Econ 403: Advanced Topics in Macroeconomics

Answer key to Problem Set #3 : Real Business Cycle Model and A Simple OLG Monetary Model

Q1: Consider the following simple RBC model: Preferences are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [bc_t^{1-\eta} + (1-b) l_t^{1-\eta}]^{\frac{1}{1-\eta}} \qquad 0 < \beta < 1 \tag{0.1}$$

where $l = 1 - n$ is leisure, technology is given by

$$y_t = e^{z_t} k_t^{\alpha} n_t^{1-\alpha} \tag{0.2}$$

and the resource constraint:

$$c_t + k_{t+1} - (1-\delta) k_t = y_t \tag{0.3}$$

a) Set up the social planner's problem for the economy and derive the first order conditions.

The social planner's problem is

$$\max_{c_t, n_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t [bc_t^{1-\eta} + (1-b)(1-n_t)^{1-\eta}] l_t^{\frac{1}{1-\eta}} \tag{0.4}$$

S.t.

$$c_t + k_{t+1} - (1-\delta) k_t = y_t = e^{z_t} k_t^{\alpha} n_t^{1-\alpha} \tag{0.5}$$

So the Lagrangian is

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ [bc_t^{1-\eta} + (1-b)(1-n_t)^{1-\eta}]^{\frac{1}{1-\eta}} + \lambda_t [e^{z_t} k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta) k_t] \right\} \tag{0.6}$$

So the first order conditions imply:

$$\frac{\partial L}{\partial c_t} = 0 \Rightarrow \frac{1}{1-\eta} [bc_t^{1-\eta} + (1-b)(1-n_t)^{1-\eta}]^{\frac{1}{1-\eta}-1} (1-\eta) bc_t^{-\eta} - \lambda_t = 0 \tag{0.7}$$

$$\frac{\partial L}{\partial n_t}=0\Rightarrow\frac{1}{1-\eta}[bc_t^{1-\eta}+(1-b)(1-n_t)^{1-\eta}]^{\frac{1}{1-\eta}-1}(1-\eta)(1-b)(1-n_t)^{-\eta}(-1)+\lambda_t(1-\alpha)e^{z_t}k_t^{\alpha}n_t^{-\alpha}=0 \tag{0.8}$$

$$\frac{\partial L}{\partial k_{t+1}}=0\Rightarrow -\lambda_t+E_t\beta\lambda_{t+1}[\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha}+(1-\delta)]=0 \tag{0.9}$$

So the optimality conditions are:

$$\frac{b}{1-b}\frac{c_t^{-\eta}}{(1-n_t)^{-\eta}}=\frac{1}{(1-\alpha)e^{z_t}k_t^{\alpha}n_t^{-\alpha}} \tag{0.10}$$

$$U_c(c_t,1-n_t)=\beta E_t U_c(c_{t+1},1-n_{t+1})[\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha}+(1-\delta)] \tag{0.11}$$

where $U_c(c_t,1-n_t)$ is the partial derivative of the instantaneous utility function with respect to c_t, and $U_c(c_t,1-n_t)=[bc_t^{1-\eta}+(1-b)(1-n_t)^{1-\eta}]^{\frac{1}{1-\eta}-1}bc_t^{-\eta}$.

Note that Equation 0.10 is the intratemporal optimality condition for the social planner, and Equation 0.11 is the intertemporal optimality condition for social planner.

b) Instead of a social planner, assume there are households that maximize utility and firms that maximize profit. Households and firms interact in competitive markets. Write down the decision problem s of the representative households and firms. Derive the first order conditions and show that they imply the same equilibrium as derived from the social planner approach.

The firms will maximize the period-to-period profit, so we have

$$\max_{k_t,n_t}\pi_t=e^{z_t}k_t^{\alpha}n_t^{1-\alpha}-w_tn_t-r_tk_t \tag{0.12}$$

First order conditions imply:

$$w_t=(1-\alpha)e^{z_t}k_t^{\alpha}n_t^{-\alpha} \tag{0.13}$$

$$r_t=\alpha e^{z_t}k_t^{\alpha-1}n_t^{1-\alpha} \tag{0.14}$$

The households maximize their life-time utility, so we have

$$\max_{c_t,n_t,k_{t+1}} E_0\sum_{t=0}^{\infty}\beta^t[bc_t^{1-\eta}+(1-b)(1-n_t)^{1-\eta}]l_t^{\frac{1}{1-\eta}} \tag{0.15}$$

S.t.

$$c_t + k_{t+1} = w_t n_t + r_t k_t + (1-\delta)k_t \tag{0.16}$$

So the Lagrangian is

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ [bc_t^{1-\eta} + (1-b)(1-n_t)^{1-\eta}]^{\frac{1}{1-\eta}} + \mu_t[w_t n_t + r_t k_t + (1-\delta)k_t - c_t - k_{t+1}] \right\} \tag{0.17}$$

Following the same steps as we solve for the social planner's problem, we can get the following optimality conditions:

$$\frac{b}{1-b}\frac{c_t^{-\eta}}{(1-n_t)^{-\eta}} = \frac{1}{w_t} \tag{0.18}$$

$$U_c(c_t, 1-n_t) = \beta E_t U_c(c_{t+1}, 1-n_{t+1})[r_{t+1} + (1-\delta)] \tag{0.19}$$

where $U_c(c_t, 1-n_t) = [bc_t^{1-\eta} + (1-b)(1-n_t)^{1-\eta}]^{\frac{1}{1-\eta}-1} bc_t^{-\eta}$.

If you substitute Equations 0.13 and 0.14 into the above households optimality conditions, you can find that you get Equations 0.10 and 0.11. Also, if you substitute Equations 0.13 and 0.14 into the budget constraint of the households, you can get the resource constraint. So the solution to the social planner's problem is the same as the solution to the decentralized economy (competitive equilibrium). So in the economy the competitive equilibrium is Pareto optimal.

c) For a given value of c_t, how does an increase in b affect the labor supply curves? explain?

From the intratemporal optimality condition, we can get the labor supply function:

$$(1-n_t)^{\eta} = \frac{1}{w_t}\frac{1-b}{b}c_t^{\eta} = \frac{1}{w_t}\left(\frac{1}{b} - 1\right)c_t^{\eta} \tag{0.20}$$

So if b increases, we should have the RHS of the above equation decreases (for given values of w_t and c_t), which implies the LHS should also decrease. If we have n_t increase, then $1-n_t$ will decrease, and $(1-n_t)^{\eta}$ will also decrease. So when b

increases, the equilibrium level of n_t will also increase, for given level of c_t and w_t. In other words, the labor supply curve should shift right.

The intuition is that $\frac{1-b}{b}$ represents the relative importance of leisure to consumption in the instantaneous utility function. When b increase, the households value consumption more and value leisure less, so they will work more to increase their consumption.