According to the postulates of quantum mechanics,a qubit is an element of $\mathbb{C}^2$ of size one, and it is possible to ignore the global phase. Thus, a qubit can be understood as an element of a one-dimensional complex projective space $\mathbb{C}\mathbf{P}^1$. By definition, the elements $\mathbb{C}\mathbf{P}^1$ are a pair of complex numbers (α, β) with equivalence

$$(\alpha, \beta) \sim \lambda(\alpha, \beta), \qquad \lambda \in \mathbb{C}, \lambda \neq 0.$$

We usually represented a qubit by any vector of magnitude one with preserved ambiguity regarding the global phase, i.e. regarding multiplication by a complex unit.

If we want to represent the element of a complex line unambiguously, several possibilities are available:

1. The pair (α, β), where we expand the requirement of the unit norm $\alpha\alpha^* + \beta\beta^* = 1$ by the assumption that α is real and non-negative. With this assumption, we actually choose the global phase, except for the situation where $\alpha = 0$, for which we choose the pair $(0, 1)$. Note that for any unit vector (α, β) one can find $\psi \in [0, \pi/2]$ such that $|\alpha| = \cos\psi$ a $|\beta| = \sin\psi$. Our choice of representative can then be written as $(\cos\psi, e^{-i\varphi}\sin\psi)$, where $\varphi \in [0, 2\pi)$ is given uniquely except for the case $(1, 0)$, where we put $\varphi = 0$ (similarly, the above convention selects $\varphi = 0$ for $(0, 1)$).

> **Note:** In the literature, φ is often chosen so that the representative is $(\cos\psi, e^{i\varphi}\sin\psi)$. We violate this convention to bring it into line with the usual concept of stereographic projection below.

2. Qubit can therefore also be represented by the pair (ψ, φ), which can be understood as polar coordinates of one half of a unit sphere. To extend such a representation to the whole sphere, let's put $\vartheta = 2\psi$. Then we have

$$(\alpha, \beta) = \left(\cos\frac{\vartheta}{2}, e^{-i\varphi}\sin\frac{\vartheta}{2}\right)$$

and qubits uniquely correspond to the set of pairs (ϑ, φ), $\vartheta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$ extended by $(0, 0)$, $(\pi, 0)$, according to the above conventions.

It turns out that the complex projective line can be represented by a real unit sphere $\mathbb{S}^2$ (in mathematics we speak of the Riemann sphere, in quantum physics we speak of the Bloch sphere).

3. Qubit can therefore also be represented by three real numbers (x, y, z), satisfying $x^2 + y^2 + z^2 = 1$, which represent the standard Cartesian coordinates of the Bloch sphere. The relation to polar coordinates is given as

$$(x, y, z) = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta),$$

under the assumption that we measure the angle φ in the plane of the axes x and y starting from the axis x and measuring the angle ϑ from the axis z.

4. The pair (α, β) can finally be replaced by a number

$$\frac{\alpha}{\beta},$$

which is a representative of the projective line

$$\left(\frac{\alpha}{\beta}, 1\right),$$

where $(1,0)$ is the improper point of the projective line, naturally called ∞. The qubits are thus represented by the *extended complex plane* $\mathbb{C} \cup \{\infty\}$.

In total, we have the following representations for elements of $\mathbb{C}\mathbf{P}^1$:

$$\begin{aligned} \left(\cos\frac{\vartheta}{2},\, e^{-i\varphi}\sin\frac{\vartheta}{2}\right) &\in \mathbb{C}^2, \\ (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta) &\in \mathbb{S}^2, \\ e^{i\varphi}\cot\frac{\vartheta}{2} &\in \mathbb{C}\cup\{\infty\}\,. \end{aligned}$$

From the point (x,y,z) on the Bloch sphere, the corresponding element of the extended complex plane can be obtained by the *stereographic projection*, let's denote it $\mathcal{S}$, where each point on the sphere corresponds to its image projected from the North Pole to the plane given by the axes x and y understood as a complex plane with the imaginary axis y (we assign the point ∞ to the north pole itself). From the similarity we simply see that

$$\mathcal{S}(x,y,z) = \frac{x+iy}{1-z}\,.$$

For $(\alpha,\beta) = \left(\cos\frac{\vartheta}{2},\, e^{-i\varphi}\sin\frac{\vartheta}{2}\right)$ and $(x,y,z) = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)$ we therefore have

$$\mathcal{S}(x,y,z) = \frac{x+iy}{1-z} = \frac{e^{i\varphi}\sin\vartheta}{1-\cos\vartheta} = e^{i\varphi}\frac{2\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}}{2\sin^2\frac{\vartheta}{2}} = e^{i\varphi}\cot\frac{\vartheta}{2} = \frac{\alpha}{\beta}\,.$$

