5. Seminar: OLG models.

The first part should have been covered in the seminar, here is the solution. It is also similar type to the excercises that might occur in the midterm. The second part – social security in the olg model – is included mostly for illustration of the main idea. If you go through it, you will see how different systems affect optimal behavior. In case of PAYG system for example, population growth starts to play an important role (it can be seen from changed budget constraint), otherwise the optimization is the same as introduced in the lecture.

1) Consider the Diamond model with logarithmic utility function and the Coub- Douglas production function. Describe how each of the following affects $\boldsymbol{k_{t+1}}$ as a function of $\boldsymbol{k_t}$:
   a) A rise in $\boldsymbol{n}$,
   b) A downward (proportional) shift of the production function (if the production function is in the form of Coub Douglas function $\boldsymbol{f(k)=B.k^{\alpha}}$, this means the fall in $\boldsymbol{B}$)

<u>Hint:</u>

You have derived (using the Lagrange function and constant relative risk aversion utility function) the following equation:

$$k_{t+1} = \frac{1}{(1+n)\cdot(1+g)}\cdot s(r_{t+1})\cdot w_t = \frac{1}{(1+n)\cdot(1+g)}\cdot s(f'(k))\cdot\left[f(k_t)-k_t\cdot f'(k_t)\right]$$

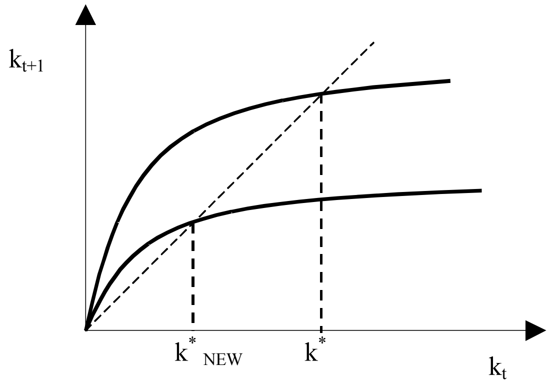
For logarithmic utility function (the limit case of the CES function with zero risk aversion) and for the Coub-Douglas production function this expression could be rewritten as:

$$k_{t+1} = \frac{1}{(1+n)\cdot(1+g)}\cdot\frac{1}{2+\rho}\cdot(1-\alpha)\cdot k_t^{\alpha} = D\cdot k_t^{\alpha}$$

This is the expression we will use in this exercise.

Solution

a) If the growth of the population n increases, the term $\boldsymbol{D}$ must fall and function $\boldsymbol{k_{t+1}}$ moves proportionally down (see the graph). Because of the fact that the saving rate of the young does not depend on $\boldsymbol{n}$, given amount of $\boldsymbol{k}$ and thus also creates the same amount of savings as before increase of $\boldsymbol{n}$. As the population increases in time t+1 the $\boldsymbol{k}$ must decrease at t+1 together with decrease of $\boldsymbol{k^*}$ and $\boldsymbol{y^*}$.



b) If we assume the production function in form $\boldsymbol{f(k)=B.k^{\alpha}}$ the function $\boldsymbol{k_{t+1}}$ changes to:

$$k_{t+1} = \frac{1}{(1+n)\cdot(1+g)}\cdot\frac{1}{2+\rho}\cdot(1-\alpha)\cdot B\cdot k_t^{\alpha}$$

Together with fall of $\boldsymbol{B}$ the function $\boldsymbol{k_{t+1}}$, falls proportionally and $\boldsymbol{k^*}$ thus decreases. The graph is thus same as in case of a).

2) Consider a Diamond model with logarithmic utility function, Coub- Douglas production function and ***g=0***

a) Pay-as-you-go social security

Suppose the government taxes each young individual amount T and uses the proceeds to pay benefits to old individuals; thus each old person receives ***(1+n).T***.

i) How, if at all, does this change affect the k_{t+1} as a function of k_t?

ii) How, if at all, does this change affect the balanced growth path value of k (k^*)?

b) Fully funded social security

Suppose the government taxes each young person amount T and uses the proceeds to purchase capital. Individuals born at t therefore receive ($1+r_{t+1}$) when they are old.

i) How, if at all, does this change affect the k_{t+1} as a function of k_t?

ii) How, if at all, does this change affect the balanced growth path value of k (k^*)?

Solution:

a)i)

The utility function is given as:

$$U_t = \ln(C_{1,t}) + \frac{1}{1+\rho} \cdot \ln(C_{2,t+1})$$

the budget constraints modify to:

$$C_{1,t} + S_t = A_t \cdot w_t - T$$

$$C_{21,t+1} = S_t \cdot (1+r_{t+1}) + (1+n) \cdot T$$, where S_t denotes saving at time t.

