Foundation of modern mathematics

5.12.2022 □□

Review of topics

① Metric space

$$X \neq \emptyset \mapsto (X, d)$$

$$d : X^2 \to \mathbb{R}^+$$

Ex. Let X be any set. How do you define a simple metric?

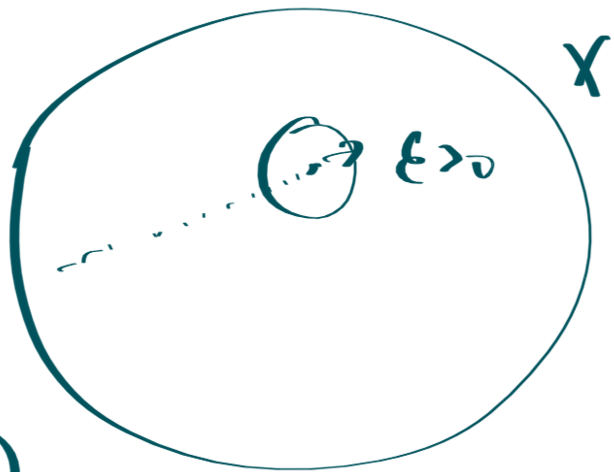
$$\forall x, y \in X \quad \begin{cases} d(x,y) = 0 & x = y \\ d(x,y) = 1 & x \neq y \end{cases}$$

Ex Is this metric complete?

(~ every Cauchy seq has a limit)

YES (because there are no non-trivial Cauchy sequences)

doesn't end with a tail composed of the same point.



(x_n)

Cauchy: $\forall \varepsilon > 0 \; \exists n_0 \; \forall m, n \geq n_0$

$$d(x_m, x_n) < \varepsilon$$

Consider our metric. and look at $\varepsilon = \frac{1}{2}$

then for (x_n) to be Cauchy

It must be that for some n_0 $d(x_m, x_n)$, $m, n \geq n_0$ is $< \frac{1}{2}$. But if $x_m \neq x_n$ then $d(x_m, x_n) = 1$

The only Cauchy seq's look like this

$$x_0\ x_1\ x_2 \ldots \underbrace{y\ y\ y\ y \ldots}_{\text{tail}} \to y$$

Corollary

Every set X can be given a complete metric.

[Q] What does it mean to say that X is not metrizable?

2 Topological spaces.

$(X, \mathcal{T})$

$X \neq \emptyset$

$\mathcal{T} \subseteq P(X)$

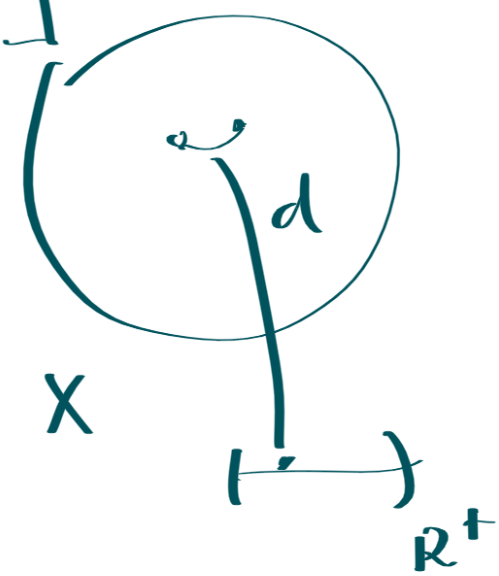
topology

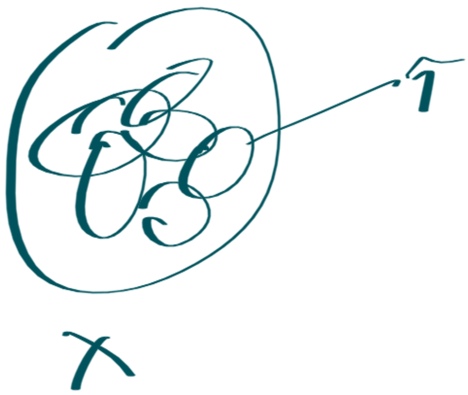
3 properties

- $X, \emptyset \in \mathcal{T}$
- $\cap$ finite
- $\cup$ any union

metric

3 properties





We call elements of $\mathcal{T}$ "open sets".

(3) Metric $\longrightarrow$ topology

$(X, d) \longrightarrow (X, \mathcal{T}_d)$

canonical topology for (X, d)

$O \in \mathcal{T}_d \iff$

$\forall x \in O \; \exists \varepsilon > 0$

$$B(x,\varepsilon) \subseteq O$$

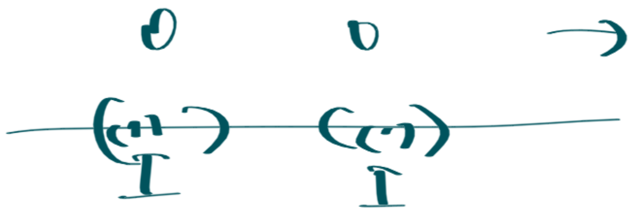
<u>Def</u> we call $\{B(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$ the <u>basis</u> of T_d.