Then the following diagram of qubit representations commute:

$$\begin{tikzcd} (\alpha,\beta) \arrow[r] \arrow[d] & \left(\cos\frac{\vartheta}{2},\, e^{-i\varphi}\sin\frac{\vartheta}{2}\right) \arrow[d] \\ \frac{\alpha}{\beta} & (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta) \arrow[l, "\mathcal{S}"'] \end{tikzcd}$$

The picture 1 shows the inverse stereographic projection $P = \mathcal{S}^{-1}$. It is given by:

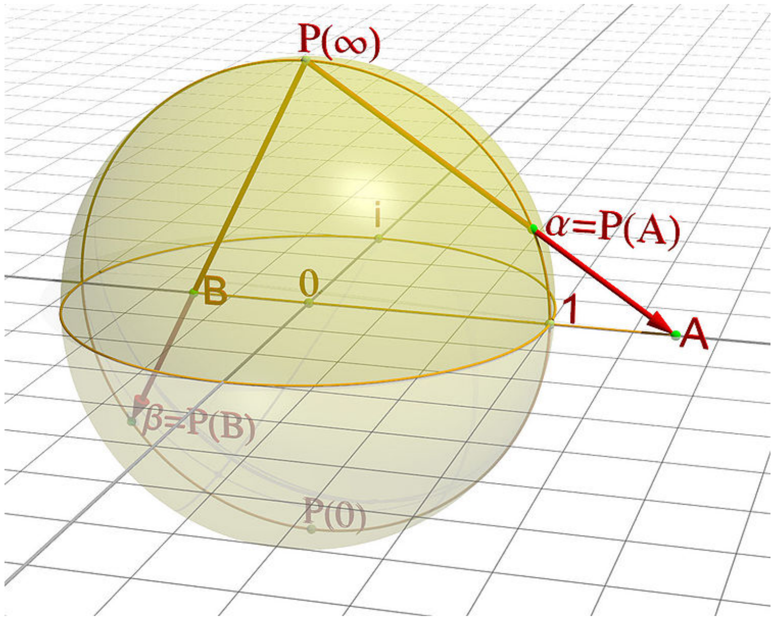
$$\mathcal{S}^{-1} : a+bi \mapsto \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right).$$

Finally, for a unit (α,β), the corresponding point on the Bloch sphere can be obtained directly as

$$(\mathrm{Re}(2\alpha\beta^*), \mathrm{Im}(2\alpha\beta^*), |\alpha|^2 - |\beta|^2).$$

5. The last important representation of the qubit ψ is using the projection operator $|\psi\rangle\langle\psi|$. It is also called *density operator* in quantum mechanics and plays an important role when working with so-called mixed systems. Note first that the density operator actually represents a unique representative, because two states differing by the global phase have the same operator. At the same time, it has decomposition using Pauli matrices, which is related to the above geometric representations. Indeed, for

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\vartheta}{2} \\ e^{i\varphi}\sin\frac{\vartheta}{2} \end{pmatrix}$$



OBRÁZEK 1. Inevrse stereographic projection (from Wikipedia)

we have

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+\cos\vartheta & e^{-i\varphi}\sin\vartheta \\ e^{i\varphi}\sin\vartheta & 1-\cos\vartheta \end{pmatrix} = \frac{1}{2}(E + xX + yY + zZ),$$

where

$$\begin{cases} x = \sin\vartheta\cos\varphi, \\ y = \sin\vartheta\sin\varphi, \\ z = \cos\vartheta\,. \end{cases}$$

Therefore, if we put $\sigma = (X, Y, Z)$, we can write

$$|\psi\rangle\langle\psi| = \frac{1}{2}(E + r_\psi \cdot \sigma),$$

where r_ψ is the representative of $|\psi\rangle$ on the Bloch sphere.