Let us express the S_t from the second budget constraint:

$$S_t = \frac{C_{21,t+1}}{(1+r_{t+1})} - \frac{(1+n)}{(1+r_{t+1})} \cdot T$$ we will substitute this into the first budget constraint:

$$C_{1,t} + \frac{C_{21,t+1}}{(1+r_{t+1})} - \frac{(1+n)}{(1+r_{t+1})} \cdot T = A_t \cdot w_t - T$$ thus

$$C_{1,t} + \frac{C_{21,t+1}}{(1+r_{t+1})} = A_t \cdot w_t - \frac{(r_{t+1}-n)}{(1+r_{t+1})} \cdot T$$

We will carry out the optimisation with the use of Lagrange function:

$$\Lambda = \ln(C_{1,t}) + \frac{1}{1+\rho} \cdot \ln(C_{2,t+1}) + \lambda \cdot \left[A_t \cdot w_t - \frac{(r_{t+1}-n)}{(1+r_{t+1})} \cdot T - C_{1,t} - \frac{C_{21,t+1}}{(1+r_{t+1})} \right]$$

$$\frac{\partial \Lambda}{\partial C_{1,t}} = \frac{1}{C_{1,t}} - \lambda ,$$ thus in optimum $\frac{1}{C_{1,t}} = \lambda$

$$\frac{\partial \Lambda}{\partial C_{2,t+1}} = \frac{1}{1+\rho} \cdot \frac{1}{C_{2,t+1}} - \lambda \cdot \frac{1}{(1+r_{t+1})},$$ thus in optimum $\frac{1}{C_{2,t+1}} = \lambda \cdot \frac{1+\rho}{(1+r_{t+1})}$

We divide both conditions

$$\frac{C_{2,t+1}}{C_{1,t}} = \frac{1+r_{t+1}}{1+\rho}$$ and substitute this into the budget constraint

$$C_{1,t} + \frac{\frac{1+r_{t+1}}{1+\rho} \cdot C_{1,t}}{(1+r_{t+1})} = A_t \cdot w_t - \frac{(r_{t+1}-n)}{(1+r_{t+1})} \cdot T$$ thus

$$C_{1,t} = \left[A_t \cdot w_t - \frac{(r_{t+1}-n)}{(1+r_{t+1})} \cdot T \right] \cdot \frac{1+\rho}{2+\rho}$$

From this we could express the savings S_t (we will substitute for $C_{1,t}$ into first budget constraint):

$$S_t = A_t \cdot w_t - T - C_{1,t} = A_t \cdot w_t - T - \left[A_t \cdot w_t - \frac{(r_{t+1} - n)}{(1 + r_{t+1})} \cdot T \right] \cdot \frac{1 + \rho}{2 + \rho} =$$

$$= A_t \cdot w_t \cdot \left(1 - \frac{1 + \rho}{2 + \rho} \right) - T \cdot \left(1 - \frac{(r_{t+1} - n)}{(1 + r_{t+1})} \cdot \frac{1 + \rho}{2 + \rho} \right) = A_t \cdot w_t \cdot \frac{1}{2 + \rho} - T \cdot \frac{(1 + r_{t+1}) \cdot (2 + \rho) - (r_{t+1} - n) \cdot (1 + \rho)}{(1 + r_{t+1}) \cdot (2 + \rho)}$$

If $r_{t+1}=n$ then the savings decrease for the same amount for which the taxes have risen, if $r_{t+1}<n$,they would fall more, if it is $r_{t+1}>n$ would fall less than increase in T.

Let us mark $Z_t = \frac{(1 + r_{t+1}) \cdot (2 + \rho) - (r_{t+1} - n) \cdot (1 + \rho)}{(1 + r_{t+1}) \cdot (2 + \rho)}$, thus $S_t = A_t \cdot w_t \cdot \frac{1}{2 + \rho} - T \cdot Z_t$.

The capital at time t+1 equals total savings at time t, thus

$K_{t+1}=S_t.L_t$ or (expressed per unit of effective labour) $k_{t+1}= K_{t+1}/(A.L_{t+1})=S_t/A.L_t/L_{t+1}= S_t/[A.(1+n)]$

Let us substitute for S_t:

$$k_{t+1} = \left[w_t \cdot \frac{1}{2 + \rho} - T \cdot Z_t / A \right] \cdot \frac{1}{(1 + n)}$$

Real wage w_t is given as $w_t=A. (1-\alpha).k_t^{\alpha}$, thus finally

$$k_{t+1} = \left[A \cdot (1 - \alpha) \cdot k_t^{\alpha} \cdot \frac{1}{2 + \rho} - T \cdot Z_t / A \right] \cdot \frac{1}{(1 + n)}$$

a)ii)

To see the effect of the introduction of the social security system (thus to see whether the imposing the tax T decreases or increases k_{t+1}) depends on the sign of Z_t. If Z_t is negative k^* increases and ice versa.

