5.6 Technological change: Models with an expanding variety of products

- **technological process:** advances in methods of production, and types and qualities of products
  - in Solow and Ramsey growing at exogenous rate x
- Our goal = explain the **origin** of x
  - expansion in the **number of varieties** of product (or new industries)
  - **quality improvements** for existing products

Model: 3 types of agents:

- final good producers: hire labor + intermediate inputs -> produce and sell output
- R&D firms: invent "new products" = intermediate inputs -> get patent on them -> sell at profit maximizing price
- households

5.6.1 Producers of final output:

- **Production function** (for firm i)

$$Y_i = AL_i^{1-\alpha}\sum_{j=1}^{N}(X_{ij})^\alpha = AL_i^{1-\alpha}\Big[X_{i1}^\alpha + X_{i2}^\alpha + \ldots + X_{iN}^\alpha\Big], \quad \alpha \in (0,1)$$

  - A - efficiency parameter
  - X_{ij} - use of jth type of specialized intermediate good
  - N - number of varieties of intermediate goods

- **Characteristics** of production function:
  - decreasing marg. product of both L_i and X_{ij}
  - constant returns to scale (in total)
  - additive separability of intermediate inputs => two intermediate goods are neither complements ($\frac{\partial MPX_{iz}}{\partial X_{iv}} > 0$) nor substitutes $\frac{\partial MPX_{iz}}{\partial X_{iv}} < 0$)
  - as $\lim_{(X_{ij} \to 0)} \frac{\partial Y_i}{\partial X_{ij}} = \infty$, firms are motivated to use all N types of intermediate goods (IG)

- **Profit maximization:**

$$\max \pi_i = AL_i^{1-\alpha}\sum_{j=1}^{N}(X_{ij})^{\alpha} - wL_i - \sum_{j=1}^{N} P_j X_{ij}$$

$$\frac{\partial Y_i}{\partial X_{ij}} = AL_i^{1-\alpha}\alpha X_{ij}^{\alpha-1} = P_j$$

$$\frac{\partial Y_i}{\partial L_i} = A(1-\alpha)L_i^{-\alpha}X_{ij}^{\alpha} = (1-\alpha)\frac{Y_i}{L_i} = w$$

$$X_{ij} = L_i\Big(\frac{\alpha A}{P_j}\Big)^{\frac{1}{1-\alpha}}$$

where $X_{ij}(P_j)$ is a **demand function** for IG X_{ij} with constant price elasticity $\frac{-1}{1-\alpha}$ (i.e. decreasing with price).

5.6.2 R&D firms:

- invention and production of new IG $\Rightarrow$ expansion of N
- **2 stage decision process**:
  1. Decision to finance invention: compare NPV of expected projects and current R&D expenditures on invention
  2. Determine the optimal price for the new intermediate good
- we solve the problem backwards

Stage 2: Optimal price once the good has been invented

- problematic motivation for research - idea is a non-rival good, everyone can use it and produce the intermediate good
- **patent:** inventor of good j retains perpetual monopoly right over the production and sale of the good X_j (his invention)

PV of profits from discovering the j^{th} intermediate good

$$V(t) = \int_t^{\infty} \underbrace{\pi_j(v)}_{\text{profit flow}} \underbrace{e^{-\bar{r}(t,v)(v-t)}}_{\text{discount factor}} dv; \qquad \bar{r}(t,v) \equiv \frac{1}{v-t}\int_t^v r(w)dw$$

- revenues: $P_j(v)X_j(v)$
- costs: ass. 1 unit of X_j = 1 unit of Y (MC = AC =1)
- profits: $\pi_j(v) = [P_j(v) - 1]X_j(v)$

$$X_j(v) = \sum_i X_{ij}(v) = \sum_i L_i\Big(\frac{\alpha A}{P_j}\Big)^{\frac{1}{1-\alpha}} = L\Big(\frac{\alpha A}{P_j}\Big)^{\frac{1}{1-\alpha}}$$

- no accumulation and no intertemporal optimization in the problem $\Rightarrow$ **static optimization**

$$\max_{P_j(v)} \quad \pi_j(v) = [P_j(v) - 1]X_j$$
$$X_j + (P_j(v) - 1)\frac{\partial X_j}{\partial P_j} = 0 \quad \Rightarrow \; 1 + \Big(1 - \frac{1}{P_j(v)}\Big)\frac{P_j(v)}{X_j}\frac{\partial X_j}{\partial P_j} = 0$$
$$P_j(v) = P_j = \frac{1}{\alpha}$$

