JEM004 Macroeconomics
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Lecture Notes
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4 Overlapping Generations - Diamond Model

4.1 Preview

Compared to Ramsey model:

- turnover in the population - old people die, new people are born
- discreet time
- different results

4.2 Assumptions

- discreet timing: $t = 0, 1, 2, \ldots$
- each individual lives for 2 periods:
  - time t: young, supplies 1 unit of time , receives wage, spends it on consumption $C_{1,t}$ and savings s_t
  - time $t+1$: old, no work, consumption $C_{2,t+1}$ financed from savings and interest income
  - CRRA utility of individual born at t:

$$U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta}; \qquad \theta > 0,\ \rho > -1$$

- growth of population: $L_t = (1+n)L_{t-1}$
  - at time t we have L_t young individuals and $L_{t-1} = \frac{L_t}{1+n}$ old individuals
- production function: $Y_t = F(K_t, A_t L_t)$
  - growth of productivity $A_t = (1+g)A_{t-1}$
  - no depreciation

– competitive markets => factors earn their marginal products [1]=>

$$
\begin{aligned}
r_t &= f'(k_t) \\
w_t &= f(k_t) - k_t f'(k_t)
\end{aligned}
$$

– creation of capital (no stock!): $K_{t+1} = L_t[w_t A_t - C_{1,t}]$

4.3 Consumer's behavior - consumption and saving decision

- Budget constraints that consumers face in each period

$$
\begin{aligned}
C_{1,t} + s_t &= A_t w_t \\
C_{2,t+1} &= (1 + r_{t+1}) s_t
\end{aligned}
$$

- Combining the two budget constraints we obtain **intertemporal budget constraint**

$$
C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = A_t w_t
$$

- Optimisation problem of each individual consumer can be therefore written as

$$
\begin{aligned}
\max_{C_{1,t}, C_{2,t+1}} U_t &= \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2,t+1}^{1-\theta}}{1-\theta} \\
s.t. \qquad & C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = A_t w_t
\end{aligned}
$$

- Mathematical way of solving this problem is by using Lagrangian

$$
\begin{aligned}
\mathfrak{L} &= \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda[A_t w_t - C_{1,t} - \frac{C_{2,t+1}}{1 + r_{t+1}}] \\
(C_{1,t}) &: C_{1,t}^{-\theta} = \lambda \\
(C_{2,t+1}) &: \frac{1}{1+\rho} C_{2,t+1}^{-\theta} = \lambda \frac{1}{1 + r_{t+1}}
\end{aligned}
$$

If we plug the first F.O.C. in the second one, we will get **Euler equation**, describing the margin between current and future consumption:

$$
\frac{C_{2,t+1}}{C_{1,t}} = \left(\frac{1 + r_{t+1}}{1 + \rho} \right)^{\frac{1}{\theta}}
$$

The same result can be derived intuitively (on Lecture).

[1]Notation: k_t - capital per effective worker.

- Now, let's plug in for $C_{2,t+1}$ from Euler equation to the budget constraint.

$$
\begin{aligned}
C_{1,t} + \frac{C_{1,t}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}}{(1+r_{t+1})} &= A_t w_t \\
C_{1,t} &= \underbrace{\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}}_{1-s(r_{t+1})} A_t w_t
\end{aligned}
$$

We see that in the first period, person will always consume a share of its income, and that this share will depend on the interest rate that he faces. Therefore, the saving rate can be expressed as:

$$
s(r_{t+1}) = \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}
$$

Simple algebra (on Lecture) will bring us to conclusion that saving rate s is increasing in interest rate r_{t+1} only if $\theta < 1$ and, on the other hand, is decreasing in interest rate if $\theta > 1$. Intuitively, for people who do not have preference for current consumption ($\theta < 1$) the substitution effect of price increase prevails, while for people with preference for current consumption ($\theta > 1$) the income effect prevails.

4.4 Dynamics of the economy

How does the capital evolve in this economy? As we know, capital in the next period is equal to the amount of savings made by young in this period, i.e.

$$
\begin{aligned}
K_{t+1} &= s(r_{t+1}) A_t L_t \\
k_{t+1} &= \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t
\end{aligned}
$$

When we substitute for r_{t+1} and w_t from the solution of firm's problem, we obtain

$$
k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]
$$

Thus, we have equation that implicitly defines k_{t+1} as a function of k_t - it determines how k_t evolves over time, given its initial value. We would like to know whether there exist steady state (i.e. situation where $k_t = k_{t+1}$ and whether, if we do not begin at such value, eventually we will converge there.

4.4.1 Special case - Logarithmic utility ($\theta = 1$) and Cobb-Douglas Production ($f(k_t) = k_t^\alpha$)

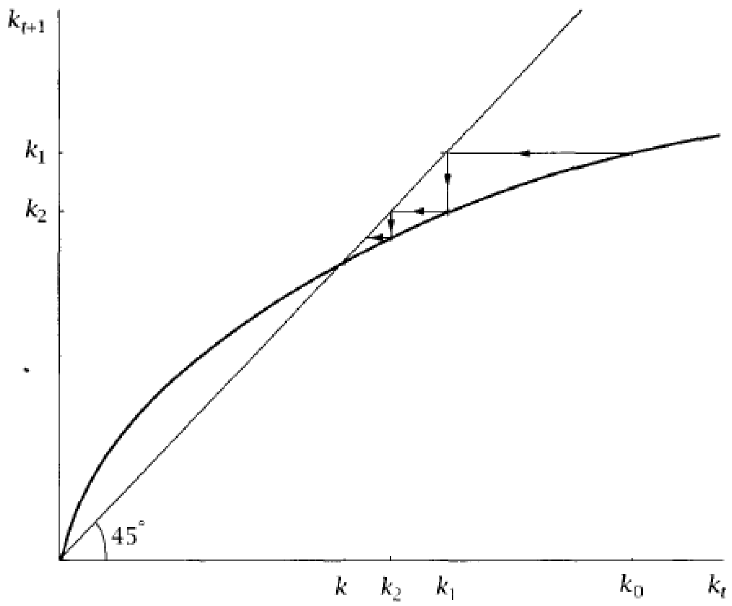
Under logarithmic utility, income and substitution effect cancel out and saving rate boils down to a constant, independent of interest rate

