Examples of the Discrete Fourier Transform

Consider the mapping $k \mapsto 2^k \mod 15$ over the group $\mathbb{Z}_{16}$. This mapping in the canonical basis (the list of values) is

$$(1, 2, 4, 8, 1, 2, 4, 8, 1, 2, 4, 8, 1, 2, 4, 8)$$

It is a periodic vector whose period divides its length. So it's actually just a vector of length four, repeated several times. Its Fourier decomposition therefore contains only those Fourier base vectors which themselves have a period four. The expression in the Fourier base is:

$$(15, 0, 0, 0, -3-6i, 0, 0, 0, -5, 0, 0, 0, -3-6i, 0, 0, 0)\,.$$

The aperiodic vector

$$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)$$

has an ugly Fourier transform

$$\begin{array}{lllll} (30, & -2-10.05i, & -2-4.83i, & -2-2.99i, & -2-2i, \\ -2-1.34i, & -2-0.83i, & -2-0.40i, & -2, -2+0.40i, & -2+0.83i, \\ -2+1.34i, & -2+2i, & -2+2.99i, & -2+4.83i, & -2+10.05i)\,. \end{array}$$

A "partly periodic" vector is obtained in Shor's algorithm by factoring the number 21 from the mapping $k \mapsto 5^k \mod 21$ over the group $\mathbb{Z}_{32}$:

$$(1, 5, 4, 20, 16, 17, 1, 5, 4, 20, 16, 17, 1, 5, 4, 20, 16, 17, 1, 5, 4, 20, 16, 17, 1, 5, 4, 20, 16, 17, 1, 5)$$

It has a period six, which, however, does not divide the length of the vector. The Fourier coefficients are

$$\begin{array}{rrrr} (56.75, & -2.75+0.11i, & -3.08+0.24i, & -3.87+0.44i, \\ -6.03+0.91i, & -21.74+3.97i, & 9.91-2.09i, & 3.65-0.85i, \\ 2.12-0.53i, & 1.45-0.37i, & 1.09-0.28i, & 0.87-0.21i, \\ 0.72-0.16i, & 0.63-0.11i, & 0.57-0.07i, & 0.54-0.03i, \\ -19.27, & 0.54+0.03i, & 0.57+0.07i, & 0.63+0.11i, \\ 0.72+0.16i, & 0.87+0.21i, & 1.09+0.28i, & 1.45+0.37i, \\ 2.12+0.53i, & 3.65+0.85i, & 9.91+2.09i, & -21.74-3.97i, \\ -6.03-0.91i, & -3.87-0.44i, & -3.08-0.24i, & -2.75-0.11i)\,. \end{array}$$