# Large cardinals and the Continuum Hypothesis

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**Abstract.** This is a survey paper which discusses the impact of large cardinals on provability of the Continuum Hypothesis (CH). It was Gödel who first suggested that perhaps "strong axioms of infinity" (large cardinals) could decide interesting set-theoretical statements independent over ZFC, such as CH. This hope proved largely unfounded for CH – one can show that virtually all large cardinals defined so far do not affect the status of CH. It seems to be an inherent feature of large cardinals that they do not determine properties of sets low in the cumulative hierarchy if such properties can be forced to hold or fail by small forcings.

The paper can also be used as an introductory text on large cardinals as it defines all relevant concepts.

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## 1 Introduction

The question regarding the size of the continuum – i.e. the number of the reals – is probably the most famous question in set theory. Its appeal comes from the fact that, apparently, everyone knows what a real number is and so the question concerning their quantity seems easy to understand. While there is much to say about this apparent simplicity, we will not discuss this issue in this paper. We will content ourselves by stating that the usual axioms of set theory (ZFC) do not decide the size of the continuum, except for some rather trivial restrictions.[1] Hence it is consistent, assuming the consistency of ZFC, that the number of reals is the least possible, i.e. the cardinal $\aleph_1$, but it can be something much larger, e.g. $\aleph_{\aleph_1}$.

The statement that the number of reals is the least possible is known as the *Continuum Hypothesis*, CH, for short:

$$\text{CH:}\ \ |\mathbb{R}| = 2^{\aleph_0} = \aleph_1.$$

CH was made famous by David Hilbert who included this problem as the first one on his list of mathematical problems for the 20th century (see for instance [5]).

Since ZFC does not decide CH, are there any natural candidates for axioms which do? That is, is there a statement $\varphi$ without apparent connection to CH which

[1] The cofinality of the size of the continuum must be uncountable.

decides CH one way or the other? In fact there are many of these, such as MA or PFA,[2] but we will require $\varphi$ to be one of a more special kind. In 1946, that is well before the development of forcing, Gödel entertained the idea of so called *stronger axioms of infinity* deciding CH (and other independent statements as well):[3]

> It is not impossible that [...] some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets ([1]).

A natural way to arrive at "true assertions" about largeness of the universe of sets is to take up analogies with natural numbers. When we compare the theory of arithmetics such as PA with the theory of sets such as ZFC, we can show that the only important strengthening of ZFC over PA is the addition of the axiom of infinity. The axiom can be formulated in many ways, but for our purposes we adopt the following definition:

(*) Axiom of Infinity: *There is an ordinal $\omega$ which is the domain of a model for the formalization of PA.*

Because of this axiom, ZFC can not only prove some logical arithmetical statements which PA itself cannot prove (unless it is inconsistent), such as $\mathrm{Con}(PA)$, but also some purely number-theoretical statements as well (such as Goodstein's theorem, see for instance [17]). Gödel suggested that perhaps by adding a stronger axiom of infinity to ZFC, this new theory might decide new statements interesting to set theoreticians.[4] Can we find such an axiom, perhaps similar to (**) or (***) below, which will decide CH?

(**) A strong Axiom of Infinity: *There is a regular cardinal $\kappa$ such that $\langle V_\kappa, \in\rangle$ is a model of the formalization of ZFC.*

or

(***) A still stronger Axiom of Infinity: *There is a regular cardinal $\kappa$ such that $\langle V_\kappa, \in\rangle$ is a model of the formalization of ZFC + (**).*

Where $V_\kappa$ is an initial part of the universe of sets (see Definition 2.1) and is the analogue of $\omega$ for sets.

**Remark.** This paper is in a sense a continuation of [6] which contains an introduction to the axioms of set theory, discusses the basic set-theoretical notions and not so briefly reviews basics of forcing. Of course, any of the standard texts such as [7] or [10] contains all the prerequisites to this article. A standard reference book for large cardinals is [8] where an interested reader can find more details.

[2] See footnote 27.

[3] As regards the intuitive "truth" of such axioms, or why they should be preferable to other types of axioms, see a discussion for instance in [2].

[4] Such extensions will always decide new statement, such as $\mathrm{Con}(ZFC)$, but these are considered too "logical" and not properly set-theoretical.

## 2 How to find large cardinals

In this section we survey large cardinals which can be considered as candidates for the stronger axioms of infinity. The selection is rather arbitrary, but does attempt to do justice to the most important concepts.

### 2.1 Inaccessible cardinals

In the presence of the Axiom of Foundation,[5] the universe $V$ is equal to the union $V = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$, where the initial segments $V_\alpha$ are defined by recursion along the ordinal numbers ORD as follows:

**Definition 2.1**

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= \mathscr{P}(V_\alpha), \\ V_\lambda &= \textstyle\bigcup_{\alpha<\lambda} V_\alpha, \; \textit{for } \lambda \textit{ limit ordinal}, \\ V &= \textstyle\bigcup_{\alpha \in \mathrm{ORD}} V_\alpha. \end{aligned}$$

If $\kappa$ satisfies (**) above, we say that $V_\kappa$ is a "natural model" of set theory. To obtain such a model in set theory, we must transgress the power of the plain ZFC theory – this is a consequence of the second Gödel theorem.

What are the properties of a cardinal $\kappa$ such that $V_\kappa$ satisfies (**) above? We postulated that it must be regular (we will later see that we cannot avoid this assumption), but what else?

**Definition 2.2** *We say that a cardinal $\mu$ is* strong limit *if for all $\nu < \mu$, $2^\nu < \mu$.*

Notice that every strong limit cardinal is also limit (i.e. does not have an immediate cardinal predecessor).

**Lemma 2.3** *Assume $\kappa$ satisfies (**). Then $\kappa$ is strong limit.*

*Proof.* Assume $\mu < \kappa$ is given. ZFC proves that there is a cardinal $\nu \geq \mu^+$ and a bijection $f : \nu \to \mathscr{P}(\mu)$. Since $\langle V_\kappa, \in \rangle$ is a model of ZFC, we have

$$\langle V_\kappa, \in \rangle \models \text{"There is a bijection between } \mathscr{P}(\mu) \text{ and some } \nu \geq \mu^+\text{"}.$$

Since $V_\kappa$ is transitive, and $\mathscr{P}(\mu) = (\mathscr{P}(\mu))^{V_\kappa}$,[6] any such bijection in $V_\kappa$ is really (in $V$) a bijection between $\mathscr{P}(\mu)$ and some ordinal $\nu$ in $V_\kappa$. As $\nu < \kappa$, $2^\mu < \kappa$. □

Notice that for a regular $\kappa$, if $\mu < \kappa$, then $\mathscr{P}(\mu) \in V_\kappa$ (see Footnote 6); however, this does not generally imply that $\kappa$ is strong limit because the existence of a bijection

[5] This axioms states that sets are "well-behaved"; for instance sets $x$ such as $x \in x$ are prohibited by this axiom.

