F' is a proper filter which strictly extends F. Then there is some $x \in F' \setminus F$, and since F' is proper, $x \wedge x'$ must be non-zero for all $x' \in F$. □

Lemma 5.9 *The following are equivalent for every proper filter F on a Boolean algebra B:*

(i) F is a maximal filter.
(ii) F is an ultrafilter.
(iii) F is prime.

Proof. (i)→(ii). Let $x \notin F$ be given. We aim to show that $-x \in F$. Since F is maximal, by Observation 5.8 there must be some $y \in F$ such that $x \wedge y = 0$, which is equivalent to $y \leq -x$, and so $-x \in F$.

(ii)→(iii). One side of the equivalence (5.19) is obvious: $x \in F$ or $y \in F$ implies $x \vee y \in F$ in every filter F. To show the converse, let $x \vee y \in F$ be given and assume that $x \notin F$. This implies that $-x \in F$ and hence $-x \wedge (x \vee y) = -x \wedge y$ must be in F. By Lemma 5.5(ii), $y \in F$.

(iii)→(i). Let $x \notin F$ be arbitrary. We want to show that there is some $y \in F$ such that $x \wedge y = 0$. Clearly, $x \vee -x \in F$ and since F is prime, this implies $x \in F$ or $-x \in F$. By our assumption, $x \notin F$, and so $-x \in F$. Hence $x \wedge -x = 0$ as required. □

Ultrafilters can be used to define useful homomorphisms:

Corollary 5.10 *Let B be a Boolean algebra and F an ultrafilter on B. Then the function $f : B \to \{0, 1\}$ defined by*

$$f(x) = 1 \text{ if } x \in F, \text{ and } f(x) = 0 \text{ if } x \notin F \tag{5.21}$$

is a homomorphism from the Boolean algebra B onto the Boolean algebra **2**.

Proof. Because F is a proper filter, we obtain $f(0) = 0$ and $F(1) = 1$, and also $f(x \wedge y) = f(x) \wedge f(y)$ due to (ii) in Lemma 5.5. The characterization of an ultrafilter in Definition 5.7 (ii) implies $f(-x) = -(f(x))$, and the characterization (iii) implies $f(x \vee y) = f(x) \vee f(y)$. □

We will now prove a theorem which implies that for every Boolean algebra B, there are many ultrafilters on B.

Theorem 5.11 (Boolean prime ideal theorem, BPI) *Let B be a Boolean algebra. Every $X \subseteq B$ with FIP can be extended to an ultrafilter.*

Proof. Let $\mathscr{F}_B$ be the set of all proper filters on B ordered by inclusion $\subseteq$. Note that a maximal proper filter from Lemma 5.9 is a maximal element of $\mathscr{F}_B$ in the ordering $\subseteq$.

We wish to apply Zorn's lemma (Principle of Maximality) to $\langle \mathscr{F}_B, \subseteq \rangle$; in order to do that we need to check that if $\mathscr{F}' \subseteq \mathscr{F}_B$ is a linearly ordered subfamily, then it has an upper bound in $\mathscr{F}_B$. Clearly, $F = \bigcup \mathscr{F}'$ is an upper bound of $\mathscr{F}'$ in the inclusion relation because $F' \subseteq F$ for every $F' \in \mathscr{F}'$. It remains to show that F is in $\mathscr{F}_B$. If x, y are in F, then $x \in F_x$ and $y \in F_y$ for some proper filters F_x and F_y in $\mathscr{F}'$; since $\mathscr{F}'$ is linearly ordered, we can assume without loss of generality $F_x \subseteq F_y$, and so $x, y \in F_y$ and $x \wedge y \in F_y$ and so $x \wedge y \in F$. If $x \in F$ and $y \geq x$, then for some F_x in $\mathscr{F}'$, $x \in F_x$ and so $y \in F_x$, and hence $y \in F$. F is clearly proper because otherwise 0 would be in some $F' \in \mathscr{F}'$, which is impossible because all elements of $\mathscr{F}'$ are proper. It follows that $\langle \mathscr{F}_B, \subseteq \rangle$ satisfies the assumption of Zorn's lemma for every Boolean algebra B.

