Commentary to the Dirac notation

- An element u of the unitary space V is denoted by $|u\rangle$.
- Each vector $u \in V$ corresponds to an element of dual space $V^\dagger$, which we denote by $\langle u|$. It is a linear form $f_u : V \to \mathbb{C}$, defined by a rule

$$f_u(v) := u \odot v\,,$$

  where $u, v \in V$. The basic motivation for the Dirac notation is the ability to write $\langle u|v\rangle$, instead of $f_u(v)$. This is a well established alternative notation for the scalar product $\odot$. Strictly speaking, the notation $\langle u|v\rangle$ is an abbreviation for $\langle u|(|v\rangle)$, while the equality $\langle u|(|v\rangle) = \langle u|v\rangle$ can be understood as the definition of the mapping $\langle u|$ (or f_u).
- If A is a linear mapping of $V \to V$, the notation $\langle u|A|v\rangle$ is an abbreviation for $f_u(A(v))$. The notation Av (in Dirac's notation $A|v\rangle$), which is an abbreviation for $A(v)$ (i.e. $A(|v\rangle)$), is commonly used. It is actually the identification of the representation A and its matrix (with respect to the given base). By writing $\langle u|A$ we mean a linear representation defined by the relation

$$(\langle u|A)(v) := \langle u|A|v\rangle = \langle u|(A|v\rangle)\,.$$

  The notation $\langle u|A|v\rangle$ can therefore be therefore put in parentheses in two different ways without changing the result. This again corresponds exactly to the associativity of the matrix multiplication (i.e. the associativity of the composition of mappings).
- By writing $|u\rangle\langle v|$ we mean a linear representation defined by the rule

$$(|u\rangle\langle v|)(w) := u(v \odot w) = (v \odot w)u.$$

  The notation $|u\rangle\langle v|w\rangle$ can again be enclosed in parentheses in two ways:

$$|u\rangle\langle v|w\rangle = (|u\rangle\langle v|)|w\rangle = |u\rangle(\langle v|w\rangle).$$

- Combining the previous points we get

$\langle z|u\rangle\langle v|w\rangle =$

$= \big(\langle z|(|u\rangle\langle v|)\big)(|w\rangle)$	operator $\langle z|(|u\rangle\langle v|)$ applied to w,
$= \langle z|\big((|u\rangle\langle v|)(|w\rangle)\big)$	linear form f_z applied to $(|u\rangle\langle v|)(w)$,
$= \langle z|\big(|u\rangle(\langle v|w\rangle)\big)$	linear form f_z applied to $f_v(w)u$,
$= \big((\langle z|u\rangle)\langle v|\big)(|w\rangle)$	linear form $f_{f_z(u)v}$ applied to w,
$= (\langle z|u\rangle)(\langle v|w\rangle)$	product of complex numbers $f_z(u)$ and $f_v(w)$.

Note: The greatest discomfort when using Dirac notation can arise when trying to write, for example, the scalar product u and $v + w$. Should we write it like this?

$$\big\langle u\big||v\rangle + |w\rangle\big\rangle$$

Or maybe even like that?

$$\big\langle |u\rangle\big||v\rangle + |w\rangle\big\rangle$$

What do thescaring symbols $\big\langle |u\rangle\big|$ or $\big||v\rangle\big\rangle$ mean?

However, if we do not use the notation blindly and realize that such a scalar product is an application of the form f_u to the vector $v + w$, an elegant notation suggests itself:

$$\langle u|(|v\rangle + |w\rangle).$$