<u>Ex</u>. $X = \mathbb{R}$, $d(x,y) = |x-y|$

the T_d is the usual topology on $\mathbb{R}$, meaning

$O \subseteq \mathbb{R}$ is open $\Leftrightarrow$

$\forall x \in O$ $\exists$ open interval $I \subseteq O$.



$\rightarrow$ O is open.

<u>Exercise</u>. (X,d), $d = \begin{cases} d(x,y) = 0 & x = y \\ d(x,y) = 1 & x \neq y \end{cases}$

what is T_d?

$$B(x,1) = \{x\}$$

$$B(x, 1/2) = \{x\}$$
$$B(x, 2) = X$$

$$\Rightarrow \mathcal{T}_d = P(X) \quad \left(\begin{array}{l} O \subseteq X \\ O = \bigcup_{x \in O} \{x\} \end{array} \right)$$

④ how does metric determine the associated topology?

more precisely: if $(X, \mathcal{T})$ is a topological space, is there always a metric d s.t. $\mathcal{T} = \mathcal{T}_d$, mean (X, d) generates $\mathcal{T}$ in the sense that

$$O \in \mathcal{T} \iff \forall x \in O \ \exists B(x, \varepsilon) \ B(x, \varepsilon) \subseteq O.$$

$$\boxed{\text{metric} \xleftarrow{?} \text{topology}}$$

NO

Def. $(X,\mathcal{T})$ is <u>first-countable</u> if every $x \in X$ has a countable basis: $\forall O \in \mathcal{T}, x \in O, \exists O_i$ $O_i \subseteq O$.

$\exists \{O_i \mid i < \omega\}$ open sets s.t. $\forall O_i$ $x \in O_i$

(O_i is an open neighbourhood of x)

Def $(X,\mathcal{T})$ is <u>second-countable</u> if $\exists \{O_i \mid i < \omega\}$ of open sets s.t.

$\forall x \in X$

1) $\{O_i \mid i < \omega\}$ is a countable base for x

2) every open set is union of sets in $\{O_i \mid i<\omega\}$.

Examples

- Second countable $\rightarrow$ first-countable

What about the converse?

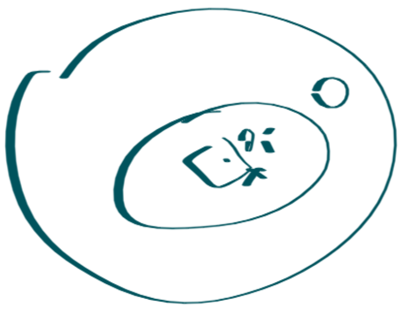
Ex

$|X| > \omega$. $(X, P(X))$

Q: $(X, P(X))$
- is it 2-nd count.?
- is it 1-st count?
- neither?

A: first-countable

$: x \mapsto \{O_i^x \mid i<\omega\}$



X

$O \subseteq X$

$: x \mapsto \{\{x\}\}$

↑ open

$\forall O \subseteq X,\ x \in O \to \{x\} \subseteq O.$

A: second-countable?

No

X is uncountable, and every $\{x\}$ is open. It means that any basis X must contain all the $\{x\}$'s. so it must have one at least $|X| \geq \omega$.

Observation

If (X, d) is a metric, then $\mathcal{T}_d$ is first countable.

Corollary

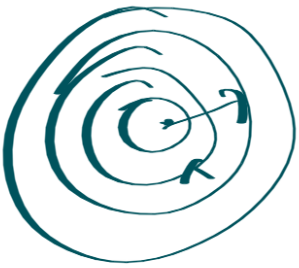
— if (X,τ) is not first-countable, it is not metrizable.

Proof of observation

given $x \in X$ we need find a local countable basis of open neighbourhoods of x

$$\{O_i^x \mid i<\omega\} \text{ the basis?}$$

example: $\{B(x, 1/n) \mid n<\omega\}$



$B(x, 1/n)$.

Example (context)

- 2^{ω_1} is not 1st countable, hence not metrizable.
- 2^{ω}, $\mathbb{R}^{\omega}$, I^{ω} are first countable, and are

metrizable

↓ Hilbert cube $\square$

Corollary

There is no limit on the size of compact metric spaces.

(5) What about separable metric spaces?
(we know that their size $\leq 2^{\omega}$)

Theorem

Suppose (X,d) is a metric space. TFAE:

1) (X,d) is separable
2) $(X, \mathcal{T}_d)$ is second countable.