[6] $(\mathscr{P}(\mu))^{V_\kappa}$ is the powerset of $\mu$ in the sense of $\langle V_\kappa, \in \rangle$. Note that for every limit ordinal $\alpha$, if $\beta < \alpha$, then $(\mathscr{P}(\beta))^{V_\alpha} = \mathscr{P}(\beta)$ because $\mathscr{P}(\beta) \subseteq V_{\beta+1}$, and so $\mathscr{P}(\beta) \in V_{\beta+2} \subseteq V_\alpha$.

between $\mathscr{P}(\mu)$ and some $\nu$ in $V_\kappa$ depends on the truth of the Replacement schema in $V_\kappa$. In fact, we state without a proof that if $\kappa$ is a regular cardinal, then all axioms of ZFC, except possibly some instances of the Schema of Replacement, are true in $\langle V_\kappa, \in\rangle$.

Lemma 2.3 motivates the following definitions:

**Definition 2.4** *We say that a cardinal $\kappa > \omega$ is* weakly inaccessible *if it is* regular *and* limit.

**Definition 2.5** *We say that a cardinal $\kappa > \omega$ is* strongly inaccessible *if it is* regular *and* strongly limit.

**Theorem 2.6** *(i) Every cardinal satisfying (**) is strongly inaccessible.*
*(ii) Every strongly inaccessible cardinal satisfies (**).*

*Proof.* Ad (i) Obvious from the definitions and Lemma 2.3.

Ad (ii) (Sketch). For every regular $\kappa$, $\langle V_\kappa, \in\rangle$ is a model of ZFC without Schema of Replacement (this is easy to check). Strong limitness is used to ensure that Replacement holds as well. □

Although it may not be immediately apparent, the weakly inaccessible cardinal is not weaker in terms of consistency strength than the strongly inaccessible cardinal. Let $\exists\kappa\ \psi_w(\kappa)$ denote the sentence "there exists a weakly inaccessible cardinal", and similarly for the strongly inaccessible $\exists\kappa\ \psi_s(\kappa)$.

**Lemma 2.7**

$$\mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ \psi_w(\kappa)) \leftrightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ \psi_s(\kappa)).$$

*Proof.* The more difficult direction is from left to right. Assume $\kappa$ is weakly inaccessible. Let $L$ be the universe of constructible set, defined by Gödel. We know that $L$ satisfies ZFC and also GCH.[7] It is immediate to see that in $L$, $\kappa$ is strongly inaccessible because being a limit cardinal together with GCH implies the desired property of strong limitness. □

Therefore by Gödel's theorem and Lemma 2.7 and Theorem 2.6(ii):

**Corollary 2.8** *If ZFC is consistent, it does not prove the existence of a weakly inaccessible cardinal.*

One can also show that if ZFC is consistent, so is the theory ZFC + "there is no strongly inaccessible cardinal", and that ZFC does not prove the implication CON(ZFC) $\rightarrow$ CON(ZFC+ "there is a strongly inaccessible cardinals").

Usually, when we talk about an *inaccessible cardinal*, we mean the strongly inaccessible, and assumption of existence of such a cardinal number is taken to be the first step in defining strong axioms of infinity. Thus we can reformulate:

$(**)_r$: (Strong Axiom of Infinity) There is a (strongly) inaccessible cardinal.

[7] The Generalized Continuum Hypothesis which states that for every cardinal $\mu$, $2^\mu = \mu^+$.

**Remark 2.9** One might wonder if we can remove the assumption of regularity in (**) and have an equivalent notion. We cannot: if $\kappa$ is strongly inaccessible, we can use the standard Löwenheim-Skolem argument to obtain an elementary substructure $\langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle$ with $\alpha > \omega$, and $\mathrm{cf}(\alpha) = \omega$. Thus $\langle V_\alpha, \in\rangle$ is a model of ZFC, but $\alpha$ is not regular.[8] That is why we need to explicitly postulate the regularity of $\kappa$ in (**).

What about (***)? Well, it is not difficult to see that if $\kappa < \kappa'$ are two strongly inaccessible cardinals, then $\langle V_{\kappa'}, \in\rangle$ is the desired model for (***). This is the case because

$$\langle V_{\kappa'}, \in\rangle \models \text{“}\kappa \text{ is a strongly inaccessible cardinal”}.$$

Thus we may reformulate:

$(***)_r$: (Still stronger Axiom of Infinity) There are two (strongly) inaccessible cardinals.

We could repeat this argument many times over, obtaining stronger and stronger axioms of infinity, in the hope of deciding more and more sentences. However, there is a limit to this recursion – so called Mahlo cardinals (see the next section).

## 2.2 Mahlo cardinals

We include this cardinal only because it is in a sense a limit to the process of arriving to a large cardinal by a process “from below”. Recall that above we have considered one, two, three, and so on inaccessible cardinals. What if we consider $\aleph_0$ or $\aleph_1$ many of them? Do we get something yet stronger? We do, but there is a natural limit to this type of strengthening of the notion of a strong axiom of infinity. Consider an inaccessible cardinal $\kappa$ such that $\kappa$ is the $\kappa$-th inaccessible cardinal – clearly, it is a limit of the process of taking more and more inaccessible cardinals as far as their number is concerned. A Mahlo cardinal is even stronger (although it may not be apparent without a more detailed look which we will not provide here):

**Definition 2.10** *A cardinal $\kappa$ is a* Mahlo cardinal *if the set of inaccessible cardinals smaller than $\kappa$ is stationary in $\kappa$.*[9]

## 2.3 Analogies with $\omega$

We said above that Mahlo cardinals are a limit to arriving to larger cardinals “from below” by repeating certain continuous processes applied to inaccessible cardinals. But what other options do we have? Mathematicians found out that it is useful to consider the usual properties of $\omega$ and try to generalize them in a suitable fashion. In fact, inaccessible cardinals can be regarded in this way – either as a generalization

[8] But it may be a singular strong limit cardinal.

[9] We will not define the notion of a stationary set here; any standard set-theoretical textbook contains this definition. Roughly speaking, a set is stationary in $\kappa$ if it intersects every continuous enumeration of unboundedly many elements below $\kappa$. In particular, every stationary subset of a regular cardinal $\kappa$ has size $\kappa$.

of the concept of a "model" for a given theory (see above in (**) and (***)), or combinatorially – notice that $\omega$ itself is *regular* and *strong limit*, i.e. no finite subset of $\omega$ is cofinal in $\omega$ and $\forall n < \omega\ 2^n < \omega$. We generalize[10] three other properties of $\omega$:[11]

- (C) $\omega$ is *compact* in the sense of the compactness theorem for the first-order predicate logic.
- (M) There is a two-valued non-trivial *measure* on $\omega$, i.e. a non-principal ultrafilter on $\omega$. This measure is $\omega$-complete: for every finite number of elements in the ultrafilter, their intersection is still in the ultrafilter.
- (R) The Ramsey property holds for $\omega$: $\omega \to (\omega)^r_k$, for $r, k < \omega$.

**Compactness (C).** The classical predicate calculus satisfies compactness: for every language and for every set of formulas $A$ (of arbitrary size) in that language if every finite subset $B \subseteq A$ has a model, so does $A$. In order to generalize this property, we consider an extension of the classical logic denoted as $L_{\kappa,\kappa}$, where $\kappa$ is a regular cardinal, as follows. A language in $L_{\kappa,\kappa}$ can have up to $\kappa$ many variables and an unlimited number of non-logical symbols (functions, constants, predicates). We also allow conjunctions and disjunctions of length $< \kappa$ and quantifications over $< \kappa$ many variables.[12] The classical logic can be denoted as $L_{\omega,\omega}$ under this notation. Now we can formulate the generalization of the compactness theorem in two ways:

- (wC) $\kappa > \omega$ is called *weakly compact* iff whenever $A$ is any collection of sentences in $L_{\kappa,\kappa}$ with at most $\kappa$ many non-logical symbols if every $B \subseteq A$ of size $< \kappa$ has a model, so does $A$.
- (sC) $\kappa > \omega$ is called *strongly compact* iff whenever $A$ is any collection of sentences in $L_{\kappa,\kappa}$ if every $B \subseteq A$ of size $< \kappa$ has a model, so does $A$.

