JEM004 Macroeconomics
IES, Fall 2010
Lecture Notes
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5 Endogenous Models of Economic Growth

5.1 Motivation

- Solow-Swan model + Ramsey model: s.s. growth rate of $Y/L = \frac{\Delta Y/L}{Y/L} = \gamma_y = g$
- x - rate of technological progress, **EXO**GENOUSLY given
- want to have growth **ENDO**GENOUS, i.e. we are able to explain it as the outcome of the decisions of agents within the model

Possible solutions:

- **AK models** - abandon diminishing returns to capital (DRC)
  - broad definition of capital (physical + **human**)
    - 1 sector: production of goods + accumulation of capital
    - 2 sectors: production of both goods and (human)capital - education
  - **learning-by-doing** + **spillovers** of knowledge
    - individual firms - DRC, aggregate level - CRC/IRC
- **R&D models** Advances in technology level determined by purposeful activity (explicitly model determinants of g)
  - expanding **variety** of products
  - **quality** improvements of existing products

5.2 AK model

We consider set-up similar to Ramsey model with 1 departure - production function is not neoclassical (concave in capital and labor), but linear.[1]

$y = f(k) = Ak, \quad A > 0$

Properties and implications:

[1] One can get this functional form easily by taking Hicks-neutral neoclassical production function - $Y_t = AK_t^{\alpha}L_t^{1-\alpha}, \quad \alpha \in (0,1)$ and just setting $\alpha = 1$ which gives $Y_t = AK_t$ or $y_t = Ak_t$.

- marg. product of capital is positive constant $f'(k) = A > 0 \quad \Rightarrow \quad r = A - \delta$
- marg. product of labor is zero $\Rightarrow w = 0$
- Inada conditions do not hold

From the standard household optimization problem we have Euler equation and law of motion for capital. Thus in equilibrium, when we plug in for r and w we obtain system of equations

$$\begin{aligned}
\dot{k} &= (A - \delta - n)k - c \\
\frac{\dot{c}}{c} &= \frac{1}{\theta}(A - \delta - \rho) \\
& \lim_{t\to\infty} k(t)e^{-(A-\delta-n)t} = 0 \quad (NPG)
\end{aligned}$$

Second equation implies that growth rate of consumption $\gamma_c = \frac{\dot{c}}{c}$ is independent of capital - i.e. given initial value $c(0)$ the consumption per capita evolves as

$$c_t = c(0\ )e^{1/\theta[A-\delta-\rho]t}.$$

We put two restrictions on the parameters:

- for $\gamma_c > 0 \quad \Rightarrow \quad A > \delta + \rho$
- for bounded (CRRA) utility $\Rightarrow \quad \rho - n > \frac{1-\theta}{\theta}(A - \delta - \rho)$

To compute the growth rate of capital and output per worker we divide law of motion for capital by k obtaining

$$\frac{\dot{k}}{k} = (A - \delta - n) - \frac{c}{k}.$$

As in S.S. $\gamma_k = \frac{\dot{k}}{k} = \text{const}$, then $\frac{c}{k} = \text{const}$ and therefore $\gamma_k = \gamma_c$. Moreover, as $y = Ak$, we finally get that

$$\gamma_y = \gamma_k = \gamma_c = \frac{1}{\theta}[A - \delta - \rho].$$

We can also get the expression for saving rate

$$s = \frac{\dot{K} + \delta K}{Y} = \frac{1}{A}\frac{\dot{K} + \delta K}{K} = \frac{1}{A}[\gamma_k + n + \delta] = \frac{1}{A}[\frac{A - \delta - \rho}{\theta} + n + \delta].$$

Therefore, the growth rates as well saving rate in the model are determined endogenously - by the parameters of the model (willingness to save of agents - θ, ρ, productivity of capital - A) rather than exogenously given growth rates. An improvement in the level of technology A raises growth rate as well as saving rate - possible target of government policies.

5.2.1 Transitional dynamics

Let us analyze how the economy will behave outside the steady state. First let us take law of motion for capital and plug in the expression for consumption growth and get

$$\dot{k} = (A - \delta - n)k - c(0\ )e^{1/\theta[A-\delta-\rho]t}$$

which if first order linear differential equation in k. After solving it (on the lecture), we obtain general solution in form

$$k(t) = \frac{c(0)}{\phi}e^{1/\theta[A-\delta-\rho]t} + \text{const}\, e^{[A-\delta-n]t}$$

where

$$\phi = [A - \delta - n] - \frac{1}{\theta}[A - \delta - \rho] = \frac{(\theta - 1)(A - \delta) + \rho - \theta n}{\theta}$$

From the condition on bounded utility function we know that $\rho - n > \frac{1-\theta}{\theta}(A - \delta - \rho)$, which implies (try yourself ;-)) $\phi > 0$.
We can determine the value of const using the NPG condition

$$\lim_{t\to\infty} k(t)e^{-(A-\delta-n)t} = 0$$

$$\lim_{t\to\infty} \left\{\frac{c(0)}{\phi}e^{\left(1/\theta[A-\delta-\rho]-[A-\delta-n]\right)t} + \text{const}\right\} = 0$$

$$\lim_{t\to\infty} \left\{\frac{c(0)}{\phi}e^{-\phi t} + \text{const}\right\} = 0$$

