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Classical and Fuzzy Two-Layered Modal Logics for Uncertainty: Translations and Proof-Theory

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ABSTRACT

This paper is a contribution to the study of two distinct kinds of logics for modelling uncertainty. Both approaches use logics with a two-layered modal syntax, but while one employs classical logic on both levels R. Fagin, J.Y. Halpern, N. Megiddo, A logic for reasoning about probabilities, Inf. Comput. 87 (1990), 78–128, and infinitely-many multimodal operators, the other involves a suitable system of fuzzy logic in the upper layer and only one monadic modality P. Hájek, L. Godo, F. Esteva, Fuzzy logic and probability, in Proceedings of the 11th Annual Conference on Uncertainty in Artificial Intelligence (UAI '95), 1995, pp. 237–244. We take two prominent examples of the former approach, the probability logics $\mathrm{Pr_{lin}}$ and $\mathrm{Pr_{pol}}$ (whose modal operators correspond to all possible linear/polynomial inequalities with integer coefficients), and three logics of the latter approach: $\mathrm{Pr}^{Ł}$, $\mathrm{Pr}^{Ł\triangle}$ and $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$ (given by the Łukasiewicz logic and its expansions by the Baaz–Monteiro projection connective $\triangle$ and also by the product conjunction). We describe the relation between the two approaches by giving faithful translations of $\mathrm{Pr_{lin}}$ and $\mathrm{Pr_{pol}}$ into, respectively, $\mathrm{Pr}^{Ł\triangle}$ and $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$, and vice versa. We also contribute to the proof theory of two-layered modal logics of uncertainty by introducing a hypersequent calculus $\mathrm{HPr}^{Ł}$ for the logic $\mathrm{Pr}^{Ł}$. Using this formalism, we obtain a translation of $\mathrm{Pr_{lin}}$ into the logic $\mathrm{Pr}^{Ł}$, seen as a logic on hypersequents of relations, and give an alternative proof of the axiomatization of $\mathrm{Pr_{lin}}$.

1. INTRODUCTION

Numerous logical systems have been proposed, and intensively studied in recent years, to cope with reasoning about uncertain events. Among them, two of the most prominent examples are the systems introduced by Fagin, Halpern, and Megiddo [11] (see also Halpern's monograph [15]), which we denote here as $\mathrm{Pr_{lin}}$ and $\mathrm{Pr_{pol}}$. These systems employ a rather sophisticated two-layered modal syntax: They start, in a first layer, by expressing classical events (i.e., propositions that can only be true or false) by means of the syntax of propositional classical logic; then, they define the atomic statements of the second syntactical layer as linear inequalities (in the case of $\mathrm{Pr_{lin}}$), or polynomial inequalities (in the case of $\mathrm{Pr_{pol}}$), of probabilities of these classical events. Each of these inequalities can be seen as the application of a multimodal operator on classical formulas. Finally, such atomic statements may be combined using classical connectives again.

The consequence relation of both logics $\mathrm{Pr_{lin}}$ and $\mathrm{Pr_{pol}}$ is then introduced semantically by means of Kripke frames enriched by a probability measure, which allows for expressing the validity of statements of these logics: in the atomic case, as the truth of inequalities involving the probability of events, i.e., of sets of worlds described by classical formulas, and, in the case of complex formulas, by using the usual semantics of classical logic.

Despite dealing with the intrinsically graded notion of probability, the semantics of these logics remains essentially bivalent. An alternative approach to reasoning about uncertain events uses the framework of mathematical fuzzy logic and takes sentences like "φ is probable" at face value, i.e., identifying its truth degree with the probability of φ. Then, one combines such formulas using connectives of a *suitable fuzzy logic*. Hence, this approach also uses a two-layered modal syntax which is, however, radically simplified. Indeed, it employs only one monadic modality (for "is probable"), instead of infinitely-many polyadic modalities, as it shifts the syntactical complexity of the atomic statements to the many-valued semantics of the fuzzy logic in question.

The original rendering of this approach [4,5] used Łukasiewicz logic Ł to govern modal formulas. The resulting logic, which we

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This paper is an extended and revised version of the conference communication [1] Besides improving the notation and streamlining the presentation of our original results, in this paper: (1) we provide inverse translations from fuzzy to classical probability logics (Theorems 2 and 4), (2) we give a hypersequent calculus of relations that axiomatizes Łukasiewicz logic in a strong sense (Theorem 5), (3) we axiomatize the probability logic based on Łukasiewicz logic with a hypersequent calculus of relations (Theorem 8), (4) using this result, we obtain simpler proofs of additional translations between fuzzy and classical probability logics (Theorem 10 and Corollary 11) and, finally, (5) we give an alternative proof of axiomatization of one of the prominent classical probability logics (Theorem 13).

denote here as $\mathrm{Pr}^{Ł}$, was given by using Kripke frames enriched by a probability measures, analogously to $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$. Later, several authors studied numerous similar logical systems by altering not only the upper logic but also the lower one (to speak about probability of fuzzy events) and even their interlinking modalities (to speak about other measures of uncertainty such as necessity, possibility, or belief functions).[1]

In this paper we will focus only on the logic $\mathrm{Pr}^{Ł}$ and two of its expansions, $\mathrm{Pr}^{Ł\triangle}$ and $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$, which use stronger fuzzy logics to govern the behavior of modal formulas in the upper syntactical layer, namely the logic $Ł_{\triangle}$ expanding Ł with the Baaz–Monteiro projection operator $\triangle$, and its further expansion $\mathrm{P}Ł_{\triangle}$ with the product conjunction.[2]

A natural question presents itself: what is the relation between the two approaches? More precisely: can $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ be translated into two-layered modal fuzzy logics and, hence, be casted into a syntactically simpler framework without losing expressivity? This paper intends to give a positive answer to this question while providing inverse translations as well, thus showing that both approaches are indeed much more closely related than it might have seemed at first sight.

The answer is, nonetheless, not as straightforward as one could expect: We present translations of $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ into, respectively the logics $\mathrm{Pr}^{Ł\triangle}$ and $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$, and vice versa. The need for the product conjunction of $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$ in the second case is hardly a surprise, since we need to take care of products in the polynomial inequalities of $\mathrm{Pr}_{\mathrm{pol}}$. However, the presence of the projection connective $\triangle$ in both cases may appear as an unexpected nuisance.

The effort toward amending this eyesore led to the second main contribution of the paper: We show that the logic $\mathrm{Pr}^{Ł}$ can be axiomatized using a particular Gentzen-style calculus, denoted as $\mathrm{HPr}^{Ł}$, which is an extension of a known calculus for Łukasiewicz logic. Unlike classical Gentzen calculi, which work with sequents, in our case we have to consider more complex syntactical structures, known as hypersequents of relations. Interestingly enough, these structures yield a rich framework that allows us to circumvent the use of the projection connective $\triangle$ and present the desired translation of $\mathrm{Pr}_{\mathrm{lin}}$ into the logic $\mathrm{Pr}^{Ł}$, seen as a logic on hypersequents of relations. Although the calculus $\mathrm{HPr}^{Ł}$ is not analytic, its existence enhances the applicability of the logic $\mathrm{Pr}^{Ł}$ and deepens our theoretic understanding of this logic; e.g., we can use it to obtain an alternative proof of the axiomatization of $\mathrm{Pr}_{\mathrm{lin}}$, which is arguably simpler that the one known from the literature [12]. Therefore, the results of this paper strengthen the overall prominence of $\mathrm{Pr}^{Ł}$ among logics of uncertainty.

[1] We refer the reader to the survey work [12] (and references therein) and to the abstract unifying framework for these logics [2].

[2] The logics of uncertainty introduced above are usually denoted using different symbols in the literature. In particular, the classical ones (or more precisely their axiomatic systems) $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ are denoted by AX^{prob} and $AX^{prob,\times}$ in Halpern's book [15] and, following the notational conventions introduced in Hájek's book [13], the fuzzy ones $\mathrm{Pr}^{Ł}$, $\mathrm{Pr}^{Ł\triangle}$, and $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$ are traditionally denoted as FP(Ł), FP($Ł_{\triangle}$), FP(PŁ), respectively. We have opted here for a uniform but neutral terminology, for ease of reference through the paper.

The paper is organized as follows: First, in Section 2, we introduce the syntax, the semantics, and axiomatizations for the logics under investigation, in a reasonably self-contained yet streamlined manner. Then, in Section 3, we present the mentioned translations of $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ into, respectively the logics $\mathrm{Pr}^{Ł\triangle}$ and $\mathrm{Pr}^{\mathrm{P}Ł\triangle}$, and vice versa. In Section 4 we introduce a hypersequent calculus $\mathrm{HPr}^{Ł}$ and prove that it axiomatizes the logic $\mathrm{Pr}^{Ł}$. In Section 5 we provide a faithful translation of the logic $\mathrm{Pr}_{\mathrm{lin}}$ into $\mathrm{Pr}^{Ł}$ (seen as a logic of hypersequents of relations) and give an alternative proof of the axiomatization of $\mathrm{Pr}_{\mathrm{lin}}$. Finally, in Section 6 we add some concluding remarks and hints at future research directions.

2. CLASSICAL AND FUZZY LOGICS OF UNCERTAINTY

2.1. Propositional Core

In this paper, we need the following four propositional logics: (1) classical logic CL cast in the language with the truth-constant $\bot$ and implication $\to$, (2) Łukasiewicz logic Ł in the same language, (3) $Ł_{\triangle}$, the expansion of Ł in the language with the additional unary connective $\triangle$ known as Baaz–Monteiro projection, and, finally, (4) $\mathrm{P}Ł_{\triangle}$, the expansion of $Ł_{\triangle}$ with the additional binary connective $\odot$ (called product conjunction). Next, we review some of the properties of these logics needed for the paper; we refer the reader to the corresponding chapters of the Handbook of Mathematical Fuzzy Logic [10] for more details and references.

We expect the reader to be familiar with the notion of formula (over an arbitrary propositional language) and the notion of evaluation in classical logic. In the case of Ł, $Ł_{\triangle}$, and $\mathrm{P}Ł_{\triangle}$, (standard) evaluations are functions from the corresponding set of formulas into the real unit interval [0, 1], such that $e(\bot) = 0$ and

$$\begin{aligned}
e(\varphi \to \psi) &= \min\{1, 1 - e(\varphi) + e(\psi)\} \\
e(\triangle\varphi) &= \begin{cases} 1 & \text{if } e(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases} \\
e(\varphi \odot \psi) &= e(\varphi) \cdot e(\psi).
\end{aligned}$$

Let L be any of these four logics and $\Psi \cup \{\varphi\}$ a set of formulas in the language of L. We say that φ is a semantical consequence of Ψ in L, in symbols $\Psi \models_{\mathrm{L}} \varphi$, if for each evaluation e such that $e(\psi) = 1$ for all $\psi \in \Psi$, we have $e(\varphi) = 1$.

We also expect the reader to be familiar with the notion of derivability relation $\vdash_{AX}$ in a (finitary) Hilbert-style axiomatic system AX; we say that AX is an axiomatization of a logic L if for each *finite* set $\Psi \cup \{\varphi\}$ of formulas, we have $\Psi \models_{\mathrm{L}} \varphi$ iff $\Psi \vdash_{AX} \varphi$. It is well known that there are numerous axiomatizations of classical logic (where the equivalence holds even for infinite sets of premises) and the three fuzzy logics considered here. We write $\vdash_{\mathrm{L}}$ when an axiomatization of a logic L is fixed or known from the context.

Let us conclude this subsection by recalling additional definable connectives of Łukasiewicz logic together with their standard semantics:

$\neg\varphi$	$\varphi \to \bot$	$1 - x$
$\varphi \vee \psi$	$(\varphi \to \psi) \to \psi$	$\max\{x, y\}$
$\varphi \wedge \psi$	$\neg(\neg\varphi \vee \neg\psi)$	$\min\{x, y\}$
$\varphi \oplus \psi$	$\neg\varphi \to \psi$	$\min\{1, x + y\}$
$\varphi \otimes \psi$	$\neg(\neg\varphi \oplus \neg\psi)$	$\max\{0, x + y - 1\}$
$\varphi \ominus \psi$	$\neg(\varphi \to \psi)$	$\max\{0, x - y\}$.