We will discuss the relationship between (wC) and (sC) later in the text.

**Measure (M).** One can find a non-principal ultrafilter on $\omega$, i.e. a set $U \subseteq \mathscr{P}(\omega)$ such that for all $A, B$ subsets of $\omega$:

- (i) If $A \in U$ and $A \subseteq B$, then $B \in U$.
- (ii) If $A, B \in U$, then $A \cap B \in U$.
- (iii) For no $n < \omega$, $\{n\} \in U$.
- (iv) For all $A$, either $A \in U$ or $\omega \setminus A \in U$.

Note that by induction, (ii) implies that if $A_0, \ldots, A_n$ are sets in $U$ for $n < \omega$, then their intersection is in $U$ – this property can be called *$\omega$-completeness* to emphasize the analogy with *$\kappa$-completeness* for a cardinal $\kappa > \omega$ introduced below. $U$ is non-principal because it is not generated by a single number (property (iii)); (iii) together with other properties implies that every set $A \in U$ is infinite.

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[10] We assume AC, the Axiom of Choice, in formulating these generalizations.

[11] Note that a priori there is no guarantee that we get anything like a large cardinal in this fashion; the generalization may turn out to be mathematically trivial and uninteresting. The fact that we do get large cardinals seems to indicate that these generalizations are mathematically relevant.

[12] For instance "$\exists_{\beta<\alpha} x_\beta \varphi$", $\alpha < \kappa$, quantifies over $\alpha$-many variables in $\varphi$.

(M) $\kappa > \omega$ is called *measurable* iff there is a $\kappa$-complete non-principal ultrafilter $U$ on $\kappa$:

(i) If $A \in U$ and $A \subseteq B$, then $B \in U$.

(ii) If $\mu < \kappa$, and $\{A_\xi \mid \xi < \mu\}$ are sets in $U$, then $\bigcap_{\xi<\mu} A_\xi$ is in $U$.

(iii) For no $\xi < \kappa$, $\{\xi\} \in U$.

(iv) For all $A$, either $A \in U$ or $\kappa \setminus A \in U$.

Such an ultrafilter $U$ is often called a *measure* because it "measures" subsets of $\kappa$ by a two-valued $\kappa$-complete measure: if $A \in U$, then measure of $A$ is 1, if $A \notin U$, then its measure is 0.

**Ramsey partitions (R).** Let $f$ be a function from $[\omega]^r$ to $k$, where $[\omega]^r$ denotes the set of all subsets of $\omega$ with exactly $r$ elements, and $k = \{0, \dots, k-1\}$ is a set of size $k$ ($r \geq 1$ and $k \geq 2$ to avoid trivialities).

**Definition 2.11** *We say that* $A \subseteq \omega$ *is* homogeneous *for* $f : [\omega]^r \to k$ *if*

$$|\mathrm{rng}(f \restriction [A]^r)| = 1.$$

Ramsey proved in 1930 that for any such $f$ there is an infinite homogeneous subset, in the arrow notation:

$$\omega \to (\omega)^r_k, \text{ for } r, k < \omega.$$

The argument is by induction on $r$, and the nontrivial step is to show

$$\omega \to (\omega)^2_2.$$

This we read that for any partition of two-element subsets of $\omega$ to two sets we can find an infinite homogeneous set. We therefore attempt to generalize:

(wR) A cardinal $\kappa > \omega$ is called *weakly Ramsey* if $\kappa \to (\kappa)^2_2$, i.e. for every partition of two-element subsets of $\kappa$ to two sets we can find a homogeneous set of size $\kappa$.

We later learn that this generalization is not getting us a new concept, so we will need to strengthen it. That is why we call this property (wR) and not (R). See the next section for the now standard definition of the Ramsey cardinal.

## 2.4 Compact, measurable, and Ramsey cardinals

As we mentioned above, there is a priori no guarantee that the cardinals defined above under (wC), (sC), (M), and (wR) are even inaccessible. However, as it turns out, they are not only inaccessible but even Mahlo. By way of illustration, we show that a measurable cardinal $\kappa$ is inaccessible.

**Theorem 2.12** *Every measurable cardinal is inaccessible.*

*Proof.* Let $U$ be a non-pricipal $\kappa$-complete ultrafilter witnessing measurability of $\kappa$. First notice that by $\kappa$-completeness and non-principality of $U$, all elements in $U$ have size $\kappa$. $\kappa$ is clearly regular, otherwise if $\{\xi_\alpha \mid \alpha < \mathrm{cf}(\kappa)\}$ is cofinal in $\kappa$ for $\mathrm{cf}(\kappa) < \kappa$, then while for each $\alpha < \mathrm{cf}(\kappa)$, $\xi_\alpha \notin U$, $\bigcup_{\alpha<\mathrm{cf}(\kappa)} \xi_\alpha = \kappa \in U$, contradicting the $\kappa$-completeness of $U$.[13] As regards strong limitness of $\kappa$, assume for contradiction that for some $\lambda < \kappa$, we have $2^\lambda \geq \kappa$, and let $f : \kappa \to \mathscr{P}(\lambda)$ be an injection. For a fixed $\alpha < \lambda$, we can consider two subsets of $\kappa$ given by $f$: $X_0^\alpha = \{\xi < \kappa \,|\, \alpha \in f(\xi)\}$ and $X_1^\alpha = \{\xi < \kappa \,|\, \alpha \notin f(\xi)\}$. For each $\alpha < \lambda$, exactly one of the two sets $X_0^\alpha$ and $X_1^\alpha$ is in $U$; let us denote this set as $X^\alpha$. By $\kappa$-completeness of $U$, $X = \bigcap_{\alpha<\lambda} X^\alpha$ must be in $U$. However $X$ can have at most one element since $f$ is an injection – if $\xi \neq \zeta$ are in $X$, then $f(\xi) \neq f(\zeta)$ and hence at some $\alpha < \lambda$, $\xi$ must be in $X_0^\alpha$ and $\zeta$ in $X_1^\alpha$ (or conversely). This contradicts the non-principality of $U$. It follows that $\kappa$ is strong limit, and hence inaccessible. □

With nice combinatorial arguments, not always trivial ones, one can show that every strongly compact cardinal is measurable, every measurable is weakly compact, and every weakly compact is Mahlo, and every Mahlo is inaccessible. Thus disparate combinatorial notions gave rise to a linearly ordered scale of cardinals.