- $P_j(v) = P_j = P = 1/\alpha > 1$ - price of intermediate goods is constant over time, same for all IG and higher than marginal costs (due to monopoly power)
- We can thus plug into the expressions for demand (for individual IG as well as aggregate), output, profit and NPV of profits:

$$X_j(v) = L\Big(\frac{\alpha A}{P_j}\Big)^{\frac{1}{1-\alpha}} = L[A\alpha^2]^{\frac{1}{1-\alpha}} = X_j = \frac{X}{N} \tag{1}$$
$$Y = AL_i^{1-\alpha}\sum_{j=1}^{N}(X_{ij})^{\alpha} = AL_i^{1-\alpha}NX_j^{\alpha} = LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}} = \frac{X}{\alpha^2} \tag{2}$$
$$\pi_j(v) = [P_j(v) - 1]X_j = \Big[\frac{1-\alpha}{\alpha}\Big]L[A\alpha^2]^{\frac{1}{1-\alpha}} = \pi_j = \pi = \frac{1}{N}\alpha(1-\alpha)Y \tag{3}$$
$$V(t) = \Big[\frac{1-\alpha}{\alpha}\Big]L[A\alpha^2]^{\frac{1}{1-\alpha}}\int_t^{\infty} e^{-\bar{r}(t,v)(v-t)}dv \tag{4}$$

Stage 1: Decision to enter R&D business

- assumption: cost of creating new IG = η units of Y, η is constant [2]
- firm decides to enter if $V(t) \geq \eta$
- **FREE ENTRY** condition: $V(t) = \eta$
  (if $V(t) < \eta$ nobody would enter - no technology growth, if $V(t) > \eta$ everybody would enter and infinite amounts of money would be invested - infeasible in equilibrium)

[2] Parameter η can be a function of N, either decreasing ("economies of scale") or decreasing (running out of ideas).

- If we differentiate free entry condition with respect to time and use expressions for $V(t)$ (4) and π (3) as well as the fact that $\bar{r}(t,v) \equiv \frac{1}{v-t}\int_t^\infty r(w)dw$ we obtain condition for the clearing of investment market.

$$\underbrace{r(t)}_{\text{rate of return on assets}} = \underbrace{\frac{\pi}{V(t)} + \frac{\dot{V}(t)}{V(t)}}_{\text{rate of return to investing in R\&D}}$$

  - $\frac{\pi}{V(t)}$ - profit rate
  - $\frac{\dot{V}(t)}{V(t)}$ - capital gain or loss from the change in the value of the research firm
  - $V(t) = \eta \Rightarrow \dot{V}(t) = 0 \Rightarrow r(t) = r = \frac{\pi}{V(t)} = \frac{\pi}{\eta}$

$$r = \left[\frac{1-\alpha}{\alpha}\right]\frac{L}{\eta}[A\alpha^2]^{\frac{1}{1-\alpha}}$$

- old and new products have the same flow of profits (same markups), i.e. aggregate value of firms that are owned by households is ηN

5.6.3 Households:

- population is constant - $n = 0$
- utility function $\int_0^\infty \frac{c^{1-\theta}-1}{1-\theta}e^{-\rho t}dt$
- Aggregate budget constraint: $d(Assets)/dt = wL + rAssets - C$
- Euler equation: $\dot{C}/C = \dot{c}/c = 1/\theta(r-\rho)$

5.6.4 Equilibrium:

- in equilibrium all firms are owned by people and their stocks are only asset available in the economy, therefore

$$Assets = \eta N \quad \Rightarrow \quad \frac{d(Assets)}{dt} = \eta \dot{N}$$

- wage: $w = (1-\alpha)\frac{Y}{L}$
- interest rate: $r = \left(\frac{1-\alpha}{\alpha}\right)\frac{L}{\eta}[A\alpha^2]^{\frac{1}{1-\alpha}} = \frac{1}{\eta N}\alpha(1-\alpha)Y$
- aggregate income of the households

$$wL + rAssets = (1-\alpha)\frac{Y}{L}L + \frac{1}{\eta N}\alpha(1-\alpha)Y\eta N = (1-\alpha^2)Y$$

- Economy wide resource constraint

$$\eta \dot{N} = Y - \underbrace{\alpha^2 Y}_{=X} - C$$

- **growth rate of economy:** constant and by assumption positive

$$\gamma_c = \frac{\dot{c}}{c} = \frac{1}{\theta}\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{L}{\eta}[A\alpha^2]^{\frac{1}{1-\alpha}} - \rho\right]$$