Whenever necessary to avoid confusions, we add " Ł" as a subscript to the connectives in order to distinguish them from the classical ones.

2.2. Five Two-Layered Modal Languages

We start by recalling the language $\mathcal{L}_{\text{lin}}$ of the logic Pr_{lin}. It is a two-layered modal language: first, in a lower layer, we have the *nonmodal* formulas which are simply those of classical propositional logic. Then, we have *basic inequality formulas* of the form

$$\sum_{i=1}^{n} a_i P(\varphi_i) \geqslant c,$$

where φ_is are nonmodal formulas and c and a_i are constants for integers (in other works in the literature real numbers [15] or also rationals [3] are used). In the extreme case in which $n = 0$ or all a_i's are 0, we have the basic inequality formula $0 \geqslant c$. The linear combination on the left-hand side of the inequality is called a *basic inequality term*. The formulas of the upper layer of $\mathcal{L}_{\text{lin}}$, called *modal* formulas, are then obtained from basic inequality formulas via the usual connectives of classical logic. Obvious abbreviations apply; in particular, we use the following:

$-\sum_{i=1}^{n} a_i P(\varphi_i)$	for	$\sum_{i=1}^{n} -a_i P(\varphi_i)$
$P(\varphi) \geqslant P(\psi)$	for	$P(\varphi) - P(\psi) \geqslant 0$
$t \leqslant c$	for	$-t \geqslant -ct$
$t < c$	for	$\neg(t \geqslant c)$
$t = c$	for	$(t \geqslant c) \wedge (t \leqslant c)$

The language $\mathcal{L}_{\text{pol}}$ is obtained by using again the language of classical logic for the lower layer, and allowing any *polynomial basic inequality terms* in the upper layer, i.e., the basic inequality formulas of $\mathcal{L}_{\text{lin}}$ have the general form

$$\sum_{i=1}^{n} a_i P(\varphi_{i,1}) \cdots P(\varphi_{i,m_i}) \geqslant c.$$

Complex formulas of the upper layer are built as in $\mathcal{L}_{\text{lin}}$, combining basic inequality formulas by means of connectives of classical logic. Note that in $\mathcal{L}_{\text{pol}}$ one can express fundamental probabilistic notions, e.g., independence of events using formulas of the kind

$$P(\varphi \wedge \psi) = P(\varphi) \cdot P(\psi).$$

Let us now turn our attention to the fuzzy approach toward logics of probability. We introduce three languages, $\mathcal{L}_P^{\text{Ł}}$, $\mathcal{L}_P^{\text{Ł}\triangle}$, and $\mathcal{L}_P^{\text{PŁ}\triangle}$, where, as before, the lower-layer formulas are those of classical logic, but instead of basic inequality formulas combined by connectives of classical logic, the modal formulas are built from simple *atomic modal formulas* of the form $P(\varphi)$ (where φ is a classical formula) using the connectives of the logic Ł, $\text{Ł}_\triangle$, or $\text{PŁ}_\triangle$, respectively. For example, $P(p \to p)$ and $P(\bot)$ are atomic modal formulas in any of these two-layered modal languages, $P(\bot) \to_{\text{Ł}} P(p \to p)$ is a nonatomic modal formula in any of them, $\triangle P(\bot)$ is a nonatomic modal formula in $\mathcal{L}_P^{\text{Ł}\triangle}$ and $\mathcal{L}_P^{\text{PŁ}\triangle}$, and $\triangle P(\bot) \odot P(p \to p)$ is a nonatomic modal formula in $\mathcal{L}_P^{\text{PŁ}\triangle}$. Observe that the two-layered syntax does not admit iterative applications of the modal operator (e.g., $P(P(p) \to P(q))$ is a not a well-formed formula) nor combinations of atomic modal formulas with nonmodal formulas in the upper level (e.g., $p \odot P(q)$ is not a well-formed formula either).

Remark 1. Note that a basic inequality formula $\sum_{i=1}^{n} a_i P(\varphi_i) \geqslant c$ of $\mathcal{L}_{\text{lin}}$ can be seen as an atomic modal formula obtained by applying an n-ary modality $\Box_{a_1,\dots,a_n,c}$, on n classical formulas $\varphi_1, \dots, \varphi_n$. In this way, one can see $\mathcal{L}_{\text{lin}}$ as an instance of an abstract two-layered modal language [2]. The same is true for $\mathcal{L}_{\text{pol}}$, but here the set of used modalities is even more complex. Thus, the five languages can be summarized in the following table:

Language	Lower l.	Modalities	Upper l.
$\mathcal{L}_{\text{lin}}$	CL	$\{ t \geqslant c : t \text{ lin}\}$	CL
$\mathcal{L}_{\text{pol}}$	CL	$\{ t \geqslant c : t \text{ poly}\}$	CL
$\mathcal{L}_P^{\text{Ł}}$	CL	$\{P\}$	Ł
$\mathcal{L}_P^{\text{Ł}\triangle}$	CL	$\{P\}$	$\text{Ł}_\triangle$
$\mathcal{L}_P^{\text{PŁ}\triangle}$	CL	$\{P\}$	$\text{PŁ}_\triangle$

Convention 1. Henceforth, we will adopt the following notational convention for distinguishing modal and nonmodal formulas in each of the logics we consider.

	Nonmodal	Modal
Formulas	$\varphi, \psi, \dots$	$\gamma, \delta, \dots$
(Multi)sets of formulas	$\Phi, \Psi, \dots$	$\Gamma, \Delta, \dots$

Note that we will use the same symbols for sets and multisets of formulas, relying on the context for resolving ambiguities.

2.3. One Semantics and Five Logics

The semantical picture for all five languages is based on Kripke models enriched by (finitely additive) probability measures. A (probabilistic) Kripke model is a triple $\mathbf{M} = \langle W, \langle e_w \rangle_{w \in W}, \mu \rangle$, where

- W is a nonempty set of worlds
- e_ws are classical propositional evaluations
- μ is a finitely additive measure over a Boolean subalgebra of the powerset algebra of W such that

$$\varphi^{\mathbf{M}} = \{w : e_w(\varphi) = 1\}$$

is a measurable set for any classical formula φ

Clearly, $\mathbf{M}$ allows us to define the truth values of nonmodal formulas in each of its worlds. The assignment of truth values of modal formulas depends on the language in question, but in all cases we evaluate modal formulas only at the level of the whole model.

For basic inequality formulas of $\mathcal{L}_{\mathrm{lin}}$ we define

$$\left\|\sum_{i=1}^{n} a_i P(\varphi_i) \geqslant c\right\|_{\mathbf{M}} = 1 \quad \text{iff} \quad \sum_{i=1}^{n} a_i \mu\left(\varphi_i^{\mathbf{M}}\right) \geqslant c.$$

The truth values of basic inequality formulas of $\mathcal{L}_{\mathrm{pol}}$ are defined analogously, and truth values of complex modal formulas in both languages are then defined using the truth-functions of classical connectives.

Recall that $\mathcal{L}_P^{Ł}$, $\mathcal{L}_P^{Ł\triangle}$, and $\mathcal{L}_P^{PŁ\triangle}$ share the same atomic modal formulas; we define their truth values simply as

$$||P(\varphi)||_{\mathbf{M}} = \mu(\varphi^{\mathbf{M}}).$$

Then, clearly, we always have $||P(\varphi)||_{\mathbf{M}} \in [0,1]$, and so we can compute the truth values of more complex modal formulas using truth functions for connectives of the corresponding logic. For example $||P(p \to p)||_{\mathbf{M}} = 1$, because $p \to p$ is a tautology of classical logic, and $||P(\bot)||_{\mathbf{M}} = 0$ because $\bot$ is a contradiction. As for nonatomic examples, one can easily compute $||P(\bot) \to_Ł P(p \to p)||_{\mathbf{M}} = 1$, $|| \triangle P(\bot)||_{\mathbf{M}} = || \triangle P(\bot) \odot P(p \to p)||_{\mathbf{M}} = 0$.

For each of the five languages we have introduced, we can define the corresponding logic as the consequence relation on the set of modal formulas given as preservation of the truth value 1 over all Kripke models; for instance, we define $\mathrm{Pr}_{\mathrm{lin}}$ as the following consequence relation between sets of modal $\mathcal{L}_{\mathrm{lin}}$-formulas and modal $\mathcal{L}_{\mathrm{lin}}$-formulas as

$$\Gamma \models_{\mathrm{Pr}_{\mathrm{lin}}} \delta \text{ iff } \quad ||\delta||_{\mathbf{M}} = 1 \text{ for each Kripke model } \mathbf{M} \text{ where } ||\gamma||_{\mathbf{M}} = 1 \text{ for each } \gamma \in \Gamma.$$

Analogously, we define the logics $\mathrm{Pr}_{\mathrm{pol}}$, $\mathrm{Pr}^{Ł}$, $\mathrm{Pr}^{Ł\triangle}$, and $\mathrm{Pr}^{PŁ\triangle}$.

2.4. Axiomatizations

An axiomatization for $\mathrm{Pr}_{\mathrm{lin}}$ proposed in the literature [11], which we will denote here as $AX_{\mathrm{Pr}_{\mathrm{lin}}}$, consists of: (1) any axiomatization of classical logic for both modal and nonmodal formulas, (2) the following three axioms and one rule,

(QU1) $P(\varphi) \geqslant 0$

(QU2) $P(\top) = 1$

(QU3) $P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) = P(\varphi)$

(QUGEN) From $\varphi \leftrightarrow \psi$ infer $P(\varphi) = P(\psi)$

and (3) the axioms to manipulate linear inequalities, meant to be instantiated with any basic inequality formula $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c$, integers c' and $d' < c$ and $d > 0$, and permutation σ:

(LQ1) $P(\varphi) \geqslant P(\varphi)$

(LQ2) $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c \leftrightarrow \sum_{i=1}^{k} a_i P(\varphi_i) + 0P(\varphi) \geqslant c$

(LQ3) $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c \leftrightarrow \sum_{i=1}^{k} a_{\sigma(i)} P(\varphi_{\sigma(i)}) \geqslant c$

(LQ4) $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c \wedge \sum_{i=i}^{k} b_i P(\varphi_i) \geqslant c' \to$ $\to \sum_{i=1}^{k} (a_i + b_i) P(\varphi_i) \geqslant c + c'$

(LQ5) $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c \leftrightarrow \sum_{i=1}^{k} d a_i P(\varphi_i) \geqslant dc$

(LQ6) $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c \vee \sum_{i=1}^{k} a_i P(\varphi_i) \leqslant c$

(LQ7) $\sum_{i=1}^{k} a_i P(\varphi_i) \geqslant c \to \sum_{i=1}^{k} a_i P(\varphi_i) > d'$

The proof that $AX_{\mathrm{Pr}_{\mathrm{lin}}}$ is indeed an axiomatization of $\mathrm{Pr}_{\mathrm{lin}}$ relies essentially on linear programming methods. The original paper [11] also shows that the satisfiability problem for $\mathrm{Pr}_{\mathrm{lin}}$ is NP-complete. On the other hand, the same paper proves that the satisfiability problem for $\mathrm{Pr}_{\mathrm{pol}}$ is in PSPACE, and provides an axiomatization for this logic, but only via a reduction to real closed field theory. Another axiomatization of $\mathrm{Pr}_{\mathrm{pol}}$, in the language $\mathcal{L}_{\mathrm{pol}}$ was found only later [18] and it includes an infinitary rule.

In contrast, the axiomatizations of $\mathrm{Pr}^{Ł}$, $\mathrm{Pr}^{Ł\triangle}$, and $\mathrm{Pr}^{PŁ\triangle}$ are much simpler: [2,12,13] they use any axiomatization of classical logic for nonmodal formulas, any axiomatization of Ł (or $Ł_\triangle$ or $PŁ_\triangle$, respectively) for modal formulas and just three additional axioms and one rule:

(A1) $(P\varphi \otimes P(\varphi \to \psi)) \to_Ł P\psi$

(A2) $P\neg\varphi \leftrightarrow_Ł \neg_Ł P\varphi$

(A3) $P(\varphi \vee \psi) \leftrightarrow_Ł [(P\varphi \ominus P(\varphi \wedge \psi)) \oplus P\psi]$

(Nec) From φ infer $P\varphi$.