What about the weakly Ramsey cardinal? With a little work, it can be shown that the definition (wR) is in fact equivalent to (wC). And so the classes of weakly compact cardinals and weakly Ramsey cardinals are the same. However, there is a way how to generalize the Ramsey property and obtain something stronger than a weakly compact cardinal:

(R) A cardinal $\kappa > \omega$ is called *Ramsey* if $\kappa \to (\kappa)_2^{<\omega}$, i.e. for every partition of all finite subsets of $\kappa$ to two sets we can find a homogeneous set $A$ of size $\kappa$.[14]

Many questions concerning these cardinals are quite difficult. For instance, it had long been open (from 1930's to 1960's) whether the least inaccessible cardinal can be measurable. By a new method using elementary embeddings and ultrapowers developed by Scott, it was proved in the 1960's that measurable cardinals are quite large – they can never by the least inaccessible, or the least weakly compact cardinal. In fact if $\kappa$ is measurable, then it is the $\kappa$-th weakly inaccessible cardinal, and more. We will touch briefly on the method of elementary embeddings in Section 3.4.

Finally, let us note that measurable cardinals were first used – before the introduction of the Cohen's method of forcing – to argue for the consistency of the statement $V \neq L$, i.e. that there exists a non-constructible set. It was Scott [15] who showed in 1961 that if there exists a measurable cardinal, then $V \neq L$. Nowadays large cardinal which imply that $V \neq L$ are called "large" large cardinals, while others are called "small" large cardinals. Inaccessible, Mahlo, and weakly compact cardinals are "small", while Ramsey, measurable and strongly compact are "large".

## 2.5 Motivation

We showed that by a natural attempt to generalize properties which hold for $\omega$, we arrive at interesting notions in set theory which form a linear scale, as regards the strength of the notions. This is often taken as a heuristical point in favour of the naturalness of the definitions. Not least because by the linearity, no two

[13] Notice that $\kappa$-completeness can be equivalently expressed as follows: whenever $\mu < \kappa$ and $\{X_\alpha \,|\, \alpha < \mu\}$ are sets not in $U$, the the union $\bigcup_{\alpha<\mu} X_\alpha$ is not in $U$, either.

[14] For every $n < \omega$, $|\mathrm{rng}(f \restriction [A]^n)| = 1$.

large cardinals are inconsistent together – so far, no large cardinal was found that prohibits the existence of some other large cardinals. The properties which can be generalized range from purely logical (such as the inaccessible cardinal witnessing (**), or (wC) and (sC)), to combinatorial (wR), (R) and measure-theoretic (M).

On the downside, all these notions substantially increase the consistency strength of the relevant theories, thus increasing the risk of introducing a contradiction. It is conceivable, but not considered probable now, that ZFC is consistent, while ZFC + "there is an inaccessible" is not. Or that ZFC + "there is a weakly compact cardinal" is consistent while ZFC + "there is a measurable cardinals" is not. See Section 4 for more discussion on consistency strength.

Such discussion are not of logical interest only. It can be shown for instance that a certain weakening of the GCH, denoted as SCH,[15] is provable in ZFC if ZFC refutes the existence of inaccessible cardinals.[16] However, with some large cardinals around, SCH cannot be proved, and is therefore independent over the theory ZFC + certain large cardinals.[17]

# 3 Large cardinals and CH

As we mentioned earlier, Gödel expressed his hopes that perhaps large cardinals could provide a natural extension of ZFC with interesting set-theoretical consequences such as determining the truth or falsity of CH. However, with the development of forcing on the way, Levy and Solovay in 1967 [11] came with arguments which are almost universal and show that truth or falsity of CH is unaffected by large cardinals. In the following sections, we assume some basic understanding of forcing on the reader's part.

## 3.1 How to force CH and ¬CH

A standard forcing notion to force CH, which we will denote as $\mathbb{P}_{\mathrm{CH}}$, is composed of functions $f : \omega_1 \to 2$ with the domain of $f$ being at most countable. The extension is by reverse inclusion. $\mathbb{P}_{\mathrm{CH}}$ adds a new subset of $\omega_1$, and collapses $2^\omega$ to $\omega_1$ in the process.[18] $\mathbb{P}_{\mathrm{CH}}$ is called the *Cohen forcing for adding a subset of* $\omega_1$.

To force ¬CH, we will use $\omega_2$ copies of the *Cohen forcing which adds a new subset of* $\omega$. Formally, a condition in $\mathbb{P}_{\neg\mathrm{CH}}$ is a function with finite domain from $\omega_2$ to 2. One can show that $\mathbb{P}_{\neg\mathrm{CH}}$ preserves cardinals and forces $2^\omega = \omega_2$.

For our purposes notice that $|\mathbb{P}_{\mathrm{CH}}| = 2^\omega$ and $|\mathbb{P}_{\neg\mathrm{CH}}| = \omega_2$, i.e. both forcings are quite small, certainly smaller than the first inaccessible.

[15] GCH, the Generalized Continuum Hypothesis, states that for all cardinals $\kappa$, $2^\kappa = \kappa^+$. SCH, the Singular Cardinal Hypothesis, states that for all singular cardinals $\kappa$, $2^\kappa = \max(2^{\mathrm{cf}(\kappa)}, \kappa^+)$.

[16] This is true for larger cardinals than just inaccessibles.

[17] For instance if ZFC + (sC) is consistent, so is ZFC + ¬SCH.

[18] Notice that for every $X \subseteq \omega$ in $V$, it is dense in $\mathbb{P}_{\mathrm{CH}}$ that there exists some $\alpha < \omega_1$ and $p$ such that $p$ restricted to $[\alpha, \alpha + \omega)$ is a characteristic function of $X$. The function defined in a generic extension which takes every $\alpha < \omega_1$ to a subset of $\omega$ given by the restriction of the generic filter to $[\alpha, \alpha + \omega)$ is therefore onto $(2^\omega)^V$. It follows that $2^\omega$ of $V$ is collapsed to $\omega_1$.

## 3.2 Inaccessible and Mahlo cardinals and CH

We have defined above two "small" forcings which can force CH and ¬CH, $\mathbb{P}_{\mathrm{CH}}$ and $\mathbb{P}_{\neg\mathrm{CH}}$, respectively. As it turns out, for the preservation of large cardinals, it suffices to assume that the forcing in question has size $< \kappa$.

**Theorem 3.1** *Let $P$ be a forcing of size $< \kappa$ and let $G$ be a $P$-generic filter. Assume $\kappa$ is inaccessible or Mahlo in $V$, then $\kappa$ is inaccessible or Mahlo, respectively, in $V[G]$. In particular, these large cardinals do not decide CH.*

*Proof.* First notice that the theorem really implies that these large cardinals do not decide CH. Suppose for contradiction that one of these cardinals decides CH; for example let us assume that ZFC + "there is an inaccessible" proves CH. Assume there is an inaccessible and force with $\mathbb{P}_{\neg\mathrm{CH}}$; we obtain a generic extension where ¬CH holds and there is still an inaccessible. This a contradiction.

Let us now turn to the proof of the rest of the theorem. By standard forcing technique, if $\lambda < \kappa$ is given, then there are just $2^{|P|\lambda}$-many nice names for subsets of $\lambda$ in $V[G]$. Since $\mu = 2^{|P|\lambda} < \kappa$ by inaccessibility of $\kappa$, we have $V[G] \models 2^\lambda \leq \mu < \kappa$, i.e. $\kappa$ remain inaccessible in $V[G]$.[19]

To argue for preservation of Mahloness, we show as a lemma that forcings with $\kappa$-cc preserve stationarity of subsets of $\kappa$.

