The Hopf fibration (an optional additional material)

If

$$\alpha = a + bi, \quad \beta = c + di,$$

then the unit vector (α, β) corresponds to

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4,$$

which is an element of $\mathbb{S}^3$, (three-dimensional) sphere in $\mathbb{R}^4$. This representation is not unique, we identify vectors that differ by a global phase. One can easily verify that $(e^{i\varphi}\alpha, e^{i\varphi}\beta)$ correspond to points

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

which form in $\mathbb{R}^4$ a circle $\mathbb{S}^1$ around the origin. This circle corresponds to a single point on the Bloch sphere. We thus get the mapping

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2,$$

called the *Hopf fibration*. In words, the three-dimensional sphere decomposes into (disjoint) circles, which correspond to points on two-dimensional sphere.