As we know that $\phi > 0$ then const $= 0$. Further

$$k(t) = \frac{c(0)e^{1/\theta[A-\delta-\rho]}}{\phi} = \frac{c(t)}{\phi}$$

That means there is **NO TRANSITIONAL DYNAMICS**. Given initial level of state variable - capital $k(0)$ we immediately determine optimal consumption $c(0) = \phi k(0)$ and output $y(0) = Ak(0)$ and all variables grow at constant rate $\gamma = \frac{1}{\theta}[A - \delta - \rho]$

5.3 AK model - one sector with human capital

- Criticism - where are the people?
- Include effect of labor input into AK model - through stock of human capital
- More skilled individual can produce higher output than unskilled
- advantage: using neoclassical production function we get AK model results

In this set up, we have standard neoclassical production with two inputs - physical capital K and human capital H

$$y = F(K,H) = KF(1,H/K) = Kf(H/K); \quad \text{kde } f'(H/K) > 0$$

One sector technology applies - i.e. one good can be used for consumption as well as for investment into human and physical capital.

Rental prices for competitive firms:

$$\begin{aligned} R_H &= \frac{\partial Y}{\partial H} = Kf'(H/K)1/K = f'(H/K) \\ R_K &= f(H/K) - Kf'(H/K)H/K^2 = f(H/K) - H/Kf'(H/K) \end{aligned}$$

Since the inputs are perfect substitutes, net returns (net of depreciation) must be equal

$$R_H - \delta_H = R_K - \delta_K \quad \Rightarrow R_H - R_K = \delta_H - \delta_K = const$$

We get

$$f'(H/K) + H/Kf'(H/K) - f(H/K) = \delta_H - \delta_K$$

and thus there is unique value of ratio H/K that will be in equilibrium. If we define $f(H/K) \equiv A$ then we have $Y = AK$ - standard AK model with all its implications - constant endogenous growth, same for all variables (H/K = const, i.e. $\gamma_H = \gamma_K$).

5.4 Learning by doing - capital spillovers (Example 1.)

- advantage - we can use neoclassical production function (firm level) $Y_i = F(K_i, A_i L_i)$and still get endogenous growth
- **Learning by doing** - individual firm level:
  - each time a firm produces, it learns how to produce more efficiently
  - therefore, accumulation of experience A is according to $\dot{A}_i = \dot{K}_i$
  - efficiency gain enters in labor part, but it is measured in terms of capital
- **Knowledge spillovers**: aggregate level
  - learning is a public good - when firm learns (produces), it affects the aggregate technology level - **learning externality**
  - accumulation of experience: $\dot{K}_i = \dot{A}_i = \dot{A} = \dot{K}$
  - if we neglect initial level of knowledge before capital production $\int_0^t \dot{A} = \int_0^t \dot{K} \quad \Rightarrow \quad A_t = K_t$
  - the production function transforms $Y_i = F(K_i, KL_i)$

Assume standard form of Cobb-Douglas production function

$$Y_i = K_i^{\alpha}(KL_i)^{1-\alpha}$$

For simplicity we assume constant L (n=0) and thus

$$y_i = k_i^{\alpha} K^{1-\alpha}; \quad \frac{\partial y_i}{\partial k_i} = \alpha k_i^{\alpha-1} K^{1-\alpha}$$

Firms' problem: firms take the aggregate level of technology (A or, as we know, K) as given and do not consider the effect of their investment (capital accumulation) on it (i.e. do not consider $K = g(\dot{k}_i)$). Therefore, the profit maximization does not take this effect into consideration.

$$\begin{aligned}
\max_{k_i} \pi_i &= k_i^{\alpha} K^{1-\alpha} - (r+\delta)k_i - w \\
F.O.C. \qquad & \alpha k_i^{\alpha-1} K^{1-\alpha} = r + \delta \\
& r = \alpha k_i^{\alpha-1} K^{1-\alpha} - \delta \\
& w = f(k_i) - f'(k_i)k_i = k_i^{\alpha} K^{1-\alpha} - \alpha k_i^{\alpha} K^{1-\alpha} = (1-\alpha) k_i^{\alpha} K^{1-\alpha}
\end{aligned}$$

Households problem is not affected by the knowledge spillovers, therefore we obtain standard Euler equation and budget constraint

$$\begin{aligned}
\gamma_c = \frac{\dot{c}}{c} &= \frac{1}{\theta}[r - \rho] \\
\dot{a} &= w + ra - c
\end{aligned}$$

In equilibrium $K = k_i L$ so we can plug in for $r = \alpha L^{1-\alpha} - \delta$

$$\begin{aligned}
\gamma_c &= \frac{1}{\theta}[\alpha L^{1-\alpha} - \delta - \rho] \\
\dot{k} &= (L^{1-\alpha} - \delta)k - c
\end{aligned}$$

Obviously, γ_c is independent of k, i.e. it is constant. Also, c, k and y grow at the same rate. If we rewrite $y_i = k_i^{\alpha} K^{1-\alpha} = \underbrace{L^{1-\alpha}}_{A} k$, we see we are again dealing with an AK type model.