**Lemma 3.2** *Assume $Q$ is a forcing notion. If $Q$ is $\kappa$-cc, then it preserves stationary subsets of $\kappa$.*

*Proof.* Let $V[E]$ be a $Q$-generic extension and $S$ stationary subset. We wish to show that $S$ is still stationary in $V[E]$: that is, we need to show that if $C \in V[E]$ is closed unbounded, then $S \cap C \neq \emptyset$. Fix a closed unbounded $C$ and let $p \in E$ force this:

$$p \Vdash \dot{C} \text{ is a closed unbounded subset of } \check{\kappa}.$$

Denote

$$D = \{\xi < \kappa \,|\, p \Vdash \check{\xi} \in \dot{C}\}.$$

Note that $D \subseteq C$ and $D \in V$. We prove that $D$ is a closed unbounded subset of $\kappa$. Now the claim follows because $D \in V$, and so $D \cap S \neq \emptyset$. To prove $D$ is closed unbounded, it suffices to argue that it is unbounded (closure is easy). Let $\alpha < \kappa$ be given. By induction construct for each $n < \omega$ a maximal antichain $A_n = \langle q_n^\xi \,|\, \xi < \alpha_n\rangle$ of elements below $p$ and an increasing sequence of ordinals $\langle \beta_n^\xi \,|\, \xi < \alpha_n\rangle$, where $\alpha_n < \kappa$ (this is possible by $\kappa$-cc), such that:

(a) $\beta_0^0 \geq \alpha$;
(b) for each $n$, $\langle \beta_n^\xi \,|\, \xi < \alpha_n\rangle$ is strictly increasing;
(c) if $m < n$ then all elements in the $\beta_n$-sequence are above the $\beta_m$-sequence;
(d) $q_n^\xi \Vdash \check{\beta}_n^\xi \in \dot{C}$.

[19] Note also that all cardinals $\geq |P|^+$, and hence also $\mu$, remain cardinals in $V[G]$.

Since for every $n < \omega$, $\alpha_n < \kappa$, $\bigcup_{n<\omega}\{\beta_n^\xi \mid \xi < \alpha_n\}$ is bounded in $\kappa$.

We show that $\delta = \sup\{\beta_n^\xi \mid n < \omega, \xi < \alpha_n\}$ is in $D$, that is $p \Vdash \check{\delta} \in \dot{C}$. By forcing theorems, it suffices to show that whenever $F$ is a $Q$-generic and $p \in F$, then $\delta \in \dot{C}^F$. Since each $A_n$ is maximal below $p$, $F \cap A_n$ is non-empty for each $n < \omega$. It follows that there is a sequence $\langle q_n \mid n < \omega\rangle$ of conditions in $F$ which force that elements of $\dot{C}$ are unbounded below $\delta$. Hence $\delta \in \dot{C}^F$ as required. □

Since our forcing $P$ has size $< \kappa$, it certainly has the $\kappa$-cc, and therefore the set of regular cardinals below $\kappa$ is still stationary in $V[G]$. That is $\kappa$ is still Mahlo in $V[G]$. □

## 3.3 Weakly compact and measurable cardinals and CH

By way of example, we show that if $P$ has size $< \kappa$, then $\kappa$ is still weakly compact or measurable in $V[G]$ if it was weakly compact or measurable, respectively, in $V$. In Theorems 3.3 and 3.4 we will give direct arguments, while in Section 3.4 we will put large cardinals into a more general picture so that we can formulate a uniform approach to preservation of large cardinals.

**Theorem 3.3** *Assume $\kappa$ is weakly compact in $V$ and $P$ has size $< \kappa$, and $G$ is $P$-generic. Then $\kappa$ is weakly compact in $V[G]$.*

*Proof.* As a fact we state that $\kappa$ is weakly compact iff

$$\kappa \to (\kappa)^n_\lambda, \text{ for every } n < \omega, \lambda < \kappa. \tag{3.1}$$

Let us fix in $V[G]$ a function $f : [\kappa]^2 \to 2$; it suffices to find in $V[G]$ a homogeneous set $X \subseteq \kappa$ of size $\kappa$. By Forcing theorem, there is $p \in P$ such that

$$p \Vdash \dot{f} : [\kappa]^2 \to 2.$$

Define back in $V$,

$$h : [\kappa]^2 \to \mathscr{P}(P \times 2)$$

by

$$h(s) = \{\langle q, i\rangle \mid q \leq p \;\&\; q \Vdash \dot{f}(\check{s}) = i\}.$$

Since $|\mathscr{P}(P \times 2)| < \kappa$, we can apply (3.1) and find a homogeneous set $X \subseteq \kappa$ for the function $h$. We claim that

$$p \Vdash \check{X} \text{ is homogeneous for } \dot{f},$$

or equivalently

$$X \text{ is homogeneous for } f \text{ in } V[G].$$

The homogeneity of $X$ for $h$ means that for all $s \in [X]^2$, $h(s)$ is equal to some fixed set of the form $A = \{\langle q, i\rangle \mid q \leq p \;\&\; q \Vdash \dot{f}(\check{s}_0) = i\}$, for some $s_0 \in [\kappa]^2$. Notice that because $p$ forces that $\dot{f}$ is a function, there can be no "contradictory pairs" $\langle q, 0\rangle$ and $\langle q, 1\rangle$ in $A$; that is for each $q \leq p$ occurring at the first coordinate of a pair in

$A$ there is unique $i(q)$ such that $\langle q, i(q)\rangle$ is in $A$. Assume $F$ is any $P$-generic with $p \in F$. For each $s \in [X]^2$, there is some $q(s) \in F$ such that $\langle q(s), i(q(s))\rangle$ is in $A$. If $s_1, s_2$ are in $[X]^2$, then $q(s_1)$ and $q(s_2)$ are compatible in $F$ by some $r$ which thus decides both $\dot{f}(\check{s}_1)$ and $\dot{f}(\check{s}_2)$; furthermore, there is a unique $i(r)$ such that $\langle r, i(r)\rangle$ is in $A$ and so $i(q(s_1)) = i(q(s_2)) = i(r)$. This proves that $p$ forces that $X$ is homogeneous for $\dot{f}$. □

**Theorem 3.4** *Assume $\kappa$ is measurable in $V$ and $P$ has size $< \kappa$. Then $\kappa$ is measurable in $V^P$.*

*Proof.* Let $G$ be a $P$-generic filter, and let $U$ be a $\kappa$-complete non-principal ultrafilter on $\kappa$ in $V$. We will show that

$$W = \{A \subseteq \kappa \mid \exists B \in U \ B \subseteq A\}$$

is a $\kappa$-complete non-principal ultrafilter in $V[G]$; we say that $W$ is generated by $U$. It is easy to show that $W$ is non-principal, closed upwards, and $\kappa$-complete – that is that is a $\kappa$-complete non-principal filter:

(i) Non-principality. Since $U$ is non-principal and every element of $W$ is above an element of $U$, the argument follows.

(ii) $\kappa$-completeness. Fix in $V[G]$ a sequence $\langle A_\xi \mid \xi < \lambda\rangle$, $\lambda < \kappa$ of sets in $W$. By definition of $W$, there is $p \in G$ such that

(3.2) $p \Vdash$ "There exists a sequence $\langle \dot{B}_\xi \mid \xi < \lambda\rangle$ of sets in $U$ such that for every $\xi < \lambda, \dot{B}_\xi \subseteq \dot{A}_\xi$."

