Self-adjoint unitary operators and the AXBXC decomposition

In the previous chapter, we introduced without explanation the extended Euler formula

$$e^{-i\frac{\omega}{2}\,\xi\cdot\sigma} = \cos\frac{\omega}{2}E + (x\ell + yj + zk)\sin\frac{\omega}{2}\,.$$

Let us now show that this formula corresponds to the definition of an operator function as defined on normal operators, i.e. as a function applied to eigenvalues.

The operators in the domain of our function are of the form

$$\xi\cdot\sigma = xX + yY + zZ,$$

where $\xi = (x, y, z)$ is a unit real vector. The following lemma shows that these matrices, together with identity, form an intersection of two popular classes of normal matrices, namely unitary and Hermitian (that is,self-adjoint) matrices. These are matrices for which $A = A^\dagger$ (Hermitian) and $A^{-1} = A^\dagger$, or $A = A^{-1} = A^\dagger$. Such is, for example, the ubiquitous Hadamard matrix. In particular, $A^2 = E$ also holds, which, obviously, is true for a diagonalizable matrix just when it has a diagonal form

$$\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}.$$

Note, however, that $A^2 = E$ also holds for some non-diagonalizable matrices, such as the matrix $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

Theorem. The matrix A is a Hermitian unitary, just when it is equal to $\pm E$, or is of the form $xX + yY + zZ$, where $(x, y, z) \in \mathbb{S}^2$.

Proof. Since all three Pauli matrices are Hermitian, $(xX+yY+zZ)^\dagger = xX+yY+zZ$ holds, so the matrix $xX+yY+zZ$ is Hermitian. The relation $(xX+yY+zZ)^2 = E$ results by a direct calculation from the fact that Pauli matrices are involutive and anticommutative, i.e. that they satisfy

$$X^2 = Y^2 = Z^2 = E, \qquad XY = -YX, \qquad YZ = -ZY, \qquad ZX = -XZ.$$

Conversely, if A is a Hermitian unitary, it has a diagonal form

$$\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix},$$

that is, $\pm E$ or $\pm Z$. If the diagonal form is $\pm E$, then $A = \pm E$ (the identity has the same form for all bases). If the diagonal form is $\pm Z$, then the determinant is -1. From the characterization of unitary matrices (see chapter *Geometry of projective unitary operators*) it is now easy to see that a unitary matrix with determinant -1, which is also Hermitian, is of the form

$$\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} = xX + yY + zZ,$$

where $x^2 + y^2 + z^2 = 1$. □

For the operator A with real eigenvalues r_1 nda r_2 the expression $e^{-i\frac{\omega}{2}A}$ denotes the operator that has a diagonal form

$$\begin{pmatrix} e^{-i\frac{\omega}{2}r_1} & 0 \\ 0 & e^{-i\frac{\omega}{2}r_2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\omega}{2}r_1 - i\sin\frac{\omega}{2}r_1 & 0 \\ 0 & \cos\frac{\omega}{2}r_2 - i\sin\frac{\omega}{2}r_2 \end{pmatrix}.$$

In addition, if r_1 i r_2 are equal to ± 1, we indeed get (in the base of eigenvectors) due to cosine evenness and sinus oddity

$$e^{-i\frac{\omega}{2}A} = \cos\frac{\omega}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\sin\frac{\omega}{2}\begin{pmatrix}r_1 & 0\\ 0 & r_2\end{pmatrix} = \cos\frac{\omega}{2}E - iA\sin\frac{\omega}{2}\,.$$

The formula is therefore correct for Hermitian unitary matrices.