As in $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$, the satisfiability problems for $\mathrm{Pr}^{Ł}$ and $\mathrm{Pr}^{PŁ\triangle}$ are also known to be NP-complete and in PSPACE, respectively [14].

3. TRANSLATING $\mathrm{Pr}_{\mathrm{lin}}$ INTO $\mathrm{Pr}^{Ł\triangle}$ AND $\mathrm{Pr}_{\mathrm{pol}}$ INTO $\mathrm{Pr}^{PŁ\triangle}$

In this section, we show that the classical probability logic $\mathrm{Pr}_{\mathrm{lin}}$ can be faithfully translated into the two-layered modal fuzzy logic $\mathrm{Pr}^{Ł\triangle}$ and vice versa, and then we extend this result to obtain translations between the logics $\mathrm{Pr}_{\mathrm{pol}}$ and $\mathrm{Pr}^{PŁ\triangle}$. Let us start by preparing two useful notational conventions.

First, for any formula φ of Ł, Ł$_\triangle$, or PŁ$_\triangle$ with (at most) n propositional variables $p_1, \dots, p_n$ we denote by f_φ the function from $[0,1]^n$ to $[0,1]$ such that, for each evaluation e, we have

$$e(\varphi) = f_\varphi(e(p_1), \dots, e(p_n)).$$

Second, for each function $f : \mathrm{R}^n \to \mathrm{R}$, we define the function $f^\# : [0,1]^n \to [0,1]$ as

$$f^\# = \min\{1, \max\{f, 0\}\}.$$

Let us start by translating $\mathrm{Pr}_{\mathrm{lin}}$ into $\mathrm{Pr}^{Ł\triangle}$. Let $t \geqslant c$ be a basic inequality formula in $\mathcal{L}_{\mathrm{lin}}$, where t stands for $\sum_{i=1}^n a_i P(\varphi_i)$, and consider the linear polynomial with integer coefficients

$$f(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i - c + 1.$$

By the well-known McNaughton Theorem (see, e.g., its formulation in Lemma 2.1.21 of chapter IX in the Handbook of Mathematical Fuzzy Logic[8]) one can algorithmically build a formula ψ of Ł over variables $p_1, \dots, p_n$, such that

$$f_\psi = f^\#.$$

Let us denote by $\psi(P(\varphi_1), \dots, P(\varphi_n))$ the formula resulting from ψ by replacing each variable p_i in ψ by $P(\varphi_i)$ and let us define

$$(t \geqslant c)^\bullet = \triangle\psi(P(\varphi_1), \dots, P(\varphi_n)).$$

Clearly, $(t \geqslant c)^\bullet$ is a formula of $\mathcal{L}_P^{Ł\triangle}$. We can easily extend it to a translation of all modal formulas from $\mathcal{L}_{\mathrm{lin}}$ by setting $\bot^\bullet = \bot_Ł$ and $(\gamma \to \delta)^\bullet = \gamma^\bullet \to_Ł \delta^\bullet$. Let us denote as $\Gamma^\bullet$ the set resulting from applying the translation to each formula in Γ.

Theorem 1. *Let $\Gamma \cup \{\delta\}$ be a set of modal formulas of $\mathcal{L}_{\mathrm{lin}}$. Then, $\Gamma \models_{\mathrm{Pr}_{\mathrm{lin}}} \delta$ iff $\Gamma^\bullet \models_{\mathrm{Pr}^{Ł}{}_\triangle} \delta^\bullet$.*

Proof. It is easy to see that all we need to prove is that, for each Kripke model $\mathbf{M}$ and each modal formula γ of $\mathcal{L}_{\mathrm{lin}}$, we have $||\gamma||_{\mathbf{M}} = 1$ iff $||\gamma^\bullet||_{\mathbf{M}} = 1$.

We prove the claim by induction over the complexity of γ. Assume that γ is a basic inequality formula $\sum_{i=1}^n a_i P(\varphi_i) \geqslant c$. Then, we can write the following sequence of equivalences: $||\gamma||_{\mathbf{M}} = 1$ iff $\sum_{i=1}^n a_i \mu(\varphi_i^{\mathbf{M}}) \geqslant c$ iff $\sum_{i=1}^n a_i ||P(\varphi_i)||_{\mathbf{M}} \geqslant c$ iff $\max\{0, \min\{1, \sum_{i=1}^n a_i ||P(\varphi_i)||_{\mathbf{M}} - c + 1\}\} = 1$ iff $f^\#(P(\varphi_1), \dots, P(\varphi_n)) = 1$ iff $f_\psi(P(\varphi_1), \dots, P(\varphi_n)) = 1$ iff $||\gamma^\bullet||_{\mathbf{M}} = 1$.

To prove the induction step, we only need to note that (1) for a basic inequality formula γ we have that (thanks to the semantics of $\triangle$) $||\gamma^\bullet||_{\mathbf{M}} < 1$ implies $||\gamma^\bullet||_{\mathbf{M}} = 0$ and (2) the Łukasiewicz implication behaves on values 0 and 1 as the classical one.

Now we will show a translation in the converse direction, from $\mathrm{Pr}^{Ł}$ into $\mathrm{Pr}_{\mathrm{lin}}$. Consider any modal formula γ of $\mathrm{Pr}^{Ł\triangle}$ and the formula $\hat{\gamma}$ in the language of Ł$_\triangle$ resulting from γ by replacing each atomic modal formula $P(\varphi_i)$ by a propositional variable p_i. It is well known [8] that

$$f_{\hat{\gamma}} = \max_{k \in K} \min_{j \in J_k} t_{k,j}$$

where for each $k \in K$ and $j \in J_k$, there is a linear function $f_{k,j}$ with integer coefficients and

$$t_{k,j} = f^\#_{k,j} \qquad \text{or} \qquad t_{k,j} = 1 - \triangle(1 - f^\#_{k,j}).$$

We define the translation γ° of the formula γ as

$$\gamma^\circ = \bigvee_{k \in K} \bigwedge_{j \in J_k} \gamma_{k,j}$$

where for $f_{k,j} = \sum_{i=1}^n a_i x_i + c$, we have

$$\gamma_{k,j} = \begin{cases} \sum_{i=1}^n a_i P(\varphi_i) \geqslant 1 - c & \text{if } t_{k,j} = f^\#_{k,j} \\ \sum_{i=1}^n a_i P(\varphi_i) < -c & \text{otherwise} \end{cases}$$

As in the previous translation, for any set of modal formulas Γ of $\mathrm{Pr}^{Ł\triangle}$, we also let $\Gamma^\circ = \{\gamma^\circ \mid \gamma \in \Gamma\}$.

Theorem 2. *Let $\Gamma \cup \{\delta\}$ be a set of modal formulas of $\mathcal{L}_P^{Ł\triangle}$. Then, $\Gamma \models_{\mathrm{Pr}^{Ł}{}_\triangle} \delta$ iff $\Gamma^\circ \models_{\mathrm{Pr}_{\mathrm{lin}}} \delta^\circ$.*

Proof. First note that, for each linear function $f = \sum_{i=1}^n a_i x_i + c$ with integer coefficients and each Kripke model $\mathbf{M}$, we have

- $\mathbf{M}$ satisfies $\sum_{i=1}^n a_i P(\varphi_i) \geqslant 1 - c$ iff
$$f^\#(||P(\varphi_1)||_{\mathbf{M}}, \dots ||P(\varphi_n)||_{\mathbf{M}}) = 1.$$
- $\mathbf{M}$ satisfies $\sum_{i=1}^n a_i P(\varphi_i) < -c$ iff
$$1 - \triangle(1 - f^\#(||P(\varphi_1)||_{\mathbf{M}}, \dots ||P(\varphi_n)||_{\mathbf{M}})) = 1.$$

Therefore, for each Kripke Model $\mathbf{M}$ and each formula γ of $\mathrm{Pr}^{Ł}$, we have the following chain of equivalent statements which is clearly all we need to prove the theorem:

- $||\gamma||_{\mathbf{M}} = 1$
- $f_{\hat{\gamma}}(||P(\varphi_1)||_{\mathbf{M}}, \dots ||P(\varphi_n)||_{\mathbf{M}}) = 1$
- There is $k \in K$ such that, for each $j \in J_k$, we have $t_{k,j}(||P(\varphi_1)||_{\mathbf{M}}, \dots, ||P(\varphi_n)||_{\mathbf{M}}) = 1$
- There is $k \in K$ such that, for each $j \in J_k$, we have that $\gamma_{k,j}$ is satisfied in $\mathbf{M}$
- The formula γ° is satisfied in $\mathbf{M}$

Now we extend the translation to $\mathrm{Pr}_{\mathrm{pol}}$ and $\mathrm{Pr}^{PŁ\triangle}$ and show that the classical probability logic $\mathrm{Pr}_{\mathrm{pol}}$ can be faithfully translated into the two-layered modal fuzzy logic $\mathrm{Pr}^{PŁ\triangle}$ and so, thanks to the known simple and finitary axiomatization of $\mathrm{Pr}^{PŁ\triangle}$, this translation provides us with an alternative indirect but simple and finitary axiomatization of $\mathrm{Pr}_{\mathrm{pol}}$.

Let $t \geqslant c$ be a basic inequality formula in $\mathcal{L}_{\mathrm{pol}}$ of the form

$$\sum_{i=1}^n a_i P(\varphi_{i,1}) \cdots P(\varphi_{i,m_i}) \geqslant c$$

As before, we consider the linear polynomial

$$f(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i - c + 1,$$

and the corresponding formula ψ_f of Ł over propositional variables $p_1, \dots, p_n$, such that

$$e(\psi_f) = \max\{0, \min\{1, f(e(p_1), \dots, e(p_n))\}\}.$$

Let us denote as $(t \geqslant c)^\bullet$ the formula resulting from $\triangle\psi_f$ by replacing each propositional variable p_i in ψ_f by $P(\varphi_{i,1}) \odot P(\varphi_{i,2}) \odot \cdots \odot P(\varphi_{i,m_i})$. Clearly, $(t \geqslant c)^\bullet$ is a formula of $\mathcal{L}_{\mathrm{Pr}^{\mathrm{PŁ}_\triangle}}$. We can easily extend it to a translation of all modal formulas from $\mathcal{L}_{\mathrm{pol}}$ by setting $\bot^\bullet = \bot_{Ł}$ and $(\gamma \to \delta)^\bullet = \gamma^\bullet \to_{Ł} \delta^\bullet$. Let us denote as $\Gamma^\bullet$ the set resulting from applying the translation to each formula in Γ.

Theorem 3. *Let* $\Gamma \cup \{\delta\}$ *be a set of formulas of* $\mathcal{L}_{\mathrm{pol}}$. *Then,* $\Gamma \vDash_{\mathrm{Pr}_{\mathrm{pol}}} \delta$ *iff* $\Gamma^\bullet \vDash_{\mathrm{Pr}^{\mathrm{PŁ}_\triangle}} \delta^\bullet$.

Proof. Again, it is enough to show that, for each Kripke model $\mathbf{M}$ and each modal formula γ of $\mathcal{L}_{\mathrm{pol}}$, we have $\|\gamma\|_{\mathbf{M}} = 1$ iff $\|\gamma^\bullet\|_{\mathbf{M}} = 1$, which is proved in the same way as in Theorem 1.

To provide the inverse translation from $\mathrm{Pr}^{\mathrm{PŁ}_\triangle}$ into $\mathrm{Pr}_{\mathrm{pol}}$, we proceed analogously to the case of $\mathrm{Pr}^{Ł_\triangle}$ into $\mathrm{Pr}_{\mathrm{lin}}$: we know [8] that for formulas $\hat{\gamma}$ of $\mathrm{PŁ}_\triangle$ we have an analogous description of $f_{\hat{\gamma}}$: the only difference is that the functions $f_{k,j}$ can now be *polynomial* with integer coefficients. Thus we have to change the definition of $\gamma_{k,j}$ accordingly; in particular for $f_{k,j} = \sum_{i=1}^{n} a_i x_{i,1} \dots x_{i,m_i} + c$, we set $\gamma_{k,j} =$

$$\begin{cases} \sum_{i=1}^{n} a_i P(\varphi_{i,1}) \cdots P(\varphi_{i,m_i}) \geqslant 1 - c & \text{if } t_{k,j} = f_{k,j}^{\#} \\ \sum_{i=1}^{n} a_i P(\varphi_{i,1}) \cdots P(\varphi_{i,m_i}) < -c & \text{otherwise.} \end{cases}$$

Having these new definitions, we can observe that the proof of Theorem 2 also gives the fatihfulness of the final translation of this section:

Theorem 4. *Let* $\Gamma \cup \{\delta\}$ *be a set of modal formulas of* $\mathcal{L}_P^{\mathrm{PŁ}_\triangle}$. *Then,* $\Gamma \vDash_{\mathrm{Pr}^{Ł_\triangle}} \delta$ *iff* $\Gamma^\circ \vDash_{\mathrm{Pr}_{\mathrm{pol}}} \delta^\circ$.