Scale effects: implications of this model would be that larger countries ($\nearrow$ L) should grow more.

Pareto inefficiency: due to the fact, that there is capital externality that firms do not internalize (in normal words, firms do not consider the impact of their investments on the aggregate level of technology in the economy), the outcome of decentralized equilibrium (where all agents act to achieve their selfish personal best and interact with other agents) is not optimal (i.e. they could have higher utility by other setting). Let us prove it.

Let us consider what would be decision of a central planner (almighty and good god of the economy :-)) that wants to maximize the utility of all the agents and knows everything (i.e. he knows how the technology accumulation works). We can put down his problem in terms of problem of a representative agent (as all agents are homogenous).

$$\begin{aligned} \max \quad & \int_0^\infty \frac{c_t^{1-\theta}-1}{1-\theta} e^{-\rho t} dt \\ s.t. \quad & \dot{k}_t = k_t^\alpha K^{1-\alpha} - c_t - \delta k_t = k_t L^{1-\alpha} - c_t - \delta k_t \end{aligned}$$

The outcome of this maximization problem results into the standard Euler equation for γ_c, where we just plug in for the marginal product of capital

$$\gamma_c = \frac{1}{\theta}[L^{1-\alpha} - \delta - \rho] = \gamma_{CP}$$

If we compare the growth rates for decentralized economy and economy under central planner, we confirm our suspicion - growth rate under central planner is higher than growth rate under decentralized equilibrium, exactly due to unexploited externality.

$$\frac{1}{\theta}[\alpha L^{1-\alpha} - \delta - \rho] < \frac{1}{\theta}[L^{1-\alpha} - \delta - \rho] \ \Leftrightarrow \ \gamma_{CP} > \gamma_{DE}$$

5.5 Public goods model of productive government services (Example 2.)

- externality that increases the marginal productivity of capital
- G is nonrival and nonexcludable (every firm makes use of all G)

$$\begin{aligned} Y_i &= AL_i^{1-\alpha} K_i^\alpha G^{1-\alpha} = L_i[Ak_i^\alpha G^{1-\alpha}] \\ G &= \tau Y \quad \text{- proportional tax on gross output = burden on firms} \end{aligned}$$

Firms have to maximize after tax income

$$\begin{aligned} \max_{k_i} \pi_i \quad &= \quad (1-\tau)Ak_i^\alpha G^{1-\alpha} - (r+\delta)k_i - w \\ F.O.C. \quad & \quad (1-\tau)\alpha Ak_i^{\alpha-1} G^{1-\alpha} = r + \delta \\ & \quad r = (1-\tau)\alpha Ak_i^{\alpha-1}G^{1-\alpha} - \delta \\ & \quad w = f(k_i) - f'(k_i)k_i = (1-\alpha)(1-\tau)k_i^{\alpha-1}K^{1-\alpha} \end{aligned}$$

Household's optimization is unchanged (taxation only affects firms). Therefore, we can directly proceed to equilibrium conditions. In equilibrium, we know that government expenditures are financed with proceedings from output produced by firms, therefore

$$\begin{aligned} G &= \tau LAk^\alpha G^{1-\alpha} = (\tau LA)^{1/\alpha} k \\ r &= (1-\tau)(\tau L)^{\frac{1-\alpha}{\alpha}} \alpha A^{1/\alpha} - \delta \\ y &= (1-\tau)A(\tau LA)^{\frac{1-\alpha}{\alpha}} = \underbrace{(1-\tau)(\tau L)^{\frac{1-\alpha}{\alpha}} A^{1/\alpha}}_{\text{const}} k \end{aligned}$$

Again, we are dealing with AK type model. We plug in to Euler equation and obtain expression for the growth rate of the economy (consumption, capital as well as output).

$$\gamma_c = \gamma_k = \gamma_y = \frac{1}{\theta}\Big[(1-\tau)(\tau L)^{\frac{1-\alpha}{\alpha}}\alpha A^{1/\alpha} - \delta - \rho\Big]$$

If the government wants to maximize the growth rate, what is the **optimal tax rate**? Maximum growth rate is attained when first derivative w.r.t τ is equal to 0.

$$\begin{aligned}
\frac{1}{\theta}\alpha A^{1/\alpha}L^{\frac{1-\alpha}{\alpha}}\left[\frac{1-\alpha}{\alpha}\tau^{\frac{1-2\alpha}{\alpha}}(1-\tau) - \tau^{\frac{1-\alpha}{\alpha}}\right] &= 0 \\
\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau} &= 1 \\
(1-\tau)(1-\alpha) &= \alpha\tau \\
\tau &= 1-\alpha
\end{aligned}$$

Therefore, equilibrium growth rate in **decentralized economy with benevolent government** is

$$\gamma_c = \gamma_k = \gamma_y = \frac{1}{\theta}\Big[(\alpha^2 A^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}}L^{\frac{1-\alpha}{\alpha}} - \delta - \rho\Big]$$