Proof (sketch)

2) → 1)

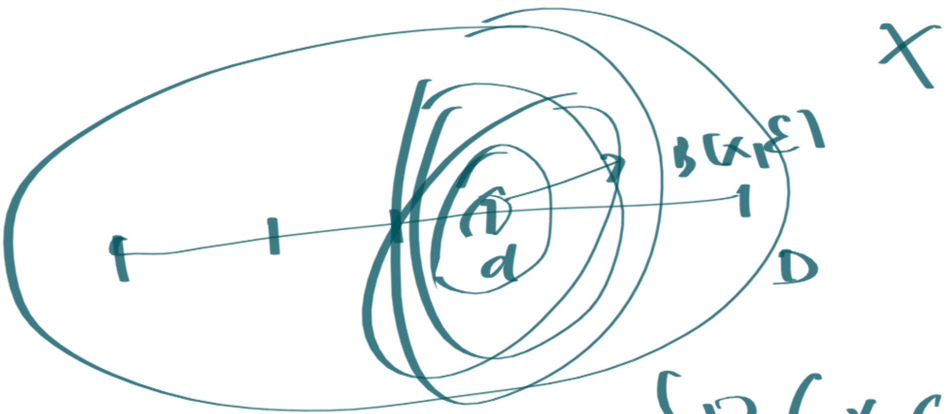
Let $\{O_i \mid i<\omega\}$ be a countable basis. We are looking for a countable dense set $D \subseteq X$

$(\forall O \in T_d, O \neq \emptyset \rightarrow D \cap O = \emptyset)$

hint: choose arbitrarily $x_n \in O_n$.

show that $\{x_n \mid n<\omega\}$ is a dense set.

1) → 2) (you need to use the metric)



$\forall d \in D$ consider $\{B(x,\varepsilon) \mid \varepsilon \in \mathbb{Q}^+\}$

Since $|D| = \omega$, and $|\mathbb{Q}| = \omega$,

$\{ B(x, \varepsilon) \mid x \in D, \varepsilon \in \mathbb{Q}^+ \}$

is countable. Check it is a basis.

$\square$

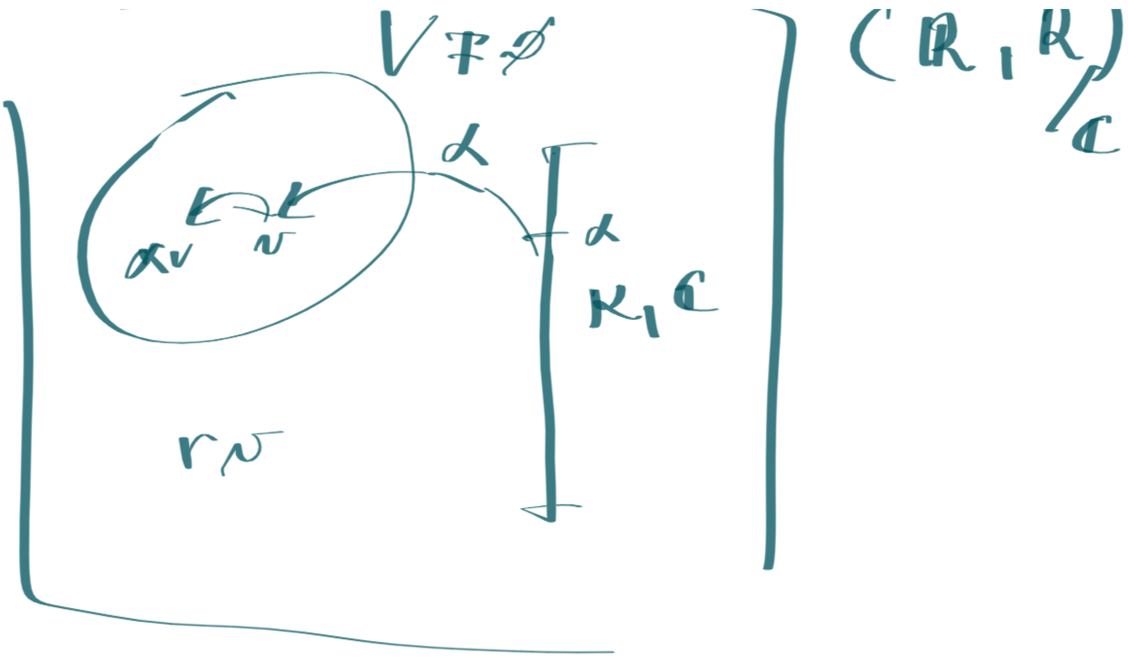
Exercise*.

Suppose (X, d) is a compact metric space, then τ_d is second-countable ($\sim (X, d)$ is a separable space)

(end of discussing topology and metric)

VECTOR SPACES

and linear transformations



transformation

$(V, F) \longrightarrow (W, F)$

In reality only certain transformation are permitted (considered)