By $|P| < \kappa$, there is for each $\xi$ and $\dot{B}_\xi$ a family $\mathscr{B}_\xi$ of size $< \kappa$ of sets in $U$ such that

$$p \Vdash \dot{B}_\xi \in \check{\mathscr{B}}_\xi.$$

By $\kappa$-completeness of $U$, for every $\xi$, $b_\xi = \bigcap \mathscr{B}_\xi$ is in $U$. The sequence $\langle b_\xi \mid \xi < \lambda\rangle$ exists in $V$, and therefore by $\kappa$-completeness of $U$ in $V$, $\bigcap_{\xi<\lambda} b_\xi$ is in $U$. It follows

$$p \Vdash \check{b}_\xi \subseteq \dot{A}_\xi \text{ for every } \xi < \lambda \text{ and } p \Vdash \bigcap_{\xi<\lambda} \check{b}_\xi \subseteq \bigcap_{\xi<\lambda} \dot{A}_\xi,$$

and hence $\bigcap_{\xi<\lambda} A_\xi$ is in $W$.

It remains to show that $W$ is an ultrafilter. Let $\dot{X}$ be a name for a subset of $\kappa$. For each $p \in P$, let

$$A_p = \{\alpha < \kappa \mid p \text{ decides whether } \check{\alpha} \in \dot{X}\}.$$

Notice that

$$D = \{p \in P \mid A_p \in U\} \text{ is dense in } P.$$

This is because for each $q \in P$,

$$\bigcup_{p \le q} A_p = \kappa$$

and by $\kappa$-completeness of $U$ and the fact that $|P| < \kappa$, there must be some $p \le q$ such that $A_p \in U$. Let $r$ be in $D \cap G$ – then $A_r \in U$ where $A_r$ can be written as a disjoint union of $A_0 = \{\alpha < \kappa \mid r \Vdash \check{\alpha} \in \dot{X}\}$ and $A_1 = \{\alpha < \kappa \mid r \Vdash \check{\alpha} \notin \dot{X}\}$. If $A_0 \in U$, then $\dot{X}^G \in W$, and if $A_1 \in U$, then $\kappa \setminus \dot{X}^G \in W$. □

## 3.4 A uniform approach

So far we have argued that inaccessible, Mahlo, weakly compact, and measurable cardinals do not decide CH. This, *per se*, is not an argument that other large cardinals cannot behave differently in this respect – after all, every argument we gave was unique to a given large cardinal concept, and not directly generalizable to other large cardinals. As it turns out, however, many large cardinals can be formulated in terms of elementary embeddings, and there is a uniform approach which shows that such cardinals do not affect CH. Among the cardinals with definitions through elementary embeddings are weakly compact cardinals, measurable cardinals, strongly compact cardinals, supercompact cardinals and many others.

**Definition 3.5** *Let $M$ and $N$ be two transitive classes. We say that $j : M \to N$ is an elementary embedding if for every formula and every $n$-tuple $m_0, \dots, m_n$ of elements in $M$, if $\varphi^M(m_0, \dots, m_n)$, then $\varphi^N(j(m_0), \dots, j(m_n))$.*

The notation $\varphi^M$ is defined recursively and subsists in replacing every occurrence of an unbounded quantifier $Qx$ with $Qx \in M$. Note that $M, N$ and $j$ may be proper classes.[20]

We say that $\kappa$ is a critical point of $j : M \to N$ if for all $\alpha < \kappa$, $j(\alpha) = \alpha$, and $j(\kappa) > \kappa$. One can show that if $j$ is not the identity it has a critical point which is always a regular uncountable cardinal in $M$.

**Theorem 3.6** *The following are equivalent for a cardinal $\kappa > \omega$:*

*(i) $\kappa$ is measurable.*
*(ii) There is an elementary embedding $j : V \to M$ with critical point $\kappa$, where $M$ is some transitive class.*

*Proof.* Ad (i)→(ii). (Sketch) A generalization due to Scott [15] of the ultrapower construction can be used to form the ultrapower of the whole universe $V$ via a $\kappa$-complete ultrafilter $U$ witnessing the measurability of $\kappa$. One can show that this construction is well defined and yields a proper class ultrapower model, denoted as $\mathrm{Ult}_U(V)$. Since the $U$ is $\omega_1$-complete, one can further show that that the ultrapower is well-founded and can therefore be collapsed using the Mostowski collapsing function. Thus there is an elementary embedding

$$j : V \to \mathrm{Ult}_U(V) \cong M$$

to a transitive isomorphic image of $\mathrm{Ult}_U(V)$. $\kappa$-completeness of $U$ is invoked to prove that $j$ is the identity below $\kappa$, and $j(\kappa) > \kappa^+$.

[20] There are some logical issues here because ZFC does not formalize satisfication for proper classes, and hence one should be careful in saying that some $\varphi$ holds in $M$, or that $j$ is elementary. The relativation $\varphi^M$ solves the issue to a certain extent, but it is not entirely optimal (for instance the property "$j$ is elementary" is a schema of infinitely many sentences). Luckily, as always with issues like these, there are ways to make these concepts completely correct from the formal point of view. See for instance [4] for a nice discussion of approaches to formalizing large cardinal concepts which refer to elementary embeddings.

Ad (ii)→(i). Let $j : V \to M$ be elementary with critical point $\kappa$. Let us define

$$U = \{X \subseteq \kappa \,|\, \kappa \in j(X)\}.$$

We will show that $U$ is a $\kappa$-complete non-principal ultrafilter. It is non-principal because for every $\alpha < \kappa$, $j(\{\alpha\}) = \{\alpha\}$ and therefore $\{\alpha\} \notin U$. $\kappa$-completeness follows by the following argument: if $\{A_\xi \,|\, \xi < \mu\}$ are sets in $U$ for $\mu < \kappa$, then

$$j(\{A_\xi \,|\, \xi < \mu\}) = \{j(A_\xi) \,|\, \xi < \mu\}$$

because $j(\mu) = \mu$ and therefore the $j$-image of the system $\{A_\xi \,|\, \xi < \mu\}$ is just the system of the $j$-images of the individual sets. Therefore

$$\kappa \in \bigcap_{\xi<\mu} j(A_\xi)$$

and hence

$$\bigcap_{\xi<\mu} A_\xi \in U.$$

Thus $U$ is a $\kappa$-complete non-principal filter. It remains to show that $U$ is an ultrafilter – but this is easy: if $X \subseteq \kappa$ is given, then $\kappa = X \cup (\kappa \setminus X)$, and so

$$\kappa \in j(\kappa) = j(X) \cup j(\kappa \setminus X)$$

by elementarity. Hence $\kappa \in j(X)$ or $\kappa \in j(\kappa \setminus X)$.

Notice that $U$ is generated by a single element – $\kappa$. But it is not principal because $\kappa$ is not in the range of $j$. If $\xi$ is in the range of $j$, then any attempt to define an ultrafilter as we did ends up with a principal ultrafilter because the singleton of $j^{-1}(\xi)$ would be in the filter. Conversely, if we defined our $U$ with any other $\xi$ in the interval $[\kappa, j(\kappa))$, we would get a non-principal $\kappa$-complete ultrafilter by an identical argument.