Let F be a proper filter extending X: $X \subseteq F$ (such an F exists by Lemma 5.20). By Zorn's lemma there is a maximal filter U in $\langle \mathscr{F}, \subseteq \rangle$ above the filter F: $F \subseteq U$. Thus U extends X and by Lemma 5.9, U is an ultrafilter. □

Notice we have proved BPI in ZFC. It is known that BPI cannot be proved in ZF alone. However, it also known that ZF + BPI is too weak to prove AC. Thus BPI is considered as a weaker choice principle, yet strong enough to prove for instance the completeness theorem for the first-order predicate logic. For more details, see [2].

Remark 5.12 The name BPI is motivated historically by the notion of an *ideal* which is a dual to a filter. For a Boolean algebra B, we say that $I \subseteq B$ is a *proper ideal* if (i) $0 \in I$, $1 \notin I$, (ii) if $x \in I$ and $y \leq x$, then $y \in I$, and finally (iii) if $x, y \in I$, then $x \vee y \in I$. It is routine to show that if F is a proper filter then $F^* = \{x \in B \mid -x \in F\}$ is a proper ideal (called the *dual ideal to* F), and conversely if I is a proper ideal then $I^* = \{x \in B \mid -x \in I\}$ is a proper filter (called the *dual filter to* I). We say that a proper ideal I is a *prime ideal* if for every $x \in B$, either $x \in I$ or $-x \in I$. It follows that the dual filter I^* is an ultrafilter if and only if I is a prime ideal.

We now have all tools necessary to prove Stone's representation theorem. Before proving the theorem, we first show a connection between atoms and certain ultrafilters to motivate the construction.

Lemma 5.13 *Let B be a Boolean algebra. Then*

(i) For every $x > 0$ in B, $F_x = \{y \in B \mid x \leq y\}$ is a proper filter. We call this filter a principal filter generated by x*.*
(ii) For every $x > 0$ in B, $F_x = \{y \in B \mid x \leq y\}$ is an ultrafilter if and only if x is an atom in B.

Proof. (i). For every $x > 0$, $\{x\}$ has FIP, and hence F_x is a proper filter by Lemma 5.6.

(ii). In the direction from left to right, we argue exactly as in Lemma 5.1(iii)→(i): Assuming F_x is an ultrafilter, fix some b such that $0 \leq b < x$. We aim to show that $b = 0$. Clearly $x = b \vee (x \wedge -b)$, and so b or $x \wedge -b$ must be in F_x by Lemma 5.9 because $x \in F_x$. However $x \not\leq b$, and so $b \notin F_x$. Hence $x \wedge -b$ must be in F_x, which implies $x \leq x \wedge -b$ by the definition of F_x, and so in particular $x \leq -b$. This implies $x \wedge b = 0$. However, by the assumption $b < x$, $x \wedge b = b$, and so $b = 0$.

In the converse direction, we also argue as in Lemma 5.1, this time we use the implication (i)→(ii). If x is an atom, then for every y either $x \leq y$ or $x \leq -y$. It follows that F_x is an ultrafilter. □

If a Boolean algebra B is atomic, there is a straightforward correspondence between the atoms and the principal ultrafilters generated by these atoms. In fact, it is easy to reprove Theorem 5.2 using the principal ultrafilters F_a for $a \in \text{At}(B)$: Modify the definition of f in (5.17) by setting $f(x)$ to be the collection of all principal ultrafilters F_a, $a \in \text{At}(B)$, which contain x as an element. If B is non-atomic, then there are no principal ultrafilters. In Theorem 5.14, the solution is to use all ultrafilters and define $f(x)$ as the collection of all ultrafilters which contain x as an element. BPI guarantees that $f(x)$ is well defined.

Theorem 5.14 (Stone) *Every Boolean algebra is isomorphic to some algebra of sets.*

Proof. Let $B = \langle B, \wedge, \vee, -, 0, 1\rangle$ be an arbitrary Boolean algebra. Let us denote by $\mathrm{Ult}(B)$ the family of all ultrafilters on B. Define a function f from B to $\mathscr{P}(\mathrm{Ult}(B))$ by

$$f(x) = \{U \in \mathrm{Ult}(B) \mid x \in U\}. \tag{5.22}$$

We claim that f is an isomorphism between B and the algebra of sets $f[B]$ ($f[B]$ is the subalgebra of the powerset algebra $\mathscr{P}(\mathrm{Ult}(B))$ determined by the range of f; see Section 2.4).