4. PROOF THEORY FOR $\mathrm{Pr}^{Ł}$

This section is devoted to the proof theory of the two-layered modal fuzzy logic $\mathrm{Pr}^{Ł}$, seen as a logic on hypersequents instead of simple formulas. In the first subsection, we will extend the known [16,17] hypersequent calculus of relations HŁ for Łukasiewicz logic to a system $\mathrm{HŁ}^{res}$ and show that it axiomatizes Łukasiewicz logic in a stronger sense. In the second subsection, we will introduce another hypersequent calculus of relations, H $\mathrm{Pr}^{Ł}$, and (using a translation into the calculus $\mathrm{HŁ}^{res}$) show that it is an axiomatization of $\mathrm{Pr}^{Ł}$.

Before delving into the details of the calculi, let us recall a few basic notions concerning multisets, that will repeatedly occur in the treatment of sequents and hypersequents.

By a *multiset* over a set A we mean a function Γ from A to the set $\mathbb{N}$ of natural numbers. By $\wp^M(A)$ we denote the set of all multisets over A. The *root set* of a multiset Γ is the set

$$|\Gamma| = \{a \in A \mid \Gamma(a) > 0\}.$$

If $a \in |\Gamma|$, we say that a is an *element* of Γ of *multiplicity* $\Gamma(a)$. A multiset Γ is *finite* if $|\Gamma|$ is finite. The *empty multiset*, i.e., the constant function 0, will be denoted by the same symbol $\varnothing$ used for the empty set—the context will always be sufficient to resolve ambiguities. Given two multisets Γ and Δ, we define their multiset union $\Gamma \uplus \Delta$ as

$$(\Gamma \uplus \Delta)(a) = \Gamma(a) + \Delta(a), \quad \text{for each } a \in A.$$

As it is customary, we use square brackets for multiset abstraction; so, e.g., $[a, a, b, c]$ will denote the multiset Γ such that $\Gamma(a) = 2$, $\Gamma(b) = \Gamma(c) = 1$, and $\Gamma(d) = 0$, for any $d \notin \{a, b, c\}$. We will denote as $[a]^n$ the multiset composed of n occurrences of a, and identify $[a]^0$ with $\varnothing$.

4.1. A Strongly Complete Hypersequent Calculus of Relations for Łukasiewicz Logic

Let us start by recalling a proof-theoretic system for Łukasiewicz logic introduced by Ciabattoni, Fermüller, and Metcalfe [17], which we denote here as HŁ (see also the monograph by Gabbay, Metcalfe, and Olivetti [16]).

The basic building blocks of such calculi are sequents of relations, i.e., syntactic objects of the kind $\Gamma \lhd \Delta$ where Γ and Δ are multisets of formulas, and $\lhd$ stands for either the symbol $\preceq$ or $\prec$.

A *hypersequent of relations* G is a finite multiset of sequents of relations, denoted as

$$\Gamma_1 \lhd_1 \Delta_1 \mid \dots \mid \Gamma_n \lhd_n \Delta_n$$

where each sequent $\Gamma_i \lhd_i \Delta_i$ belonging to $|G|$ is called a component of the hypersequent. In the following we omit the "of relations" suffix; as we do not work here with any other (hyper)sequents, there is no risk of confusion.

We also adopt the following simplifying conventions: we identify a sequent S with a hypersequent singleton $[S]$ and, if on either side of a sequent we have a multiset union of multisets of formulas, we write simply a comma instead of $\uplus$ (let us stress that we do not use this convention in other contexts, e.g., in the consequence relation defined below, where we work with a set of hypersequents).

Let us define the semantics of hypersequents and the corresponding consequence relation. We extend any evaluation e of formulas of Łukasiewicz logic to multisets of formulas by setting $e(\varnothing) = 0$ and

$$e([\varphi_1, \dots, \varphi_n]) = \sum_{i \leqslant n} (e(\varphi_i) - 1).$$

Then, we say that e satisfies a hypersquent G if there is a component $\Gamma \preceq \Delta$ (or $\Gamma \prec \Delta$) of G such that $e(\Gamma) \leqslant e(\Delta)$ (resp. $e(\Gamma) < e(\Delta)$). Given hypersequents $G, G_1, \dots, G_n$, we denote as $G_1, \dots, G_n \vDash_{Ł} G$ the fact that any evaluation e which satisfies $G_1, \dots, G_n$, satisfies G as well.

Note that an evaluation e satisfies a formula φ iff it satisfies the hypersequent $\varnothing \preceq \varphi$; indeed we have $0 = e(\varnothing) \leqslant e([\varphi]) = e(\varphi) - 1$ iff $1 = e(\varphi)$. Therefore, the consequence relation just defined on multisets actually contains the usual consequence relation on formulas; indeed, for any set of formulas $\Psi \cup \{\varphi\}$ we have

$$\Psi \vDash_{Ł} \varphi \quad \text{iff} \quad \{\varnothing \preceq \psi \mid \psi \in \Psi\} \vDash_{Ł} \varnothing \preceq \varphi.$$

Hypersequent calculi will be used to axiomatize the consequence relation on hypersequents. Given a calculus HAX, a derivation of a hypersequent G from hypersequents $G_1, \dots, G_n$ in HAX is just a labeled tree, where the root is G, each node is labeled by a rule of HAX, and the leaves are either axioms or one of $G_1, \dots, G_n$. As before, by $G_1, \dots, G_n \vdash_{HAX} G$ we mean that there exists such a derivation.

It is known [16] that the hypersequent calculus HŁ displayed in Table 4.1[3] axiomatizes the tautologies of Łukasiewicz logic, i.e., for any hypersequent G, we have

$$\vDash_{\text{HŁ}} G \qquad \text{iff} \qquad \vdash_{\text{HŁ}} G.$$

Let us now consider the extension of HŁ with the rule

$$\frac{G \mid \Gamma \preceq \Delta \quad G \mid \Delta \prec \Gamma}{G}(res)$$

and denote the resulting calculus as HŁ^{res}. The rule (res) clearly makes HŁ^{res} not analytic,[4] but it is needed for obtaining a calculus which, as proved in the next theorem, captures as well the consequence relation (with finite sets of premises) of the Łukasiewicz logic on formulas.

Theorem 5. *For each hypersequent G and sequents $S_1, \dots, S_n$, we have:*

$$S_1, \dots, S_n \vdash_{\text{HŁ}^{res}} G \qquad \textit{iff} \qquad S_1, \dots, S_n \vDash_{\text{HŁ}} G.$$

Proof. The left-to-right direction holds in view of the soundness of HŁ and of the soundness of the rule (res), which is easily provable. For the converse direction, let us first introduce the following notation: for any multisets Γ and Δ of formulas, we let

$$(\Gamma \prec \Delta)^\neg = \Delta \preceq \Gamma$$
$$(\Gamma \preceq \Delta)^\neg = \Delta \prec \Gamma$$

From $S_1, \dots, S_n \vDash_{\text{HŁ}} G$, we obtain

$$\vDash_{\text{HŁ}} G \mid S_1^\neg \mid \dots S_n^\neg.$$

Indeed, if this were not the case, we would be able to find an evaluation which would satisfy neither G nor any of the $S_i^\neg$. On the other hand, any evaluation which does not satisfy $S_i^\neg$, will satisfy S_i. Hence, we would obtain a counterexample to $S_1, \dots, S_n \vDash_{\text{HŁ}} G$.

Now, by the (weak) completeness of HŁ, we obtain that $\vdash_{\text{HŁ}} G \mid S_1^\neg \mid \dots \mid S_n^\neg$. By repeated applications of the rule (res) to the latter hypersequent and the sequents $S_1, \dots, S_n$, we get the desired proof of G from $S_1, \dots, S_n$ in the calculus HŁ^{res}.

[3] Note that we also include the rules for the connectives $\vee$, $\wedge$, and $\neg$, although they are in principle derivable from those for $\to$ and $\bot$ (recall that the former connectives are definable via the latter ones in Łukasiewicz logic). Also, note that when the symbol $\lhd$ occurs in a rule of the calculus, it has to be read as two rules, one for each uniform instantiation of $\lhd$.

[4] Let us recall that a calculus is said to be analytic when, for all of its rules, all the formulas occurring in the premises already occur in the conclusion. The prototypical example of rule that breaks the analyticity of a proof-system is the (cut) rule; a variant thereof can actually be shown to be derivable in our calculus HŁ^{res}.

4.2. A Hypersequent Calculus for $\text{Pr}^{\text{Ł}}$

In this subsection, we will introduce our hypersequent calculus $\text{H Pr}^{\text{Ł}}$ for the logic $\text{Pr}^{\text{Ł}}$, seen as a consequence relation on hypersequents built over *modal* formulas of $\mathcal{L}_P^{\text{Ł}}$. For ease of reference, we say that a sequent $\Gamma \lhd \Delta$ is (modal) $\text{Pr}^{\text{Ł}}$-sequent, classical sequent, or Ł-sequent whenever Γ and Δ are multisets of (modal) formulas of $\mathcal{L}_P^{\text{Ł}}$, formulas of classical logic, or formulas Łukasiewicz logic, respectively. Furthermore, we say that a sequent $\Gamma \lhd \Delta$ is an *atomic* (modal) $\text{Pr}^{\text{Ł}}$-sequent, classical sequent, or Ł-sequent whenever Γ and Δ are multisets of *atomic* (modal) formulas of $\mathcal{L}_P^{\text{Ł}}$, formulas of classical logic, or formulas of Łukasiewicz logic, respectively. We extend these conventions to hypersequents in the obvious way.

The semantics of modal $\text{Pr}^{\text{Ł}}$-hypersequents is defined in the expected way: Given a (probabilistic) Kripke model $\mathbf{M}$ and a multiset of modal formulas $[\gamma_1, \cdots \gamma_n]$ of $\text{Pr}^{\text{Ł}}$, we let

$$||[\gamma_1, \cdots \gamma_n]||_{\mathbf{M}} = \sum_{i \leqslant n} (||\gamma_i||_{\mathbf{M}} - 1)$$

and say that $\mathbf{M}$ satisfies a modal $\text{Pr}^{\text{Ł}}$-hypersequent G if $||\Gamma||_{\mathbf{M}} \leqslant ||\Delta||_{\mathbf{M}}$ (resp. $||\Gamma||_{\mathbf{M}} < ||\Delta||_{\mathbf{M}}$) for some component $\Gamma \preceq \Delta$ (resp. $\Gamma \prec \Delta$) of G; the consequence relation $G_1, \dots, G_n \vDash_{\text{Pr}^{\text{Ł}}} G$ is then defined as expected. As in the case of Łukasiewicz logic, for every set $\Gamma \cup \{\delta\}$ of modal $\mathcal{L}_P^{\text{Ł}}$-formulas, we have

$$\Gamma \vDash_{\text{Pr}^{\text{Ł}}} \delta \quad \text{iff} \quad \{\varnothing \preceq \gamma \mid \gamma \in \Gamma\} \vDash_{\text{Pr}^{\text{Ł}}} \varnothing \preceq \delta.$$

To prove that $\text{H Pr}^{\text{Ł}}$ axiomatizes $\text{Pr}^{\text{Ł}}$, we will make an essential use of the analogous result that HŁ^{res} axiomatizes Ł and thus our axiomatization results will share the restriction to premises being sequents (which again suffices to capture $\text{Pr}^{\text{Ł}}$ seen as a consequence relation on formulas). The proof is based on a hypersequent variant of the translation of $\text{Pr}^{\text{Ł}}$ into Ł, which is at the core of various proofs of completeness of $\text{Pr}^{\text{Ł}}$ (the original idea is due to Hájek, Godo, and Esteva [4] and was further developed in subsequent works) [2,12,13]. In particular, we reduce the validity of modal $\text{Pr}^{\text{Ł}}$-hypersequents to the validity of certain consequences over Ł-hypersequents and thus (due to our axiomatization result) also to the derivability in HŁ^{res}. Then, we complete the proof by translating Ł-hypersequents back into modal $\text{Pr}^{\text{Ł}}$-hypersequents and showing that certain extra premises, corresponding to the axioms of probability, are derivable in $\text{HPr}^{\text{Ł}}$.