The importance of $U$, as generated by $\kappa$, is that $U$ is *normal*, but this goes beyond the scope of this paper. □

The above theorem provides a new tool to show that a measurable cardinal is preserved by forcing. It suffices to show that in a generic extension $V[G]$, there exists an elementary embedding with critical point $\kappa$. The following lemma is very useful for finding elementary embeddings in the generic extensions.

**Lemma 3.7 (Silver)** *Assume $j : M \to N$ is an elementary embedding between transitive classes $M, N$. Let $P \in M$ be a forcing notion and let $G$ be $P$-generic over $M$.*[21] *Assume further that $H$ is $j(P)$-generic over $N$ such that*

$$\{j(p) \,|\, p \in G\} \subseteq H. \tag{3.3}$$

*Then there exists elementary embedding $j^* : M[G] \to N[H]$ such that:*

*(i) $j^*$ restricted to $M$ is equal to $j$,*
*(ii) $j^*(G) = H$.*

[21] This means that $G$ meets every dense open set which is an element of $M$.

*We call $j^*$ a* lifting *of $j$ to $M[G]$.*

*Proof.* We first show how to define $j^*$. Let $x$ be an element of $M[G]$ and let $\dot{x}$ be a name for $x$ so that $\dot{x}^G = x$. We set

$$j^*(\dot{x}^G) = (j(\dot{x}))^H.$$

This definition is correct because by elementarity $j(\dot{x})$ is a $j(P)$-name; further if $\dot{y}$ is another name for $x$, $\dot{y}^G = \dot{x}^G = x$, then there is some $p \in G$ such that $p \Vdash \dot{y} = \dot{x}$. By elementarity,

$$j(p) \Vdash j(\dot{x}) = j(\dot{y}).$$

By (3.3), $j(p) \in H$ and therefore $(j(\dot{x}))^H = (j(\dot{y}))^H$.

$j^*$ is elementary by the following implications:

$$\text{(3.4)} \quad \varphi^{M[G]}(x, \ldots) \to \exists p \in G\ p \Vdash \varphi(\dot{x}, \ldots) \to \exists p \in G\ j(p) \Vdash \varphi(j(\dot{x}), \ldots) \to \varphi^{N[H]}(j^*(x), \ldots),$$

where the last implication follows by (3.3).

Ad (i). For $x \in M$, $j^*(x) = (j(\check{x}))^H = j(x)$, by the properties of the canonical name $\check{x}$.

Ad (ii). Let $\dot{g}$ be a canonical name for the generic filter, i.e. a name which always interprets by the generic filter. Then $\dot{g}^G = G$, and $j^*(G) = (j(\dot{g}))^H = H$. □

Silver's "lifting lemma" allows us to reprove Theorem 3.4 in a more straightforward way:

**Theorem 3.8** *Assume $\kappa$ is measurable in $V$, $P$ has size $< \kappa$ and $G$ is $P$-generic. Then $\kappa$ is measurable in $V[G]$.*

*Proof.* By Theorem 3.6, there is an embedding $j : V \to M$ with critical point $\kappa$ (this embedding exists, i.e. is definable, in $V$). Since $j$ is the identity below $\kappa$, one can easily show that $V_{\kappa+1} = (V_{\kappa+1})^M$ and $j(x) = x$ for every $x \in V_\kappa$. In particular

$$j(P) = P$$

because $|P| < \kappa$ implies that we can assume $P \in V_\kappa$.[22] It follows by Silver's lemma, when we substitute $G$ for $H$, that there exists a lifting

$$j^* : V[G] \to M[G],$$

where $j^*(G) = G$. Since $j^*$ is definable in $V[G]$, it shows by Theorem 3.6 that $\kappa$ is still measurable in $V[G]$. □

[22] If $|P| < \kappa$, then there is an isomorphic copy of $P$ which is in $V_\kappa$.

### 3.5 Other large cardinals

Many large cardinals can be formulated in terms of elementary embeddings and hence the proof in Theorem 3.8 can be straightforwardly generalized to argue that these cardinals do not decide CH. Here is a list of some more known large cardinals defined using elementary embeddings satisfying certain properties, where $\kappa > \omega$:

- $\kappa$ is *weakly compact* iff $\kappa$ is inaccessible and for every transitive model $M$ of ZF without the powerset axiom such that $\kappa \in M$, $M$ is closed under $< \kappa$-sequences and $|M| = \kappa$, there is an elementary embedding $j : M \to N$, $N$ transitive, with critical point $\kappa$.
- $\kappa$ is *strongly compact* iff for every $\gamma > \kappa$ there is an elementary embedding $j : V \to M$ with critical point $\kappa$, $j(\kappa) > \gamma$, and for any $X \subseteq M$ with $|X| \leq \gamma$, there is a $Y \in M$ such that $Y \supseteq X$ and $(|Y| < j(\kappa))^M$.
- $\kappa$ is *supercompact* iff for every $\gamma > \kappa$ there is an elementary embedding $j : V \to M$ with critical point $\kappa$, $j(\kappa) > \gamma$, and ${}^\gamma M \subseteq M$.[23]
- $\kappa$ is *strong* iff for every $\gamma > \kappa$ there is an elementary embedding $j : V \to M$ with critical point $\kappa$, $j(\kappa) > \gamma$, and $V_\gamma \subseteq M$.

Even Ramsey cardinals can be formulated in terms of elementary embeddings, see for instance [12]. All the cardinals considered so far are linearly ordered in terms of strength: for instance every supercompact is strongly compact, and every strongly compact is strong.

Note that by a celebrated result by Kunen [9], there can be, in ZFC, no cardinal $\kappa$ such that there exists an elementary embedding $j : V \to V$ with critial point $\kappa$. This sets an upper bound on the large cardinal concept which we can consider.[24]

## 4 On the consistency strength

Large cardinals are interesting set-theoretical objects with beautiful combinatorics and surprising connections among themselves; for instance many of these can be defined in apparently disparate ways – using elementary embedding, satisfaction in various structures, or by partition properties. However, this does not fully explain the willingness with which large cardinals are almost universally accepted by the set theoreticians. To explicate the wider role of large cardinals we need to introduce the notion of a consistency strength over ZFC.

**Definition 4.1** *A statement $\sigma$ in the language of set theory is* stronger in terms of consistency *then another statement $\sigma'$ if*

$$\mathrm{CON}(\mathrm{ZFC} + \sigma) \to \mathrm{CON}(\mathrm{ZFC} + \sigma').$$

[23] ${}^\gamma M \subseteq M$ is true if for every sequence of length $\gamma$ of elements in $M$, the whole sequence is in $M$. This a non-trivial requirement because the sequence itself is in general only in $V$, and not in $M$.

[24] Rather surprisingly, it is still open whether this limiting result holds in ZF.