We first verify that f is a homomorphism between the algebras B and $\mathscr{P}(\mathrm{Ult}(B))$, which in particular guarantees that $f[B]$ is closed under operations and constants in $\mathscr{P}(\mathrm{Ult}(B))$ and therefore determines a subalgebra $f[B]$ (see Observation 2.17(i)). Clearly $f(1) = \mathrm{Ult}(B)$ and $f(0) = \emptyset$. $f(x \wedge y) = f(x) \cap f(y)$ by Lemma 5.5(ii); $f(-x) = \mathrm{Ult}(B) \setminus f(x)$ by the definition of an ultrafilter, and $f(x \vee y) = f(x) \cup f(y)$ by Lemma 5.9(iii).

By Observation 2.17(ii), the proof is finished once we show that f is an embedding. Let $x \neq y$ be two elements in B, and so $x \not\leq y$ or $y \not\leq x$. Assume without loss of generality that $x \not\leq y$; then $x \wedge -y \neq 0$, and therefore the two-element set $\{x, -y\}$ has FIP. By BPI, there is an ultrafilter U which contains x and $-y$. It follows that $U \in f(x)$, but $U \notin f(y)$, and hence $f(x) \neq f(y)$. □

5.3 Stone's topological duality

Definition 5.15 *Let (X, τ) be a topological space.*

- *(i) We say that (X, τ) is* totally disconnected *if every two distinct element x, y in X can be separated by clopen sets, for all $x \neq y$ in X there are two disjoint clopen sets O_x and O_y such that $x \in O_x$ and $y \in O_y$.*
- *(ii) We say that (X, τ) is a* Boolean space *if it is a compact totally disconnected space.*

For the following Theorem 5.16, recall the notation from Theorem 5.14. Also recall that if (X, τ) is a topological space, then $\mathrm{CO}(X, \tau)$ denotes the clopen subsets of X, and $\mathrm{CO}(X, \tau)$ denotes the clopen algebra with the domain $\mathrm{CO}(X, \tau)$.

Theorem 5.16 *For every Boolean algebra B there exists a Boolean space $(\mathrm{Ult}(B), \tau)$ such that B is isomorphic to $\mathrm{CO}(\mathrm{Ult}(B), \tau)$. It follows that every*

Boolean algebra is up to isomorphism a clopen algebra of some topological space.

Proof. Let f be the isomorphism between B and $f[B]$ from Theorem 5.14. Notice that $f[B]$ is closed under intersections: if $f(x)$ and $f(y)$ are in $f[B]$, then $f(x) \cap f(y) = f(x \wedge y)$, and $f(x \wedge y)$ is in $f[B]$. In particular, $f[B]$ is a basis of a certain topology τ on $\mathrm{Ult}(B)$: $A \in \tau$ if and only if A is a union of some sets in $f[B]$. We show that $(\mathrm{Ult}(B), \tau)$ is as desired (τ is called the *Stone's topology* on $\mathrm{Ult}(B)$).

First notice that for each $x \in B$, $f(x)$ is clopen: $f(x)$ is open because it is in the base of the topology τ; it is closed because the complement $\mathrm{Ult}(B) \setminus f(x)$ is open since it is equal to $f(-x)$. Now it follows easily that the space it totally disconnected: if $F \neq G$ are in $\mathrm{Ult}(B)$, there is some $x \in B$ such that $x \in F$ and $-x \in G$. Then $f(x)$ and $f(-x)$ separate F, G.