$$s = \frac{1}{(1+\rho)+1} = \frac{1}{2+\rho}$$

Under Cobb-Douglas production production function, wage is determined as $w_t = (1 - \alpha)k_t^{|}alpha$ and the final law of motion for capital looks

$$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)k_t^\alpha =: Dk_t^\alpha$$

The relationship between k_{t+1} and k_t is depicted in the Figure 4.4.2. The 45degrees line depicts necessary condition for equilibrium, i.e. situation in which $k_{t+1} = k_t$. We see, that in this special case we have two equilibria - one unstable and "uninteresting" (at $k^* = 0$, and one stab;e with positive capital level. The equilibrium has similar properties as in the Solow and Ramsey model - when in equilibrium, both capital and output grow at the rate $n + g$, and capital and output per capita grow at the rate g.



To understand the dynamics, let's consider the case when discount rate ρ has decreased. People therefore don't discount the future so heavily and are willing to save more - saving rate increases. The whole schedule for k_{t+1} shifts upwards, and economy converges to a new equilibrium with higher level of output.

4.4.2 General case

If we relax the assumptions, it turns out that despite the relative simplicity of the model, we could attain wide range of possible behavior of the economy. We will just discuss some more interesting cases. Let us rewrite the law of motion for capital:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \underbrace{s(f'(k_{t+1}))}_{\text{saved fraction of labor income}} \underbrace{\frac{[f(k_t) - k_t f'(k_t)]}{f(k_t)}}_{\text{labor share of income}} \underbrace{f(k_t)}_{\text{income}}$$

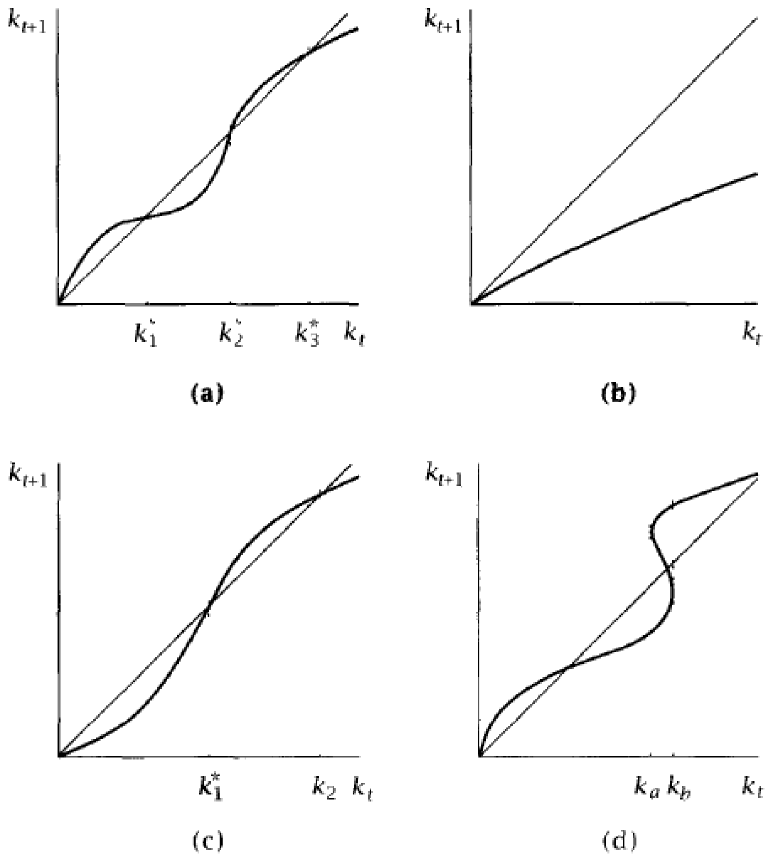


Figure 2 shows some of the possible forms for the relation between k_{t+1} and k_t. Panel a) shows case with multiple equilibria (k_1^* and k_3^*) are stable. This could happen e.g. when $\theta > 1$, i.e. people save larger fraction of income as k_t increases, because at the same time $f'(k_t)$ decreases. Panel b) shows a case in which k_{t+1} is always less than k_t and k converges to zero regardless of its initial value. Panel c) shows a case of multiple equilibria, where k converges to 0 if the initial level is low, but to a strictly positive level if the initial level is sufficiently high. Panel d) shows as case in which k_{t+1} is not uniquely determined; in fact, in the range k_a to k_b there are three possible values of k_{t+1}. This set-up raises the possibility of self-fulfilling prophecies and, consequently, the economy can exhibit fluctuations even without presence of exogenous disturbances.

Summary of findings

- As in the Solow or Ramsey model, eventually, in the equilibrium growth rate is determined by rate of technological progress g.
- unbounded growth of is not possible. $k_{t+1} < k_t$ for big values of k_t and with rising k $f'(k_t)$ converges to 0
- Otherwise similar countries can end up in different equilibrium, but we would have to make fairly strong assumptions on initial k.