Note that the translation does not depend on the proposed calculus $\text{HPr}^{\text{Ł}}$, so let us deal with it first. We start by defining for each classical formula φ its equivalence set

$$\overline{\varphi} = \{\psi \mid \vdash_{\text{CL}} \psi \leftrightarrow \varphi\}.$$

Now, for any atomic modal formula $P(\varphi)$, we let $P(\varphi)^* = p_{\overline{\varphi}}$, where $p_{\overline{\varphi}}$ is a fresh propositional variable in the language of Ł, and for complex modal formulas, we let $(\gamma_1 \to_{\text{Ł}} \gamma_2)^* = \gamma_1^* \to_{\text{Ł}} \gamma_2^*$ and $\bot_{\text{Ł}}^* = \bot_{\text{Ł}}$. We also extend the translation to multisets of formulas in the expected way, i.e., $[\gamma_1, \dots, \gamma_n]^* = [\gamma_1^*, \dots, \gamma_n^*]$. Then, given a hypersequent $G = \Gamma_1 \lhd_1 \Delta_1 \mid \dots \mid \Gamma_n \lhd_n \Delta_n$, we define $G^* = \Gamma_1^* \lhd_1 \Delta_1^* \mid \dots \mid \Gamma_n^* \lhd_n \Delta_n^*$.

Finally, we include a translation of the axioms of probability into Ł-sequents. In order to keep the translation finite, we need to make it relative to a given finite set V of propositional variables. Let us define the set AX^*_V as the union of the following sets of Ł-sequents (by V_φ we denote the set of variables occurring in φ):

$$TAUT_V = \{\emptyset \leq p_{\overline{\varphi}} \mid V_\varphi \subseteq V \text{ and } \vdash_{CL} \varphi\}$$
$$CONTR_V = \{p_{\overline{\varphi}} \leq \bot \mid V_\varphi \subseteq V \text{ and } \varphi \vdash_{CL} \bot\}$$
$$ADD_V = \{p_{\overline{\varphi\vee\psi}}, p_{\overline{\varphi\wedge\psi}} \leq p_{\overline{\varphi}}, p_{\overline{\psi}} \mid V_\varphi, V_\psi \subseteq V\}$$
$$\cup\{p_{\overline{\varphi}}, p_{\overline{\psi}} \leq p_{\overline{\varphi\vee\psi}}, p_{\overline{\varphi\wedge\psi}} \mid V_\varphi, V_\psi \subseteq V\}$$

By AX_V we denote the corresponding set of $\mathrm{Pr}^{Ł}$-sequents, obtained by replacing each propositional variable $p_{\overline{\varphi}}$ by the atomic modal formula $P(\varphi)$.

Lemma 6. *Let V be a set of propositional variables. Then, for any modal* $\mathrm{Pr}^{Ł}$*-hypersequent G containing only variables from V, we have*

$$\models_{\mathrm{Pr}^{Ł}} G \qquad \text{iff} \qquad AX^*_V \models_{Ł} G^*.$$

Proof. We prove the right-to-left direction counterpositively. Assume that $\not\models_{\mathrm{Pr}^{Ł}} G$, i.e., there is a Kripke model $\mathbf{M}$ such that, for each component $\Gamma \leq \Delta$ (resp. $\Gamma < \Delta$) of G, we have $||\Gamma||_{\mathbf{M}} > ||\Delta||_{\mathbf{M}}$ (resp. $||\Gamma||_{\mathbf{M}} \geqslant ||\Delta||_{\mathbf{M}}$). Now, let $\hat{e}$ be an evaluation of Łukasiewicz logic such that $\hat{e}(p_{\overline{\varphi}}) = ||P(\varphi)||_{\mathbf{M}}$ for each φ. This evaluation is well defined, since $p_{\overline{\varphi}} = p_{\overline{\psi}}$ means that $\vdash_{CL} \varphi \leftrightarrow \psi$, hence $||P(\varphi)||_{\mathbf{M}} = ||P(\psi)||_{\mathbf{M}}$. It is straightforward to check that $\hat{e}$ satisfies all of the sequents in AX^*_V, and none of the components of G^*, i.e., it provides a counterexample to $AX^*_V \models_{Ł} G^*$.

For the left-to-right direction, let $\hat{e}$ be an evaluation satisfying the Ł-sequents from AX^*_V that does not satisfy G^*.

From the former assumption we know that, for each φ, ψ with $V_\varphi, V_\psi \subseteq V$, we have

- $\hat{e}(p_{\overline{\top}}) = 1$
- $\hat{e}(p_{\overline{\bot}}) = 0$
- $\hat{e}(p_{\overline{\varphi\vee\psi}}) + \hat{e}(p_{\overline{\varphi\wedge\psi}}) = \hat{e}(p_{\overline{\varphi}}) + \hat{e}(p_{\overline{\psi}})$.

Let W be the set of classical evaluations and consider the subset of the powerset of W defined as

$$B_V = \{\{w \mid w(\varphi) = 1\} \mid \varphi \text{ a formula and } V_\varphi \subseteq V\}.$$

Clearly, B_V is the domain of a Boolean subalgebra $|(DT - Algorithm)||(DT - Algorithm)|\boldsymbol{B}|(/DT - Algorithm)||(/DT - Algorithm)|_V$ of the powerset algebra of W. Then, we define a function $\mu' : B_V \to [0, 1]$ as

$$\mu'(\{w \mid w(\varphi) = 1\}) = \hat{e}(p_{\overline{\varphi}}).$$

Due to the properties of $\hat{e}$ listed above, we know that μ' is a finitely additive probability measure on $|(DT - Algorithm)||(DT - Algorithm)|\boldsymbol{B}|(/DT - Algorithm)||(/DT - Algorithm)|_V$ and so, by the Horn–Tarski theorem [6,7], we know that there is a finitely additive probability measure μ on the powerset algebra of W such that $\mu(X) = \mu'(X)$ for each $X \in B_V$.

Then, $\mathbf{M} = \langle W, \langle w\rangle_{w\in W}, \mu\rangle$ is a Kripke model (the measurability condition is trivial as all subsets of W are μ-measurable) and we only need to check that G is not satisfied in $\mathbf{M}$. This is a routine check, since $||P(\varphi)||_{\mathbf{M}} = \hat{e}(p_{\overline{\varphi}})$ and so $||\Gamma||_{\mathbf{M}} = \hat{e}(\Gamma^*)$ for each multiset Γ of modal formulas occurring in G.

The calculus $\mathrm{HPr}^{Ł}$ that we propose as axiomatization of the logic $\mathrm{Pr}^{Ł}$ is composed of all the rules in Table 1, which are applicable to classical hypersequents, and all the rules in Table 2, which consist of

- variants of all of the rules in Table 1, plus the rule (res) applicable to modal $\mathrm{Pr}^{Ł}$-hypersequents,
- the axiom

$$\overline{\leq p \mid p \leq \bot}\,(cl)$$

where the propositional variable p belongs to the language of classical logic CL, i.e., to the nonmodal formulas of $\mathrm{Pr}^{Ł}$,

- and the rule

$$\frac{\varphi_1, \dots, \varphi_n \lhd \psi_1, \dots, \psi_m, [\bot]^l}{P\varphi_1, \dots, P\varphi_n \lhd P\psi_1, \dots, P\psi_m, [\bot_Ł]^l}\,(gen)$$

where $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m$ are nonmodal.

A few additional words are needed on the rule (gen). First, note that the multiset $[\bot]^l$ can also be empty (the multiplicity l is allowed to be 0), i.e., the presence of $\bot$ is not required for the application of the rule. Second, it can only be applied to classical sequents (i.e., hypersequents with only one component) and produces a modal $\mathrm{Pr}^{Ł}$-sequent.

To establish the announced axiomatizability result, we only need to prepare one crucial yet easy-to-prove lemma.

Lemma 7. *Let G be a modal* $\mathrm{Pr}^{Ł}$*-hypersequent such that* $\models_{\mathrm{Pr}^{Ł}} G$*. Then,* $AX_V \vdash_{\mathrm{HPr}^{Ł}} G$ *and there is a derivation of G from* AX_V *which does not use the rule* (gen)*.*

Proof. By Lemma 6 and Theorem 5 we know that $AX^*_V \vdash_{\mathrm{HŁ}^{res}} G^*$. Replacing each translated atom $p_{\overline{\varphi}}$ in the latter proof by an atomic modal formula $P(\varphi)$, and replacing each rule used in HŁ by its modal counterpart in $\mathrm{HPr}^{Ł}$, we obtain a proof of G from AX_V in $\mathrm{HPr}^{Ł}$ which does not make use of the rule (gen).

Theorem 8. *For each modal* $\mathrm{Pr}^{Ł}$*-hypersequent G and modal* $\mathrm{Pr}^{Ł}$*-sequents* $S_1, \dots, S_n$ *we have*

$$S_1, \dots, S_n \vdash_{\mathrm{H\,Pr}^{Ł}} G \qquad \textit{iff} \qquad S_1, \dots, S_n \models_{\mathrm{Pr}^{Ł}} G.$$

Proof. We prove the claim without premises; the extension to the full claim is then done using the rule (res) as in the proof of Theorem 5.

The soundness is easy. For the completeness direction, assume $\models_{\mathrm{Pr}^{Ł}} G$. Then, by the previous lemma, $AX_V \vdash_{\mathrm{HPr}^{Ł}} G$ and thus it suffices to show that, for each $S \in AX_V$, we have $\vdash_{\mathrm{HPr}^{Ł}} S$.

This can be obtained by suitable applications of the rule (gen). First, observe that, for $V_\varphi, V_\psi \subseteq V$, the following sequents

$$\varphi \vee \psi, \varphi \wedge \psi \leq \varphi, \psi \quad \text{and} \quad \varphi, \psi \leq \varphi \vee \psi, \varphi \wedge \psi$$

Table 1 | Hypersequent calculus of relations $\mathrm{H}\text{Ł}$ for Ł.

$$\frac{}{\varnothing \preceq \varnothing}(emp) \qquad \frac{}{\varphi \preceq \varphi}(id)$$

$$\frac{}{\bot \prec \varnothing}(\bot \prec) \qquad \frac{}{\bot \preceq \varphi}(\bot)$$

$$\frac{G}{G \mid H}(ew) \qquad \frac{G \mid H \mid H}{G \mid H}(ec)$$

$$\frac{G \mid \Phi_1, \Phi_2 \preceq \Psi_1, \Psi_2}{G \mid \Phi_1 \preceq \Psi_1 \mid \Phi_2 \preceq \Psi_2}(split_{\preceq}) \qquad \frac{G \mid \Phi_1, \Phi_2 \preceq \Psi_1, \Psi_2}{G \mid \Phi_1 \preceq \Psi_1 \mid \Phi_2 \prec \Psi_2}(split_{\text{Ł}})$$

$$\frac{G \mid \Phi_1 \lhd \Psi_1 \quad G \mid \Phi_2 \lhd \Psi_2}{G \mid \Phi_1, \Phi_2 \lhd \Psi_1, \Psi_2}(mix)$$

$$\frac{G \mid \Phi \lhd \Psi}{G \mid \Phi, \varphi \lhd \Psi}(wl) \qquad \frac{G \mid \Phi \preceq \Psi}{G \mid \Phi, \bot \prec \Psi}(w_{\bot})$$

$$\frac{G \mid \Phi, \psi \lhd \varphi, \Psi \mid \varphi \preceq \psi \quad G \mid \Phi \lhd \Psi \mid \psi \prec \varphi}{G \mid \Phi, \varphi \to \psi \lhd \Psi}(\to l) \qquad \frac{G \mid \Phi \lhd \Psi \quad G \mid \Phi, \varphi \lhd \psi, \Psi \mid \varphi \preceq \psi}{G \mid \Phi \lhd \varphi \to \psi, \Psi}(\to r)$$

$$\frac{G \mid \Phi, \varphi \lhd \Psi \mid \Phi, \psi \lhd \Psi}{G \mid \Phi, \varphi \wedge \psi \lhd \Psi}(\wedge l) \qquad \frac{G \mid \Phi \lhd \varphi, \Psi \quad G \mid \Phi \lhd \psi, \Psi}{G \mid \Phi \lhd \varphi \wedge \psi, \Psi}(\wedge r)$$

$$\frac{G \mid \Phi, \varphi \lhd \Psi \quad G \mid \Phi, \psi \lhd \Psi}{G \mid \Phi, \varphi \vee \psi \lhd \Psi}(\vee l) \qquad \frac{G \mid \Phi \lhd \varphi, \Psi \mid \Phi \lhd \psi, \Psi}{G \mid \Phi, \lhd \varphi \vee \psi, \Psi}(\vee r)$$

$$\frac{G \mid \Gamma, \bot \lhd \varphi, \Psi}{G \mid \Phi, \neg \varphi \lhd \Psi}(\neg l) \qquad \frac{G \mid \Phi, \varphi \lhd \bot, \Psi}{G \mid \Phi \lhd \neg \varphi, \Psi}(\neg r)$$