Next we show that the space is compact, and therefore Boolean. Let $\mathscr{C}$ be an open cover of $\mathrm{Ult}(B)$; without loss of generality, we can take $\mathscr{C}$ to be composed of the basic sets, i.e. $\mathscr{C} = \{f(x) \,|\, x \in A\}$ for some $A \subseteq B$. Assume for contradiction that no finite subset of $\mathscr{C}$ covers the whole space $\mathrm{Ult}(B)$. This means that for every $n \in \omega$, and every sequence $x_1, \ldots, x_n$ of elements in A,

$$f(x_1) \cup \ldots \cup f(x_n) = f(x_1 \vee \ldots \vee x_n) \neq \mathrm{Ult}(B),$$

or equivalently

$$x_1 \vee \ldots \vee x_n \neq 1 \;\Leftrightarrow\; -x_1 \wedge \ldots \wedge -x_n \neq 0. \tag{5.23}$$

It follows by the righthand side of (5.23) that the family $-A = \{-x \,|\, x \in A\}$ has FIP. Let U be an ultrafilter extending $-A$. We show that U is not covered by $\mathscr{C}$, which contradicts the fact that $\mathscr{C}$ is a cover. Assume that $U \in f(x)$ for some $x \in A$. Since U extends $-A$, $-x \in U$. However, since $U \in f(x)$, it also holds $x \in U$. This is a contradiction, and therefore $(\mathrm{Ult}(B), \tau)$ is compact.

Finally we show that the function f is an isomorphism between B and $\mathrm{CO}(\mathrm{Ult}(B), \tau)$. By the fact that every $f(x)$, $x \in B$, is clopen in τ, $f[B] \subseteq \mathrm{CO}(\mathrm{Ult}(B), \tau)$; by Theorem 5.14, $f : B \to \mathrm{CO}(\mathrm{Ult}(B), \tau)$ is an embedding. It remains to show it is onto, i.e. $f[B] = \mathrm{CO}(\mathrm{Ult}(B), \tau)$. Let X be a clopen set. As it is open, X is a union of some sets in the base: $X = \bigcup_{x \in A} f(x)$ for some $A \subseteq B$. As X is closed, $Y = \mathrm{Ult}(B) \setminus X$ is open. It follows that $\mathscr{C} = \{f(x) \,|\, x \in A\} \cup \{Y\}$ is an open cover of $\mathrm{Ult}(B)$. By compactness, there are finitely many $x_1, \dots, x_n$ from A such that

$$X = f(x_1) \cup \dots f(x_n) = f(x_1 \vee \dots \vee x_n).$$

It follows that X is in the range of f, and we are done. □

It may be interesting to review the relationship between the subfamilies of $\mathscr{P}(\mathrm{Ult}(B))$ appearing in Theorems 5.14 and 5.16: We have

$$f[B] = \mathrm{CO}(\mathrm{Ult}(B), \tau) \subseteq \tau \subseteq \mathscr{P}(\mathrm{Ult}(B)).$$

Stone's duality makes it possible to study general Boolean algebra by topological tools.

6 Atomless countable Boolean algebras

6.1 Uniqueness

In this section we show that up to isomorphism there is exactly one atomless countable Boolean algebra.

Definition 6.1 *Let B be a Boolean algebra and z a positive element in B. We say that $P \subseteq B$ is a* partition of z *if*

(i) $0 \notin P$,
(ii) $\bigvee P = z$,
(iii) For all $x \neq y$ in P, $x \wedge y = 0$.

If $z = 1$, we say that P is a partition of B.

Note that if $P \subseteq B$ is infinite, then $\bigvee P$ may or may not exist in B. We will only consider finite partitions, and therefore $\bigvee P$ will always exist.

If P and Q are two partitions of B, we say that P *refines* Q, in symbols $P \ll Q$, if for every $x \in P$ there is some $y \in Q$ such that $x \leq y$. Note that if $P \ll Q$, then for every $y \in Q$, the set $\{x \in P \mid x \leq y\}$ is a partition of y (Exercise).

Definition 6.2 *Let B a Boolean algebra and P a finite partition of B. Let $B\langle P\rangle$ denote the following subset of B:*

$$B\langle P\rangle = \{b \in B \mid (\exists X \subseteq P)\, b = \bigvee X\} \qquad (6.24)$$

Lemma 6.3 *Let B a Boolean algebra and P a finite partition of B. Then $B\langle P\rangle$ is an atomic subalgebra of B and P is the set of all atoms in $B\langle P\rangle$. In particular the size of $B\langle P\rangle$ is $2^{|P|}$.*

Proof. It is enough to show that $B\langle P\rangle$ is closed under the constants and operations in B.

$B\langle P\rangle$ contains 0 because $\bigvee\emptyset = 0$, and 1 because $\bigvee P = 1$.