  - Y and N grow at the same rate (from 8)
  - from economy wide resource constraint, if C grows at the constant rate and then N, Y has to grow at the same rate - i.e. γ_c is the common growth rate of the economy item determinants of growth rate γ
    * higher willingness to save by household ($\searrow \rho, \searrow \theta$) implies $\nearrow \gamma$
    * better technology $\nearrow A$ implies $\nearrow \gamma$
    * higher costs of new product $\eta \Rightarrow \searrow r \Rightarrow \searrow \gamma$
    * **scale effect:** larger the economy ($\nearrow L$) $\Rightarrow \nearrow \gamma$ (new invention can be used in the entire economy)

5.6.5 Welfare implications:

Let us show that outcome of decentralized equilibrium is not Pareto optimal by analyzing the solutions of the **central planner**. Central planner maximizes the utility of representative household given the economy's budget constraint

$$Y = AL^{1-\alpha}N^{1-\alpha}X^{\alpha} = C + \eta\dot{N} + X$$

The Hamiltonian and F.O.C.'s from this problem (control variables - c, X, state variable N) are

$$\begin{aligned}
H &= u(c)e^{-\rho t} + \lambda\frac{1}{\eta}(AL^{1-\alpha}N^{1-\alpha}X^{\alpha} - Lc - N) \\
\frac{\partial H}{\partial c} = 0 &: \quad u'(c)e^{-\rho t} = \frac{\lambda}{\eta} && (5) \\
\frac{\partial H}{\partial X} = 0 &: \quad \frac{\lambda}{\eta}(AL^{1-\alpha}N^{1-\alpha}\alpha X^{\alpha-1}) = 0 && (6) \\
\frac{\partial H}{\partial X} = -\frac{\dot{\lambda}}{\eta} &: \quad \frac{\lambda}{\eta}(AL^{1-\alpha}(1-\alpha)N^{-\alpha}X^{\alpha}) = -\frac{\dot{\lambda}}{\eta} && (7)
\end{aligned}$$

From combination of (5) and (6) we get the equilibrium demand for intermediate goods and resulting output, and from (5) and (7) we get and expression for the growth rate of economy. I introduce also the values from decentralized equilibrium for comparison.

$$\begin{aligned} X_{SP} &= LN[A\alpha]^{\frac{1}{1-\alpha}} \\ Y_{SP} &= LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} \\ \gamma_{SP} &= \frac{1}{\theta}\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{L}{\eta}[A\alpha]^{\frac{1}{1-\alpha}} - \rho\right] \end{aligned} \qquad \begin{aligned} X_{DE} &= LN[A\boldsymbol{\alpha^2}]^{\frac{1}{1-\alpha}} \\ Y_{DE} &= LNA^{\frac{1}{1-\alpha}}\alpha^{\frac{\mathbf{2}\alpha}{1-\alpha}} \\ \gamma_{DE} &= \frac{1}{\theta}\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{L}{\eta}[A\alpha^2]^{\frac{1}{1-\alpha}} - \rho\right] \end{aligned}$$

As $\lambda \in (0,1)$, and $X_{DE} = X_{SP}\,\alpha^{1/(1-\alpha)}$ social planner allocates more resources into the purchase of intermediate goods. It is same amount as would be demanded if the price (in DE) was at the level of marginal costs. Therefore, SP equilibrium achieves higher level of output as decentralized equilibrium, and thus greater consumption. Moreover, DE has lower growth rate than the central planner case. This is due to the fatc that monopoly creates a gap between social and private returns.

How can a government correct for this? Possible policies:

1. **Subsidies to purchases of intermediate good**
   - financed through nondistortionary tax (lump sum)
   - 1 unit of X will cost αP_j (the rest is covered from subsidy), although the price set by monopoly is still $1/\alpha$
   - amount demanded: $X = LN\left(\frac{A\alpha}{\alpha P}\right)^{1/1-\alpha} = X = LN\Big(A\alpha\Big)^{1/1-\alpha} = X_{SP}$
   - **static gain:** given N - we get demand for intermediate goods and final output like in the SP case - i.e. consumption is higher
   - **dynamic gain** N grows at efficient (higher) rate - the whole economy is growing faster
2. **Subsidies to final product**
   - producers receive revenue of $1/\alpha$ units for each sold Y
3. **Subsidies to R&D**
   - lower the value of η - can achieve dynamic gains - higher growth rate γ and returns to assets
   - however, monopoly still persist - X and Y are still too low and not socially optimal (static gain is not achieved)