$$Z_t = \frac{(1 + r_{t+1}) \cdot (2 + \rho) - (r_{t+1} - n) \cdot (1 + \rho)}{(1 + r_{t+1}) \cdot (2 + \rho)} = \frac{(1 + r_{t+1}) \cdot (1 + 1 + \rho) - (r_{t+1} - n) \cdot (1 + \rho)}{(1 + r_{t+1}) \cdot (2 + \rho)} =$$

$$= \frac{(1 + r_{t+1}) + (1 + r_{t+1}) \cdot (1 + \rho) - (r_{t+1} - n) \cdot (1 + \rho)}{(1 + r_{t+1}) \cdot (2 + \rho)} = \frac{(1 + r_{t+1}) + (1 + \rho) \cdot [1 + r_{t+1} - r_{t+1} + n]}{(1 + r_{t+1}) \cdot (2 + \rho)} =$$

$$= \frac{(1 + r_{t+1}) + (1 + \rho) \cdot (1 + n)}{(1 + r_{t+1}) \cdot (2 + \rho)} > 0$$

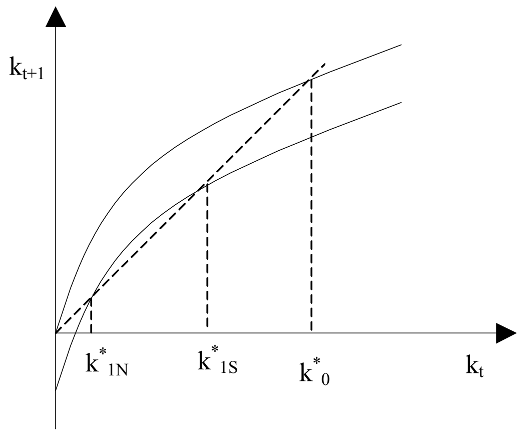
Imposing the tax thus leads to the decrease of the k_{t+1} curve and to the decrease of k^*.

What exactly happens to the $k_{t+1} =f(k_t)$ curve? For $k\to 0$ the $f^{/}(k)\to\infty$ (assuming that the Inada conditions hold) and thus also $r_{t+1}\to\infty$. Let us calculate the limit of Z_t for $k\to 0$:

$$\lim_{k\to 0} Z_t = \lim_{k\to 0} \frac{(1 + r_{t+1}) + (1 + \rho) \cdot (1 + n)}{(1 + r_{t+1}) \cdot (2 + \rho)} = \lim_{k\to 0} \frac{(1 + f^{/}(k)) + (1 + \rho) \cdot (1 + n)}{(1 + f^{/}(k)) \cdot (2 + \rho)} = \lim_{k\to 0} \frac{f^{//}(k)}{(2 + \rho).f^{//}(k)} = \frac{1}{2 + \rho}$$

(non-adjusted limit could not be calculated directly, as the fraction in the limit gives uncertain expression ∞/∞. We could use the l´Hospital rule- we could differentiate both the numerator and the denominator with respect to k, the limit than arises straight ahead)

For $k_t\to 0$ thus the curve k_{t+1} shifts from 0 down to $-T/[A.(2+\rho).(1+n)]$.



After this parallel shift of the $\boldsymbol{k_{t+1}}$ curve they are two possible points of equilibrium- the point for $\boldsymbol{k^*_{1N}}$ and for $\boldsymbol{k^*_{1S}}$. However the point for $\boldsymbol{k^*_{1N}}$ gives unstable equilibrium. Moreover it diverges left to k lower than 0.

b)i)

The budget constraint for the second periond changes to

$$C_{21,t+1} = S_t \cdot (1 + r_{t+1}) + (1 + r_{t+1}) \cdot T \text{ , thus}$$

$$S_t = \frac{C_{21,t+1}}{(1 + r_{t+1})} - T$$

If we substitute for $\boldsymbol{S_t}$ into the budget constraint for the first period we get:

$$C_{1,t} + \frac{C_{21,t+1}}{(1 + r_{t+1})} - T = A_t \cdot w_t - T \qquad \text{thus}$$

$$C_{1,t} + \frac{C_{21,t+1}}{(1 + r_{t+1})} = A_t \cdot w_t$$

This budget constraint is the same as the budget constraint in the situation before imposing the tax T, the consumption of the representative consumer thus does not change.

The private savings $\boldsymbol{S_t}$ decrease for $\boldsymbol{T}$, the overall capital does not change ($\boldsymbol{K_{t+1}=L_t.(S_t+T)}$), the same holds for the original curve $\boldsymbol{k_{t+1}}$. The only complication could arise if the $\boldsymbol{T}$ exceeds original intended $\mathbf{S_t}$ in the situation without the tax (the private savings could not be negative, the economic subjects thus could not decrease their private saving by the appropriate amount). This would lead to the over capitalisation of the economy and to decrease of the utility of the consumers. However the effective capital markets should overcome this problem as well- in case of the increase of T over originally intended S_t the economic subjects simply make their private saving negative by borrowing.