Let $a = \bigvee X$ and $b = \bigvee Y$ in $B\langle P\rangle$, for some X, Y subsets of P.

- The operation $\vee$.
  $a \vee b = \bigvee X \vee \bigvee Y = \bigvee(X \cup Y)$, which is clearly in $B\langle P\rangle$.
- The operation $\wedge$.

  $$a \wedge b = \bigvee X \wedge \bigvee Y = \bigvee\{x \wedge y \,|\, x \in X \;\&\; y \in Y\}. \tag{6.25}$$

  The last identity holds by distributivity.

  It remains to show that $W = \{x \wedge y \,|\, x \in X, y \in Y\}$ is a subset of P; then (6.25) can be used to conclude that $B\langle P\rangle$ is closed under the operation $\wedge$. But clearly, for arbitrary $x \in X$ and $y \in Y$, $x \wedge y$ is either 0 if $x \neq y$, or x if $x = y$ since P is a partition. It follows that $W = X \cap Y$, and so $W \subseteq P$.
- The operation $-$.
  We first prove that if x is in P, then

  $$-x = \bigvee(P \setminus \{x\}) \in B\langle P\rangle. \tag{6.26}$$

  Let us denote $\bigvee(P \setminus \{x\}) = y$. To show (6.26), it is enough to argue that $x \wedge y = 0$ and $x \vee y = 1$; then we can conclude by Observation 2.11 that $y = -x$. But this is easy: $x \wedge y = \bigvee\{x \wedge z \,|\, z \in P \setminus \{x\}\} = 0$, and $x \vee y = \bigvee P = 1$.

  Now we can conclude that $B\langle P\rangle$ is closed under the complement: $-a = -\bigvee X = \bigwedge\{-x \,|\, x \in X\}$. By (6.26), $-x$ is in $B\langle P\rangle$ for every $x \in X$, and since $B\langle P\rangle$ is closed under the operation $\wedge$ (6.25), we conclude that $\bigwedge\{-x \,|\, x \in X\}$ is also in $B\langle P\rangle$.

$B\langle P\rangle$ is finite (even if B is infinite) because P is finite: every element $b \in B\langle P\rangle$ is determined by a subset $X \subseteq P$, and there only $2^{|P|}$ of these, hence $|B\langle P\rangle| \leq 2^{|P|}$. We show that in fact the size of $B\langle P\rangle$ is equal to $2^{|P|}$. $B\langle P\rangle$ is atomic with the set P as atoms: each $x \in P$ is an atom and if $b = \bigvee X$ for $X \subseteq P$ is not in P, then for every $x \in X$ we have $0 < x < b$, and hence b is not an atom. By Corollary 5.3, $B\langle P\rangle$ is isomorphic to $\mathscr{P}(P)$, and so in particular the size of $B\langle P\rangle$ is $2^{|P|}$. □

Note that if B is a Boolean algebra and a partition P refines a partition Q, then $B\langle Q\rangle \subseteq B\langle P\rangle$: if $\bigvee X$ is an element of $B\langle Q\rangle$ for some $X \subseteq Q$, then $\bigvee\{y \in P \mid (\exists x \in X)\, y \leq x\}$ in $B\langle P\rangle$ is equal to $\bigvee X$.

Recall that by Corollary 5.3 an isomorphism between finite Boolean algebras is determined by any bijection between their sets of atoms. In the present context, we can formulate this result as follows:

Lemma 6.4 *Let A, B be Boolean algebras and $P \subseteq A$ a partition of A and $Q \subseteq B$ a partition of B. If $|P| = |Q|$, then any bijection f between P and Q extends to an isomorphism $\bar{f}$ between $A\langle P\rangle$ and $B\langle Q\rangle$.*

Proof. Assume that $f : P \to Q$ is a bijection. We extend f to $\bar{f}$ by setting

$$\bar{f}(\bigvee X) = \bigvee\{f(x) \mid f(x) \in X\}, \tag{6.27}$$

for every $X \subseteq P$. In order to verify that $\bar{f}$ is the desired isomorphism between $A\langle P\rangle$ and $B\langle Q\rangle$, it suffices to show that $\bar{f}$ is an isomorphism between the canonical partial orders (see Observation 2.18). Clearly $\bar{f}(0) = 0$, and $\bar{f}(1) = 1$.