Table 2 | Additional rules for the hypersequent calculus $\mathrm{HPr}^{\text{Ł}}$ for $\mathrm{Pr}^{\text{Ł}}$.

$$\frac{\varphi_1, \dots, \varphi_n \lhd \psi_1, \dots, \psi_m, [\bot]^l}{P\varphi_1, \dots, P\varphi_n \lhd P\psi_1, \dots, P\psi_m, [\bot_{\text{Ł}}]^l}(gen) \qquad \frac{}{p \preceq \bot \mid \preceq p}(cl)$$

$$\frac{}{\varnothing \preceq \varnothing}(emp) \qquad \frac{}{\gamma \preceq \gamma}(id)$$

$$\frac{}{\bot_{\text{Ł}} \prec \varnothing}(\bot \prec) \qquad \frac{}{\bot_{\text{Ł}} \preceq \gamma}(\bot)$$

$$\frac{G}{G \mid H}(ew) \qquad \frac{G \mid H \mid H}{G \mid H}(ec)$$

$$\frac{G \mid \Gamma_1, \Gamma_2 \preceq \Delta_1, \Delta_2}{G \mid \Gamma_1 \preceq \Delta_1 \mid \Gamma_2 \preceq \Delta_2}(split_{\preceq}) \qquad \frac{G \mid \Gamma_1, \Gamma_2 \preceq \Delta_1, \Delta_2}{G \mid \Gamma_1 \preceq \Delta_1 \mid \Gamma_2 \prec \Delta_2}(split_{\text{Ł}})$$

$$\frac{G \mid \Gamma_1 \lhd \Delta_1 \quad G \mid \Gamma_2 \lhd \Delta_2}{G \mid \Gamma_1, \Gamma_2 \lhd \Delta_1, \Delta_2}(mix) \qquad \frac{G \mid \Gamma \preceq \Delta \quad G \mid \Delta \prec \Gamma}{G}(res)$$

$$\frac{G \mid \Gamma \lhd \Delta}{G \mid \Gamma, \gamma \lhd \Delta}(wl) \qquad \frac{G \mid \Gamma \preceq \Delta}{G \mid \Gamma, \bot_{\text{Ł}} \prec \Delta}(w_{\bot})$$

$$\frac{G \mid \Gamma, \delta \lhd \gamma, \Delta \mid \gamma \preceq \delta \quad G \mid \Gamma \lhd \Delta \mid \delta \prec \gamma}{G \mid \Gamma, \gamma \to \delta \lhd \Delta}(\to l) \qquad \frac{G \mid \Gamma \lhd \Delta \quad G \mid \Gamma, \gamma \lhd \delta, \Delta \mid \gamma \preceq \delta}{G \mid \Gamma \lhd \gamma \to \delta, \Delta}(\to r)$$

$$\frac{G \mid \Gamma, \gamma \lhd \Delta \mid \Gamma, \delta \lhd \Delta}{G \mid \Gamma, \gamma \wedge \delta \lhd \Delta}(\wedge l) \qquad \frac{G \mid \Gamma \lhd \gamma, \Delta \quad G \mid \Gamma \lhd \delta, \Delta}{G \mid \Gamma \lhd \gamma \wedge \delta, \Delta}(\wedge r)$$

$$\frac{G \mid \Gamma, \gamma \lhd \Delta \quad G \mid \Gamma, \delta \lhd \Delta}{G \mid \Gamma, \gamma \vee \delta \lhd \Delta}(\vee l) \qquad \frac{G \mid \Gamma \lhd \gamma, \Delta \mid \Gamma \lhd \delta, \Delta}{G \mid \Gamma, \lhd \gamma \vee \delta, \Delta}(\vee r)$$

$$\frac{G \mid \Gamma, \bot_{\text{Ł}} \lhd \gamma, \Delta}{G \mid \Gamma, \neg \gamma \lhd \Delta}(\neg l) \qquad \frac{G \mid \Gamma, \gamma \lhd \bot_{\text{Ł}}, \Delta}{G \mid \Gamma \lhd \neg \gamma, \Delta}(\neg r)$$

are derivable in HŁ, hence also in $\mathrm{HPr}^{Ł}$. Adding to their derivations an application of the rule of (gen) results in a derivation of the corresponding sequent from AX_V in $\mathrm{HPr}^{Ł}$.

Next, let us now show that the sequent $\varnothing \preceq P(\varphi)$ is derivable in $\mathrm{HPr}^{Ł}$, for any classical tautology φ. First, note that if φ is a classical tautology, letting $p_1, \dots, p_n$ be the variables occurring in φ, we have that $p_1 \vee \neg p_1, \dots p_n \vee \neg p_n \models_{[0,1]_Ł} \varphi$; hence, in particular,

$$\varnothing \preceq p_1 \vee \neg p_1, \dots, \varnothing \preceq p_n \vee \neg p_n \models_Ł \varnothing \preceq \varphi$$

and, by the finite strong completeness of $\mathrm{HŁ}^{(res)}$, there is a derivation d of $\varnothing \preceq \varphi$ from the premises $\varnothing \preceq p_1 \vee \neg p_1, \dots, \varnothing \preceq p_n \vee \neg p_n$ in the calculus $\mathrm{HŁ}^{(res)}$. Recall that all the rules of this calculus belong to $\mathrm{HPr}^{Ł}$ as well, and hence we obtain our desired derivation of $\varnothing \preceq P(\varphi)$ in $\mathrm{HPr}^{Ł}$ by appending, after the conclusion $\varnothing \preceq \varphi$ of d, an application of the rule (gen), and before each premise $\varnothing \preceq p_i \vee \neg p_i$ the following:

$$\dfrac{\dfrac{\dfrac{}{\emptyset \preceq p_i \mid p_i \preceq \bot}\,(cl)}{\emptyset \preceq p_i \mid \emptyset \preceq \neg p_i}\,(\neg r)}{\emptyset \preceq p_i \vee \neg p_i}\,(\vee r)$$

Finally, let us consider the sequents in $CONTR_V$. Applying an argument similar to the one for $TAUT_V$, we have that, for any non-modal formula φ such that $\varphi \vdash_{\mathrm{CL}} \bot$, the sequent

$$\varphi \preceq \bot$$

is derivable in $\mathrm{H\,Pr}^{Ł}$. A derivation of $P(\varphi) \preceq \bot_Ł$ is then obtained by adding an application of (gen) to the derivation $\varphi \preceq \bot$.

5. ALTERNATIVE PROOF OF COMPLETENESS OF $\mathrm{Pr}_{\mathrm{lin}}$ AND TRANSLATION INTO $\mathrm{Pr}^{Ł}$

In this section, we will show that the logic $\mathrm{Pr}_{\mathrm{lin}}$ can be semantically translated into the logic $\mathrm{Pr}^{Ł}$ (seen as consequence relation of hypersequents) and obtain an alternative translation into $\mathrm{Pr}^{Ł\triangle}$ (seen as consequence on formulas). We will also use a converse of this translation and the fact that $\mathrm{HPr}^{Ł}$ axiomatizes $\mathrm{Pr}^{Ł}$ to obtain an alternative proof of the fact that the axiomatic system $AX_{\mathrm{Pr}_{\mathrm{lin}}}$ introduced in Subsection 2.4 is indeed an axiomatization of $\mathrm{Pr}_{\mathrm{lin}}$, i.e., we show that

$$\Gamma \vdash_{AX_{\mathrm{Pr}_{\mathrm{lin}}}} \delta \qquad \text{iff} \qquad \Gamma \models_{\mathrm{Pr}_{\mathrm{lin}}} \delta.$$

Recall that modal formulas of $\mathrm{Pr}_{\mathrm{lin}}$ are combinations of basic inequality formulas using connectives of classical logic. Following the usual classical terminology, let us call the basic inequality formulas and their negation *literals* and their disjunctions *clauses*. Then, we know that each modal formula of $\mathrm{Pr}_{\mathrm{lin}}$ is equivalent to a conjunction of certain clauses.

Let us start our work in this section by showing that the clauses of $\mathrm{Pr}_{\mathrm{lin}}$ can be faithfully translated into atomic modal $\mathrm{Pr}^{Ł}$-hypersequents. First, consider a basic inequality formula γ of the form

$$\sum_{i=1}^{n} a_i P(\varphi_i) \geqslant c$$

and note that γ can be equivalently replaced (modulo a suitable permutations) by another inequality

$$\sum_{i=1}^{m} a_i P(\varphi_i) \leqslant \sum_{i=m+1}^{n} a_i P(\varphi_i) - c$$

where all the a_is are nonnegative.

Next, we define

$$\begin{aligned}
\Gamma_\gamma &= [P(\varphi_1)]^{a_1}, \dots, [P(\varphi_m)]^{a_m} \\
\Delta_\gamma &= [P(\varphi_{m+1})]^{a_{m+1}}, \dots, [P(\varphi_n)]^{a_n} \\
s(\gamma) &= \sum_{i=1}^{m} a_i - \sum_{i=m+1}^{n} a_i + c \\
\gamma^H &= \begin{cases} \Gamma_\gamma \preceq \Delta_\gamma, [\bot]^{s(\gamma)} & \text{if } s(\gamma) \geqslant 0 \\ \Gamma_\gamma, [\bot]^{-s(\gamma)} \preceq \Delta_\gamma & \text{otherwise.} \end{cases}
\end{aligned}$$

Given a basic inequality formula γ and given $\gamma^H = \Gamma \preceq \Delta$, we define $(\neg\gamma)^H$ as $\Delta \prec \Gamma$. Finally, given any clause $\delta = \gamma_1 \vee \dots \vee \gamma_n$, we define δ^H as the hypersequent $\gamma_1^H \mid \dots \mid \gamma_n^H$.

Lemma 9. *Let* $\mathbf{M}$ *be a Kripke model and* δ *a clause in* $\mathcal{L}_{\mathrm{lin}}$. *Then,* $\mathbf{M}$ *satisfies* δ *iff it satisfies* δ^H.

Proof. Let us first assume that δ is a basic inequality formula of the form

$$\sum_{i=1}^{m} a_i P(\varphi_i) \leqslant \sum_{i=m+1}^{n} a_i P(\varphi_i) - c$$

where all a_is are nonnegative. We know that $\mathbf{M}$ satisfies δ iff the corresponding inequality holds with $P(\varphi_i)$ replaced by $x_i = \mu(\varphi_i^{\mathbf{M}})$. Recall that, using the definition of $s(\gamma)$, we can write that equivalently as

$$\sum_{i=1}^{m} a_i x_i - \sum_{i=1}^{m} a_i \leqslant \sum_{i=m+1}^{n} a_i x_i - \sum_{i=m+1}^{n} a_i - s(\delta)$$

Note that for each nonnegative integer k we have $\|[\bot]^k\|_{\mathbf{M}} = -k$ and we also have

$$\begin{aligned}
\|\Gamma_\delta\|^{\mathbf{M}} &= \sum_{i=1}^{m} a_i x_i - \sum_{i=1}^{m} a_i \\
\|\Delta_\delta\|^{\mathbf{M}} &= \sum_{i=m+1}^{n} a_i x_i - \sum_{i=m+1}^{n} a_i
\end{aligned}$$

Thus, indeed, δ is satisfied in $\mathbf{M}$ iff δ^H is (we only have to distinguish if $s(\delta)$ is negative or not and move it to the appropriate side of the inequality, which clearly coincides with definition of δ^H).