*We denote here this relation by*

$$\sigma' \leq_c \sigma.$$

*Statements are called* equiconsistent *if*

$$\text{CON}(\text{ZFC} + \sigma) \leftrightarrow \text{CON}(\text{ZFC} + \sigma').$$

For instance, GCH $\equiv_c \neg$CH $\equiv_c V = L \equiv_c V \neq L \equiv_c \Diamond \equiv_c$ "There are no $\omega_1$-Souslin trees".[25] Moreover we have

(4.5) $$\text{CON}(\text{ZF} - \text{ Axiom of Foundation }) \rightarrow \text{CON}(\text{ZFC} + \sigma)$$

for any $\sigma$ from the class $[\text{GCH}]_{\equiv_c}$.[26]

Note that the relation of equiconsistency $\equiv_c$ is an equivalence relation, and the relation $\leq_c$ is an ordering on the equivalence classes given by $\equiv_c$. What is the structure of this ordering? In principle, it might be highly non-linear. However, large cardinal concepts can be used to show that it is in fact mostly linear: for many combinatorial statements $\sigma$ and $\sigma'$ considered in practice, we either have $\sigma \leq_c \sigma'$ or $\sigma' \leq_c \sigma$. The key here is that large cardinal concepts themselves are linearly ordered under $\leq_c$, and very often one can show that a statement $\sigma$ is equiconsistent with a certain large cardinal axiom.

By way of example, considered the following three statements (see [7] for the definitions of the concepts mentioned):

(A) (Over ZF) All sets of reals are Lebesgue measurable.

(B) (Over ZFC) Every $\omega_2$-tree has a cofinal branch.

(C) (Over ZFC) SCH fails.

A priori, they might be incomparable under $\leq_c$; however, one can prove:

**Theorem 4.2 (Solovay [18], Shelah [16])**

$$\text{(A)} \equiv_c \text{ "there exists an inaccessible cardinal".}$$

**Theorem 4.3 (Mitchell [13])**

$$\text{(B)} \equiv_c \text{ "there exists a weakly compact cardinal".}$$

**Theorem 4.4 (Mitchell [14], Gitik [3])**

$$\text{(C)} \equiv_c \text{ "there exists a measurable cardinal of Mitchell order } o(\kappa) = \kappa^{++}\text{".}$$

**Corollary 4.5** *GCH $<_c$ (A) $<_c$ (B) $<_c$ (C).*

[25] This can be shown using Gödel's class of constructible sets $L$, or by forcing.

[26] Notice that by (4.5), $[\text{GCH}]_{\equiv_c}$ is equal to $[\nu]_{\equiv_c}$ for any $\nu$ such that ZFC $\vdash \nu$.

The above theorems are proved using two complementary methods: (i) forcing over a model with the given large cardinal, and (ii) technique of inner models to find a large cardinal (in some model of set theory) from the given combinatorial statement. For instance Theorem 4.3 is proved by iterating a certain forcing notion (such as the Sacks forcing at $\omega$) along $\kappa$, where $\kappa$ is weakly compact: this gives

$$\text{(B)} \leq_c \text{ “there exists a weakly compact cardinal”}.$$

Conversely, one can show that if (B) holds, then $\omega_2$ of $V$ is a weakly compact cardinal in $L$, and hence there is a model with weakly compact cardinal. This gives:

$$\text{“there exists a weakly compact cardinal”} \leq_c (B).$$

Problems arise when the large cardinal in question is inconsistent with $L$ (such as a measurable cardinal), then to obtain the consistence of the large cardinals, a generalization of $L$ must be defined which allows large cardinals. This is the field of *inner model theory*. So far, inner models were devised for infinitely many Woodin cardinals (Woodin cardinals are much stronger than measurable cardinals in terms of consistency strength), but not – crucially – for strongly compact or supercompact cardinals. This inability to find suitable inner models for such large cardinals is one of the most pressing problems in current set theory. Because of this, the following is still open for a certain important combinatorial statement denoted as PFA (Proper Forcing Axiom):[27]

**Open question.** We know: PFA $\leq_c$ “there exists a supercompact cardinal”. Does the converse hold as well, i.e. is PFA equiconsistent with a supercompact cardinal?

There is a general agreement that this is the case, but we cannot prove it.[28]

The following is also long open, probably for the similar reason as the case of PFA:

**Open question.** By definition, every supercompact cardinal is strongly compact. We also know that $\kappa$ can be measurable + strongly compact but not supercompact. However, we do not know, but consider probable: Are strongly compact and supercompact cardinals equiconsistent?

## 5 Conslusion

Large cardinals considered in this article do not decide CH one way or another. In fact no commonly considered large cardinals decide CH, which can be shown by similar methods.[29] However, notice that we cannot prove a statement such as “no large cardinal decides CH” because in this statement we quantify over a vague domain of “large cardinals” and hence such a statement is not in the language of set

[27] PFA, a strengthening of MA – the Martin's Axiom –, implies $2^\omega = \omega_2$ and thus decides CH. However, PFA itself is not a large cardinal axiom in the strict sense. Also, PFA trivially implies $2^\omega > \omega_1$ the way it is set up, so what is surprising is that it also implies $2^\omega \leq \omega_2$, and not that it implies failure of CH. MA, on the other hand, is consistent with any reasonable value of $2^\omega > \omega_1$.

[28] We do know that PFA implies consistency of many Woodin cardinals, and so PFA is sandwiched between “many Woodins” and “supercompact”. But this gap is quite substantial.

[29] We consider $\kappa$ to be large when it is at least inaccessible. If we drop this requirement, the situation is more complex.

theory. It may be, but it is not considered probable, that a new large cardinal will be devised which will be more susceptible to effects of small forcings. At present, no such cardinal is known.

## References


[1] S. Feferman, J. Dawson, S. Kleene, G. Moore, and J. Van Heijenoort, editors. *Kurt Gödel. Collected Works. Volume II.* Oxford University Press, 1990.

[2] Sy-David Friedman and Tatiana Arrigoni. Foundational implications of the inner model hypothesis. *Annals of Pure and Applied Logic*, 163:1360–1366, 2012.

[3] Moti Gitik. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Annals of Pure and Applied Logic*, 43:209–234, 1989.

[4] Joel David Hamkins, Greg Kirmayer, and Norman Lewis Perlmutter. Generalizations of the Kunen inconsistency. To appear in *Annals of Pure and Applied Logic*.

[5] David Hilbert. Mathematical problems. *Bull. Amer. Math. Soc.*, 8:437–479, 1902.

[6] Radek Honzik. Quick guide to independence results in set theory. *Miscellanea Logica*, 7:89–131, 2009.

[7] Tomáš Jech. *Set Theory.* Springer, 2003.

[8] Akihiro Kanamori. *The Higher Infinite.* Springer, 2003.

[9] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *The Journal of Symbolic Logic*, 407–413:21–46, 1971.

[10] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs.* North Holland, 1980.

[11] Azriel Levy and Robert M. Solovay. Measurable cardinals and the continuum hypothesis. *Israel Journal of Mathematics*, 5:234–248, 1967.

[12] W. J. Mitchell. Ramsey cardinals and constructibility. *The Journal of Symbolic Logic*, 44(2):260–266, 1979.

[13] William J. Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5(1):21–46, 1972/1973.

[14] William J. Mitchell. The core model for sequences of measures. I. *Math. Proc. Camb. Phil. Soc.*, 95:229–260, 1984.

[15] Dana S. Scott. Measurable cardinals and constructible sets. *Bulletin de l'Académie Polonaise des Sciences*, 9:521–524, 1961.

[16] S. Shelah. Can you take Solovay's inaccessible away? *Israel Journal of Mathematics*, 48:1–47, 1984.

[17] Antonín Sochor. *Klasická matematická logika.* Karolinum, 2001.

[18] Robert M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92:1–56, 1970.