If $b \leq b'$, where $b = \bigvee X$ and $b' = \bigvee X'$, note that the following statements are equivalent: $\bigvee X \leq \bigvee X' \;\Leftrightarrow\; X \subseteq X' \;\Leftrightarrow\; \{f(x) \mid x \in X\} \subseteq \{f(x') \mid x' \in X\}$ $\Leftrightarrow\; \bigvee\{f(x) \mid x \in X\} \leq \bigvee\{f(x') \mid x' \in X'\}$. □

The above analysis of Boolean algebras determined by partitions can be used to construct an isomorphism between atomless countable Boolean algebras.

Theorem 6.5 *All countable atomless Boolean algebras are isomorphic.*

Proof. Let A and B be two countable atomless Boolean algebras. We will construct an isomorphism between A and B.

Let $\langle a_i \mid 0 \leq i < \omega\rangle$ be an enumeration of elements of A. Set $P_0 = \{1\}$. We will construct a sequence of partitions of A, $\langle P_i \mid i < \omega\rangle$, such that $P_{i+1} \ll P_i$ and $a_i \in A\langle P_{i+1}\rangle$. We will require that every element $x \in P_i$ is partitioned into exactly two elements in P_{i+1}.

Assuming P_i has been constructed, we define P_{i+1}. For every $x \in P_i$ we add into P_{i+1} exactly two elements x_0 and x_1 which partition x. We distinguish two cases. (i) If $0 < x \wedge a_i < x$, set $x_0 = x \wedge a_i$ and $x_1 = x \wedge (-a_i)$. (ii) Otherwise choose an arbitrary partition $\{x_0, x_1\}$ of x. Notice that x_0 and x_1 always exist because A is atomless. By construction,

$$a_i = \bigvee\{x \wedge a_i \mid x \in P_i \;\&\; 0 < (x \wedge a_i) < x\} \vee \bigvee\{x' \in P_{i+1} \mid (\exists x \in P_i)\, x \leq a_i \;\&\; x' \leq x\}, \tag{6.28}$$

and so $a_i \in A\langle P_{i+1}\rangle$ as required.

As every element in P_i is refined by two elements in P_{i+1}, $|P_i| = 2^i$ for every $i \in \omega$. Since the construction eventually captures every a_i,

$$A = \bigcup_{i\in\omega} A\langle P_i\rangle. \tag{6.29}$$

Given an enumeration $\langle b_i \mid 1 \leq i < \omega\rangle$ of B, we can repeat the above construction and obtain a sequence of partitions $\langle Q_i \mid i \in \omega\rangle$ such that

$$B = \bigcup_{i\in\omega} B\langle Q_i\rangle, \tag{6.30}$$

where the size of Q_i is 2^i and Q_{i+1} refines Q_i for every i (by refining every element in Q_i into exactly two elements).

By induction, we construct a sequence of partial isomorphisms $\langle \bar{f}_i \mid i \in \omega\rangle$ such that $\bar{f}_i \subseteq \bar{f}_{i+1}$ for every $i \in \omega$ and $\bar{f}_i$ is an isomorphism between $A\langle P_i\rangle$ and $B\langle Q_i\rangle$. Define $\bar{f}_0$ to be the isomorphism between $\{0^A, 1^A\}$ and $\{0^B, 1^B\}$. Assuming $\bar{f}_i$ has already been constructed, in order to define $\bar{f}_{i+1}$, we first define a suitable bijection f_{i+1} between P_{i+1} and Q_{i+1}, and then extend f_{i+1} to $\bar{f}_{i+1}$ by Lemma 6.4. For every $x \in P_i$, let $\{x_0, x_1\}$ be the partition of x in P_{i+1} and $\{y_0, y_1\}$ the partition of $\bar{f}_i(x)$ in Q_{i+1}. Define f_{i+1} as follows: for every $x \in P_i$,

$$f_{i+1}(x_0) = y_0 \text{ and } f_{i+1}(x_1) = y_1.$$

Let $\bar{f}_{i+1}$ be as in Lemma 6.4. It remains to show that $\bar{f}_i \subseteq \bar{f}_{i+1}$. For $\bigvee X \in A\langle P_i\rangle$, let us denote by X^* the set $\{y \in P_{i+1} \,|\, (\exists x \in X)\, y < x\}$. Clearly, $\bigvee X = \bigvee X^*$, and $\bar{f}_i(\bigvee X) = \bar{f}_{i+1}(\bigvee X^*)$.