For rotations around the main axes we get, according to the previous formula,

$$R_X(\omega) := R_{(1,0,0)}(\omega) = E\cos\frac{\omega}{2} - iX\sin\frac{\omega}{2} = \begin{pmatrix}\cos\frac{\omega}{2} & -i\sin\frac{\omega}{2}\\ -i\sin\frac{\omega}{2} & \cos\frac{\omega}{2}\end{pmatrix},$$

$$R_Y(\omega) := R_{(0,1,0)}(\omega) = E\cos\frac{\omega}{2} - iY\sin\frac{\omega}{2} = \begin{pmatrix}\cos\frac{\omega}{2} & -\sin\frac{\omega}{2}\\ \sin\frac{\omega}{2} & \cos\frac{\omega}{2}\end{pmatrix},$$

$$R_Z(\omega) := R_{(0,0,1)}(\omega) = E\cos\frac{\omega}{2} - iZ\sin\frac{\omega}{2} = \begin{pmatrix}\exp(-i\frac{\omega}{2}) & 0\\ 0 & \exp(i\frac{\omega}{2})\end{pmatrix}.$$

The explicit form of the representative of the projective class of unitary matrices corresponding to the rotation by the angle ω around the axis $\xi = (x, y, z)$ is

$$R_\xi(\omega) = \begin{pmatrix}\cos\frac{\omega}{2} - iz\sin\frac{\omega}{2} & -i(x-iy)\sin\frac{\omega}{2}\\ -i(x+iy)\sin\frac{\omega}{2} & \cos\frac{\omega}{2} + iz\sin\frac{\omega}{2}\end{pmatrix}.$$

Since $XYX = -Y$ and $XZX = -Z$, we get a useful relationship

$$XR_Y(\omega)X = XEX\cos\frac{\omega}{2} - iXYX\sin\frac{\omega}{2} = R_Y(-\omega)$$

and similarly

$$XR_Z(\omega)X = R_Z(-\omega).$$

Now we can prove the theorem we need to construct a controlled operator for the general U.

Theorem. Each unitary operator U is projectively equivalent to the operator $AXBXC$, where $ABC = E$.

Proof. We know that the U operator is projectively equivalent to a form operator

$$\begin{pmatrix}\cos\frac{\vartheta}{2} & -e^{i(\psi-\varphi)}\sin\frac{\vartheta}{2}\\ e^{i\varphi}\sin\frac{\vartheta}{2} & e^{i\psi}\cos\frac{\vartheta}{2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & e^{i\varphi}\end{pmatrix}\begin{pmatrix}\cos\frac{\vartheta}{2} & -\sin\frac{\vartheta}{2}\\ \sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i(\psi-\varphi)}\end{pmatrix},$$

which is projectively equivalent to an operator

$$R_Z(\varphi)R_Y(\vartheta)R_Z(\psi)R_Z(-\varphi)\,.$$

Using the above derived properties of conjugation by the operator X we get

$$
\begin{aligned}
&R_Z(\varphi)R_Y(\vartheta)R_Z(\psi)R_Z(-\varphi) = \\
&\quad = R_Z(\varphi)R_Y\left(\frac{\vartheta}{2}\right)R_Y\left(\frac{\vartheta}{2}\right)R_Z\left(\frac{\psi}{2}\right)R_Z\left(\frac{\psi}{2}\right)R_Z(-\varphi) = \\
&\quad = R_Z(\varphi)R_Y\left(\frac{\vartheta}{2}\right)\left(XR_Y\left(-\frac{\vartheta}{2}\right)X\right)\left(XR_Z\left(-\frac{\psi}{2}\right)X\right)R_Z\left(\frac{\psi}{2}\right)R_Z(-\varphi) = \\
&\quad = \left(R_Z(\varphi)R_Y\left(\frac{\vartheta}{2}\right)\right)X\left(R_Y\left(-\frac{\vartheta}{2}\right)R_Z\left(-\frac{\psi}{2}\right)\right)X\left(R_Z\left(\frac{\psi}{2}\right)R_Z(-\varphi)\right)
\end{aligned}
$$

and now it is enough to put

$$
A = R_Z(\varphi)R_Y\left(\frac{\vartheta}{2}\right), \quad B = R_Y\left(-\frac{\vartheta}{2}\right)R_Z\left(-\frac{\psi}{2}\right), \quad C = R_Z\left(\frac{\psi}{2}\right)R_Z(-\varphi)\,.
$$

□