QUANTUM INFORMATION

MFF UK

(0) Find density matrix of the state $|+\rangle$ (the usual uniform combination of $|0\rangle$ and $|1\rangle$) and of the mixed state corresponding to picking either $|0\rangle$ or $|1\rangle$ with the same probability.

(1) Prove that positive operators are Hermitian and so are normal, as well.

(2) Verify that equality $tr(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$ holds, where $|\psi\rangle$ is a unit vector. To prove this, it may be of help to first show that the trace of a matrix B can be computed as $\sum_i \langle a_i|B|a_i\rangle$, where $|a_i\rangle$ form an orthonormal basis.

Recall that a mixed state is defined as a convex combination of pure states, where we represent the state $|\psi\rangle$ by its density matrix $|\psi\rangle\langle\psi|$.

(3) Show that a convex combination of mixed states is again a mixed state. Then, verify that given a mixed state ρ and an evolution matrix U, the mixed state of the system after applying U is $U\rho U^\dagger$.

(4) Recall that a *projective measurement* is defined as a Hermitian operator P, where the measurement values to eigenvectors of P. Show that this type of measurement can be viewed as a particular case of a more general measurement defined by a set of operators $\{M_i\}$ satisfying $\sum_i M_i^\dagger M_i = I$.

(5) Show that, given a mixed state ρ and measurement $\{M_i\}$, all the relevant statistics regarding the outcome of measurement can be described in terms of ρ only.

(6) Let ρ^{AB} be a density matrix of a mixed state in a combined system. Show that ρ^A is a density matrix, i.e. it is positive with eigenvalues summing up to 1.

(7) * Derive a formula for computing density matrix of a state $|a\rangle \otimes |b\rangle$.

(8) Compute density matrix of a state $\frac{1}{\sqrt{2}}(|0\rangle \otimes |+\rangle) + \frac{1}{\sqrt{2}}(|1\rangle \otimes |-\rangle)$. Then, compute the reduced density matrix onto the first system of the given state.

(9) Let $\rho^{AB} = \sum_i c_i \rho^{a_i} \otimes \rho^{b_i}$ be a state of a composite system. Let $\{M_i\}_i$ be a measurement operators of A. Show that it is the case that the result of measuring ρ^A by $\{M_i\}_i$ is the same as the result of measuring ρ^{AB} by $\{M_i \otimes I\}_i$.

All the random variables considered below are assumed to be discrete and finitely supported.

(10) Prove that $H(X,Y)$ is upper bounded by $H(X) + H(Y)$ and that $H(X) - H(X|Y) = H(Y) - H(Y|X)$.