We finish the proof by arguing that

$$F = \bigcup_{i\in\omega} \bar{f}_i \tag{6.31}$$

is the desired isomorphism between A and B. Let $a_i \leq_A a_j$ in A be given, we wish to show that this is equivalent to $F(a_i) \leq_B F(a_j)$ in B. Setting $k = \max(i, j) + 1$, both a_i and a_j are elements of $A\langle P_k\rangle$. Since $\bar{f}_k$ is an isomorphism, we can conclude that $F(a_i) = \bar{f}_k(a_i) \leq_B \bar{f}_k(a_j) = F(a_j)$ in the subalgebra $B\langle Q_k\rangle$, and also in B. The converse direction is the same. □

Example. The canonical example of a countable atomless Boolean algebra is the Clopen algebra of the Cantor space, $\mathrm{CO}(2^\omega, \tau)$. Another interesting example is the Lindenbaum-Tarski algebra of theories such as PA and ZFC.

Let T be a theory and κ an infinite cardinal. If T has models of size κ and all models of T of size κ are isomorphic, we say that T is κ-categorical. Theorem 6.5 implies that the theory of atomless Boolean algebras is $\aleph_0$-categorical. Categoricity is useful in showing that certain theories are complete:

Theorem 6.6 *The first order theory of atomless Boolean algebras is complete.*

Proof. Let T denote the first order theory which contains the usual axioms of Boolean algebras plus the formula which says that there is no element which has an atom below it:

$$(\forall x \neq 0)(\exists y) 0 < y < x. \tag{6.32}$$

We will show that for every sentence σ

$$\mathsf{T} \vdash \sigma \;\Leftrightarrow\; \mathrm{CO}(2^\omega, \tau) \models \sigma. \tag{6.33}$$

In particular for every σ, either $\mathsf{T} \vdash \sigma$ or $\mathsf{T} \vdash \neg\sigma$. The direction from left to right is obvious because $\mathrm{CO}(2^\omega, \tau)$ is a model of T. For the converse, assume for contradiction $\mathrm{CO}(2^\omega, \tau) \models \sigma$ and $\mathsf{T} \nvdash \sigma$. It follows $\mathsf{T} \cup \{\neg\sigma\}$ is consistent and has a countable model M, which is atomless by (6.32). By Theorem 6.5, M must be isomorphic to $\mathrm{CO}(2^\omega, \tau)$. However $M \models \neg\sigma$ and $\mathrm{CO}(2^\omega, \tau) \models \sigma$. Contradiction. □

6.2 Antichains in Boolean algebras

An antichain is a certain subset of a Boolean algebra which provide structural information about the algebra. For instance by showing that every infinite Boolean algebra contains an infinite antichain, we can easily prove that any infinite complete Boolean algebra has size at least 2^ω, so in particular the algebra $\mathrm{CO}(2^\omega, \tau)$ is not complete.

Definition 6.7 *Let B be a Boolean algebra. We say that two elements $x \neq y$ in B are* disjoint *if*

$$x \wedge y = 0.$$

We say that a subset A of B is an antichain *if all x in A are non-zero and* pairwise disjoint*, i.e. if $x \neq y$ are in A, then x and y are disjoint.*

It is easy to see that if x, y are disjoint, then they are incomparable in the canonical ordering of the Boolean algebra.

The complement operation in a Boolean algebra B allows us to define a binary operation of *subtraction*:

$$x - y = x \wedge -y,$$

for arbitrary x, y in B.

We will use this operation in the following lemma.