The case of δ being a negated literal or a clause then easily follows using the related definitions of satisfiability of formulas of $\mathcal{L}_{\mathrm{lin}}$ and $\mathrm{Pr}^{Ł}$-hypersequents.

Theorem 10. *Let* $\Gamma \cup \{\delta\}$ *be a finite set of formulas of* $\mathcal{L}_{\mathrm{lin}}$ *and* $\delta_1 \wedge \dots \wedge \delta_m$ *a conjunctive normal form of* $(\wedge_{\gamma \in \Gamma} \gamma) \to \delta$. *Then*

$$\Gamma \models_{\mathrm{Pr}_{\mathrm{lin}}} \delta \quad \textit{iff} \quad \models_{\mathrm{Pr}^{Ł}} \delta_i^H \textit{ for each } i = 1, \dots, m.$$

Proof. First note that, due to a classical reasoning, we have $\Gamma \vDash_{\mathrm{Pr_{lin}}} \delta$ iff for each i we have $\vDash_{\mathrm{Pr_{lin}}} \delta_i$ and so the proof follows from Lemma 9.

We can use this result to provide an alternative translation from $\mathrm{Pr_{lin}}$ into $\mathrm{Pr}^{Ł\triangle}$. It is well known [16] that any Ł-hypersequent G can be interpreted (using again essentially McNaughton theorem) as a formula $I(G)$ of $Ł_\triangle$ (the operation $\triangle$ is essential to capture the sequents of the form $\Gamma \prec \Delta$) such that $\vDash_Ł G$ if and only if $\vDash_{Ł_\triangle} I(G)$. Recalling that the atomic modal formulas of $\mathrm{Pr}^Ł$ and $\mathrm{Pr}^{Ł\triangle}$ are the same, we can replace the propositional atoms by such formulas and extend the previous result to probability logics: $\vDash_{\mathrm{Pr}^Ł} G$ if and only if $\vDash_{\mathrm{Pr}^{Ł\triangle}} I(G)$. Therefore, we easily obtain:

Corollary 11. *Let $\Gamma \cup \{\delta\}$ be a finite set of formulas of $\mathcal{L}_{\mathrm{lin}}$ and $\delta_1 \wedge \cdots \wedge \delta_m$ a conjunctive normal form of $(\wedge_{\gamma \in \Gamma} \gamma) \to \delta$. Then,*

$$\Gamma \vDash_{\mathrm{Pr_{lin}}} \delta \quad \text{iff} \quad \vDash_{\mathrm{Pr}^{Ł\triangle}} I(\delta_1^H) \wedge \cdots \wedge I(\delta_m^H).$$

We will now provide a converse translation, from atomic modal $\mathrm{Pr}^Ł$-hypersequents to formulas of $\mathrm{Pr_{lin}}$. This time we will proceed syntactically: we will show that, if an atomic modal hypersequent is provable in $\mathrm{HPr}^Ł$, its translation is derivable in $AX_{\mathrm{Pr_{lin}}}$. This, together with the previous semantical translation, will provide us with an alternative completeness proof for $\mathrm{Pr_{lin}}$.

Consider a multiset Γ of atomic $\mathrm{Pr}^Ł$-formulas (i.e., formulas of the form $P(\varphi)$ or $\bot_Ł$), and recall that we denote by $\Gamma(\alpha)$ the number of occurrences of the formula α in Γ. We define a linear term t_Γ:

$$t_\Gamma = \sum_{\alpha \in |\Gamma|,\, \alpha \neq \bot_Ł} \Gamma(\alpha)\alpha$$

Let Γ, Δ be two multisets of atomic $\mathrm{Pr}^Ł$-formulas; we define the translation of atomic sequents $\Gamma \preceq \Delta$ and $\Gamma \prec \Delta$ as follows:

$$\begin{aligned} c(\Gamma \preceq \Delta) &= \sum_{\alpha \in |\Gamma|} \Gamma(\alpha) - \sum_{\beta \in |\Delta|} \Delta(\beta) \\ (\Gamma \preceq \Delta)^\sharp &= t_\Gamma \leqslant t_\Delta + c(\Gamma \preceq \Delta) \\ (\Gamma \prec \Delta)^\sharp &= \neg (\Delta \preceq \Gamma)^\sharp \end{aligned}$$

and, for any atomic modal $\mathrm{Pr}^Ł$-hypersequent $H = \Gamma_1 \lhd_1 \Delta_1 \mid \dots \mid \Gamma_n \lhd_n \Delta_n$, we define

$$H^\sharp = (\Gamma_1 \lhd_1 \Gamma_1)^\sharp \vee \cdots \vee (\Gamma_n \lhd_n \Gamma_n)^\sharp.$$

Let us now show that the translation $(\cdot)^\sharp$ is actually an inverse of the translation $(\cdot)^H$, i.e., for any clause δ of $\mathcal{L}_{\mathrm{lin}}$, we have $(\delta^H)^\sharp = \delta$. We show the claim for δ being a basic inequality formula, the generalization to clauses being easy. Let us assume, without loss of generality, that δ is of the form

$$\sum_{i=1}^m a_i P(\varphi_i) \leqslant \sum_{i=m+1}^n a_i P(\varphi_i) - c$$

with $a_i > 0$ for $i = 1, \dots, n$. We only handle the case when

$$s(\delta) = \sum_{i=1}^m a_i - \sum_{i=m+1}^n a_i + c \geqslant 0,$$

since the case where $s(\delta) < 0$ is similar. Therefore, by the definition of the translation $(\cdot)^H$,

$$\delta^H = \Gamma_\delta \preceq \Delta_\delta, [\bot]^{s(\delta)}$$

where

$$\begin{aligned} \Gamma_\delta &= [P(\varphi_1)]^{a_1}, \dots, [P(\varphi_m)]^{a_m} \\ \Delta_\delta &= [P(\varphi_{m+1})]^{a_{m+1}}, \dots, [P(\varphi_n)]^{a_n} \end{aligned}$$

Let us first note that

$$\begin{aligned} c(\delta^H) &= \sum_{\alpha \in |\Gamma|} \Gamma(\alpha) - \sum_{\beta \in |\Delta|} \Delta(\beta) - s(\delta) \\ &= \sum_{i=1}^m a_i - \sum_{i=m+1}^n a_i - \left(\sum_{i=1}^m a_i - \sum_{i=m+1}^n a_i + c \right) \\ &= -c \end{aligned}$$

Hence, we obtain

$$\begin{aligned} (\delta^H)^\sharp &= t_{\Gamma_\delta} \leqslant t_{\Delta_\delta \uplus [\bot_Ł]^{s(\delta)}} + c(\delta^H) \\ &= \sum_{i=1}^m a_i P(\varphi_i) \leqslant \sum_{i=m+1}^n a_i P(\varphi_i) - c \\ &= \delta \end{aligned}$$

We are now ready for showing our crucial lemma.

Lemma 12. *Let H_0 be an atomic modal $\mathrm{Pr}^Ł$-hypersequent. If $\vDash_{\mathrm{Pr}^Ł} H_0$, then $\vdash_{AX_{\mathrm{Pr_{lin}}}} H_0^\sharp$.*

Proof. Due to Lemma 7, we know that there is a derivation d of H_0 from the set of sequents AX_V in the calculus $\mathrm{HPr}^Ł$ such that d does not use the rule (*gen*). Note that, except for (*res*) all the rules of the calculus are analytic, hence they cannot have premises using nonatomic modal hypersequents, if the conclusions are atomic. On the other hand, inspecting the proof of the mentioned Lemma 7, we also know that the rule (*res*) needs to be applied in d only to premises containing atomic modal formulas. Therefore, all $\mathrm{Pr}^Ł$-hypersequents occurring in d have to be atomic and modal. The proof is done by showing that for each such hypersequent G_0 (i.e., in particular, for the final hypersequent H_0), we have $\vdash_{AX_{\mathrm{Pr_{lin}}}} G_0^\sharp$.

We proceed by induction on the length of d. Let us first consider the case that G_0 is one of the axioms of $\mathrm{HPr}^Ł$, i.e., we need to find proofs of the following formulas:

- $P(\varphi) \leqslant P(\varphi)$ (if G_0 is an instance of an axiom (*id*)): this is just axiom (LQ1).
- $0 \leqslant P(\varphi)$ (if G_0 is an instance of an axiom ($\bot$)): this is just axiom (QU1).
- $0 \leqslant 0$ (if G_0 is an instance of axiom (*emp*)): this is just axiom (LQ1).
- $\neg(0 \leqslant -1)$ (if G_0 is an instance of axiom ($\bot \prec$)): it follows using axioms (LQ1) and (LQ7).

Next we deal with the case $G_0 \in AX_V$, i.e., we need to find proofs of the following formulas:

- $1 \leqslant P(\varphi)$ whenever $\vdash_{CL} \varphi$: Clearly, in this case $\vdash_{\mathrm{Pr_{lin}}} \varphi \leftrightarrow \top$ and so, by the axiom (QU2) and the rule (QUGEN), we obtain $\vdash_{\mathrm{Pr_{lin}}} 1 \leqslant P(\varphi)$.
- $P(\varphi) \leqslant 0$ whenever $\varphi \vdash_{CL} \bot$. Using (QU3) we have that $P(\top \wedge \varphi) + P(\top \wedge \neg\varphi) = P(\top)$. By (QU1) and (QUGEN), we get $P(\varphi) + P(\neg\varphi) = 1$. On the other hand, since $\neg\varphi$ is a tautology, by the previous point we have $P(\neg\varphi) = 1$, hence we finally obtain a derivation of $P(\varphi) = 0$.
- $P(\varphi \vee \psi) + P(\varphi \wedge \psi) \leqslant P(\varphi) + P(\psi)$ and $P(\varphi) + P(\psi) \leqslant P(\varphi \vee \psi) + P(\varphi \wedge \psi)$: We prove both inequalities at once using two instances of axiom (QU3):

$$P(\varphi) = P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi)$$
$$P(\varphi \vee \psi) = P((\varphi \vee \psi) \wedge \psi) + P((\varphi \vee \psi) \wedge \neg\psi)$$

As the first one is equivalent (using the rule (QUGEN), properties of classical logic, and rules for manipulation of equalities in $\mathrm{Pr_{lin}}$) to

$$P(\varphi \vee \psi) = P(\psi) + P(\varphi \wedge \neg\psi),$$

the claim easily follows by simple manipulation of equalities in the logic $\mathrm{Pr_{lin}}$.

Now assume that G_0 is a consequence of some of the rules of $\mathrm{H\,Pr^{Ł}}$. Note that, as d is a derivation of the atomic modal hypersequent H_0, we do not need to check the case of logical rules and have to deal with the structural ones only. The case of rules (ew) and (ec) is simple. Indeed, whenever $G_0 = G \mid H$, then $G_0^\sharp = G^\sharp \vee H^\sharp$ and so easily get $\vdash_{AX_{\mathrm{Pr_{lin}}}} G_0^\sharp$ from either of the two possible induction assumptions: $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp$ or $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee H^\sharp \vee H^\sharp$.