Theorem 6.8 *Let B be a complete Boolean algebra which has an antichain of size ω. Then the size of B is at least 2^ω.*

Proof. Let $A = \{a_i \,|\, i < \omega\}$ be a countable antichain in B. We will construct a bijection i from the powerset of ω into B, thus showing

$$|\mathscr{P}(\omega)| = 2^\omega \leq |B|. \tag{6.34}$$

Given X a subset of ω, define a subset of A_X of A by $A_X = \{a_n \,|\, n \in X\}$. Define $i : \mathscr{P}(\omega) \to B$ by

$$i(X) = \bigvee A_X, \text{ for every } X \subseteq \omega.$$

Since B is complete, the supremum $\bigvee A_X$ exists for every A_X, and so the definition of i makes sense.

We will show that if $X \neq Y$ are two subsets of ω, then $i(X) \neq i(Y)$, thus showing (6.34). Assume for contradiction that there are $X \neq Y$ such that $x = \bigvee A_X = \bigvee A_Y$. Without loss of generality, let i be some element of ω such that $i \in X$ and $i \notin Y$. We will show that $x - a_i$ is strictly smaller than x, but it is still an upper bound of A_Y, which contradicts the assumption that x is the supremum of A_Y.

The element x is an upper bound of A_Y, and so $x \geq a_j$ for every $a_j \in A_Y$. Since $i \notin Y$, and $a_j \wedge a_i = 0$ for every $j \neq i$, $x - a_i$ is still an upper bound of A_Y:

$$a_j \leq x - a_i, \text{ or equivalently } a_j \wedge x \wedge -a_i = a_j, \text{ for every } j \in Y.$$

Since $a_i \leq x$ and $a_i > 0$, we obtain

$$x - a_i < x,$$

as desired. □

In order to apply this result to $\mathrm{CO}(2^\omega, \tau)$, we need to check if $\mathrm{CO}(2^\omega, \tau)$ contains an infinite antichain. In fact, a more general result is true:

Lemma 6.9 *Every infinite Boolean algebra B contains:*

(i) An infinite decreasing sequence.
(ii) An infinite antichain.

Proof. Assume first that B is not atomic. Then there exists an element $x > 0$ in B for which there exists no atom $a < x$. This implies that there exists an infinite decreasing sequence $x = x_0 > x_1 > x_2 > \ldots$, which shows (i). From this sequence one can define an antichain as follows: set for each $i \in \omega$,

$$a_i = x_i - x_{i+1}.$$

Then $\{a_i \mid i \in \omega\}$ is an infinite antichain, which shows (ii).

Now assume that B is atomic. Realize that B must contain an infinite number of atoms: the function f in (5.17) in Theorem 5.2 is injective because B is atomic, and thus $\mathscr{P}(\mathrm{At}(B))$ and also $\mathrm{At}(B)$ must be infinite. It follows that $\mathrm{At}(B)$ is an infinite antichain, which shows (ii). Let $\{a_i \mid i \in \omega\}$ be a countable subset of $\mathrm{At}(B)$; use this set to define a decreasing chain as follows: set for each $i \in \omega$,

$$x_i = -\bigvee\{a_n \mid n \leq i\}.$$

Then $-a_0 = x_0 > x_1 > x_2 > \ldots$ is a decreasing chain, which shows (i). □

Corollary 6.10 $\mathrm{CO}(2^\omega, \tau)$ *is not complete.*

Proof. Follows by Theorem 6.8 and Lemma 6.9. □

We have shown that the least size of an infinite complete Boolean algebra is 2^ω. Notice that there are complete Boolean algebra of size 2^ω: for instance the powerset algebra $\mathscr{P}(\omega)$.

Remark 6.11 By Theorem 6.5, the atomless countable Boolean algebra is unique. By Fact 4.14, the completion of the atomless countable Boolean algebra exists and is also unique up to isomorphism. This unique completion of the atomless countable Boolean algebra is called the *Cohen algebra*, and is sometimes denoted by $\mathbf{C}$. By Theorem 6.8, $\mathbf{C}$ has size at least 2^ω; in fact, it can be shown that

$$|\mathbf{C}| = 2^\omega.$$

Cohen algebra is named after P. Cohen who proved in 1962 that if ZF is consistent, so is $\mathsf{ZFC} + \neg\mathsf{CH}$. The method of proof he used is called *forcing*. In modern terminology, forcing can be defined with respect to a complete Boolean algebra. The original Cohen's proof can be viewed as a forcing with the Cohen algebra $\mathbf{C}$.

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