Let us now consider an instance of the rule $(split_{\preceq})$ (the case of $(split_{Ł})$ is handled analogously), i.e., the case where $G_0 = G \mid \Gamma_1 \preceq \Delta_1 \mid \Gamma_2 \preceq \Delta_2$ and the premise is $G \mid \Gamma_1, \Gamma_2 \preceq \Delta_1, \Delta_2$. By induction hypothesis, we have $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee \epsilon$, where $\epsilon = (\Gamma_1, \Gamma_2 \preceq \Delta_1, \Delta_2)^\sharp$. Without loss of generality, let $\epsilon_1 = (\Gamma_1 \preceq \Delta_1)^\sharp = t_1 \leqslant c_1$ and $\epsilon_2 = (\Gamma_2 \preceq \Delta_2)^\sharp = t_2 \leqslant c_2$. We have then

$$\epsilon = t_1 + t_2 \leqslant c_1 + c_2$$

We want to prove that

$$\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee (t_1 \leqslant c_1) \vee (t_2 \leqslant c_2).$$

First, recalling that ϵ is the same as the formula $-t_1 - t_2 \geqslant -c_1 - c_2$, and $t_2 \leqslant c_2$ is the same as $-t_2 \geqslant -c_2$, we have that

$$(\epsilon \wedge t_1 \geqslant c_1) \to t_2 \leqslant c_2$$

is an instance of axiom (LQ4). On the other hand, we have that $t_1 \leqslant c_1 \vee t_1 \geqslant c_1$ is an instance of axiom (LQ6). By classical reasoning, we obtain thus a derivation in $AX_{\mathrm{Pr_{lin}}}$ of

$$\varepsilon \to t_1 \leqslant c_1 \vee t_2 \leqslant c_2$$

hence, a derivation in $AX_{\mathrm{Pr_{lin}}}$ of $G_0^\sharp \vee \varepsilon_1 \vee \varepsilon_2$ from $G_0^\sharp \vee \epsilon$.

For the rule (mix), we have $G_0 = G \mid \Gamma_1, \Gamma_2 \preceq \Delta_1, \Delta_2$ and, by the induction hypothesis, we have $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee (\Gamma_1 \preceq \Delta_1)^\sharp$ and $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee (\Gamma_2 \preceq \Delta_2)^\sharp$. Note that, from axiom (LQ4), we know that $(\Gamma_1 \preceq \Delta_1)^\sharp \wedge (\Gamma_2 \preceq \Delta_2)^\sharp \to (\Gamma_1, \Gamma_2 \preceq \Delta_1, \Delta_2)^\sharp$ and so the claim follows using a simple classical reasoning.

For the rules (wl), we have either $G_0 = G \mid \Gamma, \gamma \preceq \Delta$ or $G_0 = G \mid \Gamma, \gamma \prec \Delta$ and, by the induction hypothesis, we have either $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee (\Gamma \preceq \Delta)^\sharp$ or $\vdash_{\mathrm{Pr_{lin}}} G^\sharp \vee (\Gamma \prec \Delta)^\sharp$. We deal with the first case with the additional assumption that $\gamma = P(\varphi)$; the second case and the case $\gamma = \bot_Ł$ are analogous. Assume that $(\Gamma \preceq \Delta)^\sharp = t \leqslant c$ and note that $(\Gamma, \gamma \preceq \Delta)^\sharp = t + P(\varphi) \leqslant c + 1$. So, the following instance of (LQ4):

$$(-P(\varphi) \geqslant -1) \wedge (-t \geqslant -c) \to -t - P(\varphi) \geqslant -c - 1$$

together with the known fact that $\vdash_{AX_{\mathrm{Pr_{lin}}}} P(\varphi) \leqslant 1$, completes the proof, using a simple classical reasoning.

For the rule $(w_\bot)$, we have $G_0 = (G \mid \Gamma, \bot_Ł \prec \Delta)$ and, by the induction hypothesis, we know that $\vdash_{AX_{\mathrm{Pr_{lin}}}} G^\sharp \vee (\Gamma \preceq \Delta)^\sharp$. Let us compute that

$$(\Gamma, \bot \prec \Delta)^\sharp = \neg(t_\Delta \leqslant t_\Gamma - c(\Gamma \preceq \Delta) - 1)$$
$$= (t_\Delta - t_\Gamma > -c(\Gamma \preceq \Delta) - 1)$$

and note that this formula follows from $(\Gamma \preceq \Delta)^\sharp = t_\Delta - t_\Gamma \geqslant -c(\Gamma \preceq \Delta)$ using axiom (LQ7). Thus, again, a classical reasoning completes the proof.

Finally, we deal with the rule (res), i.e., when $G_0 = G$ and the premises are $G \mid \Gamma \preceq \Delta$ and $G \mid \Delta \prec \Gamma$. Thus, by the induction hypothesis, we have $\vdash_{\mathrm{Pr_{lin}}} G^\sharp \vee (\Gamma \leqslant \Delta)^\sharp$ and $\vdash_{\mathrm{Pr_{lin}}} G^\sharp \vee (\Delta \prec \Gamma)^\sharp$. Note that the latter is equivalent to $\vdash_{\mathrm{Pr_{lin}}} G^\sharp \vee \neg((\Gamma \leqslant \Delta)^\sharp$ which due to classical reasoning entails $\vdash_{\mathrm{Pr_{lin}}} G^\sharp$.

Using this lemma, together with Theorems 10 and 8, we obtain the promised alternative proof of axiomatization of $\mathrm{Pr_{lin}}$.

Theorem 13. *Let $\Gamma \cup \{\delta\}$ be a finite set of formulas of $\mathcal{L}_{\mathrm{lin}}$. Then,*

$$\Gamma \vdash_{AX_{\mathrm{Pr_{lin}}}} \delta \quad \textit{iff} \quad \Gamma \models_{\mathrm{Pr_{lin}}} \delta.$$

Proof. The left-to-right direction is easy to check. For the right-to-left direction, by Theorem 10, we obtain that $\models_{\mathrm{Pr^{Ł}}} \delta_i^H$ for each $i = 1, \dots, m$, where $\delta_1 \wedge \cdots \wedge \delta_m$ is a conjunctive normal form of $(\wedge_{\gamma \in \Gamma} \gamma) \to \delta$. Then, by Lemma 12, since $(\delta_i^H)^\sharp = \delta_i$, we get that $\vdash_{AX_{\mathrm{Pr_{lin}}}} \delta_i$ for each $i = 1, \dots, m$. As axioms and rules of $AX_{\mathrm{Pr_{lin}}}$ for modal formulas are those of classical logic, we first obtain $\vdash_{AX_{\mathrm{Pr_{lin}}}} (\wedge_{\gamma \in \Gamma} \gamma) \to \delta$ and, thus, also $\Gamma \vdash_{AX_{\mathrm{Pr_{lin}}}} \delta$.

6. CONCLUSION

In this paper we have given a precise answer to the question about the relationship between the logics of uncertainty introduced and studied by Fagin, Halpern, Meggido, and others, and those developed in the area of mathematical fuzzy logic by Hájek, Godo, Esteva, and others. Indeed, we have shown that $\mathrm{Pr_{lin}}$ and $\mathrm{Pr_{pol}}$ can be faithfully translated, respectively, into the two-layered modal fuzzy logics $\mathrm{Pr}^{Ł\triangle}$ and $\mathrm{Pr}^{PŁ\triangle}$, and vice versa. Moreover, we have contributed to the proof theory of these logics by offering a hypersequent calculus of relations $\mathrm{H\,Pr}^{Ł}$ for the logic $\mathrm{Pr}^{Ł}$ (which could be easily extended to a calculus for $\mathrm{Pr}^{Ł\triangle}$). Interestingly enough, we have obtained two benefits from the formalism of hypersequents of relations for the study of $\mathrm{Pr}_{\mathrm{lin}}$: it allowed us to provide another translation into a fuzzy logic without using the $\triangle$ connective, and it gave us a new proof of the axiomatization of $\mathrm{Pr}_{\mathrm{lin}}$. Therefore, this paper has provided some further evidence that the mathematical fuzzy logic approach to reasoning about uncertain events is a fruitful one that, in a sense, can encompass other popular approaches.

Let us mention several possible future lines of research. First, regarding proof theory, a crucially important open question is whether the calculus $\mathrm{H\,Pr}^{Ł}$ can be reformulated as an analytic one, i.e., without the rule (res) or any variant thereof. Note that, however, in light of our completeness theorem, any valid $\mathrm{Pr}^{Ł}$-hypersequent G has a proof in $\mathrm{H\,Pr}^{Ł}$ with a well-structured form: a part of the proof using only the modal rules, a part using only the nonmodal rules and the rule (gen), and finally a series of applications of the nonanalytic rule (res), in order to obtain a derivation of G. This consideration might be instrumental in using the calculus for establishing interesting computational complexity bounds and/or finding conditions under which the rule (res) can be eliminated.

Regarding the proof theory of other two-layered modal fuzzy logics, we believe that the crucial trick used for proving completeness of the calculus, i.e., the translation of modal hypersequents into propositional ones, could be put to use to obtain complete hypersequent calculi in a much more general framework [2,12]. On the other hand, this method can be exploited in its full power only when we already possess a hypersequent calculus for the logic handling the modal formulas. Since, to the best of our knowledge, such a calculus is lacking for the logic PŁ, we do not see an easy way to extend our approach for obtaining analogous results for the logics $\mathrm{Pr}^{PŁ\triangle}$ and $\mathrm{Pr}_{\mathrm{pol}}$.

Moreover, we plan to continue the investigation of translations between logics of uncertainty: in particular we believe that also other classical logics dealing with measures of uncertainty different from probability, such as, e.g., plausibilities or belief functions [12,15], are amenable to similar translations into suitable two-layered modal fuzzy logics.

Finally, we plan to develop the existing abstract two-layered formalism [2] in two directions: (1) to provide, in the style of abstract algebraic logic, general completeness theorems of two-layered modal logics obtained by combination of arbitrary members of a fairly wide family of nonclassical logics and (2) show that such general results subsume most (if not all) completeness theorems provided so far in the literature for particular systems.

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REFERENCES

[1] P. Baldi, P. Cintula, C. Noguera, Translating classical probability logics into modal fuzzy logics, in: M. Stepnicka (Ed.), Proceedings of the 11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), Atlantis Press, 2019, pp. 342–349.

[2] P. Cintula, C. Noguera, Modal logics of uncertainty with two-layer syntax: a general completeness theorem, in: U. Kohlenbach, P. Barceló, R. de Queiroz (Eds.), Logic, Language, Information, and Computation, Springer, Berlin, Heidelberg, Germany, 2014, pp. 124–136.

[3] R. Fagin, J.Y. Halpern, Reasoning about knowledge and probability, J. ACM. 41 (1994), 340–367.

[4] P. Hájek, L. Godo, F. Esteva, Fuzzy logic and probability, in Proceedings of the 11th Annual Conference on Uncertainty in Artificial Intelligence (UAI '95), 1995, pp. 237–244.

[5] P. Hájek, D. Harmancová, Medical fuzzy expert systems and reasoning about beliefs, in: P. Barahona, M. Stefanelli, J. Wyatt (Eds.), Artificial Intelligence in Medicine, Springer, Berlin, Heidelberg, Germany, 1995, pp. 403–404.

[6] A. Horn, A. Tarski, Measures on Boolean algebras, Trans. Am. Math. Soc. 64 (1948), 467–497.

[7] T. Kroupa, Representation and extension of states on MV-algebras, Arch. Math. Logic. 45 (2006), 381–392.

[8] S. Aguzzoli, S. Bova, B. Gerla, Free algebras and functional representation for fuzzy logics, in: P. Cintula, P. Hájek, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, vol. 37 and 38 of Studies in Logic, Mathematical Logic and Foundations, College Publications, London, England, 2011, pp. 713–791.

[9] L. Běhounek, P. Cintula, P. Hájek, Introduction to mathematical fuzzy logic, in: P. Cintula, P. Hájek, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, vol. 37 and 38 of Studies in Logic, Mathematical Logic and Foundations, College Publications, London, England, 2011, pp. 1–101.

[10] P. Cintula, P. Hájek, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, vol. 37 and 38 of Studies in Logic, Mathematical Logic and Foundations, College Publications, London, England, 2011.

[11] R. Fagin, J.Y. Halpern, N. Megiddo, A logic for reasoning about probabilities, Inf. Comput. 87 (1990), 78–128.

[12] T. Flaminio, L. Godo, E. Marchioni, Reasoning about uncertainty of fuzzy events: an overview, in: P. Cintula, C. Fermüller, L. Godo (Eds.), Understanding Vagueness: Logical, Philosophical, and Linguistic Perspectives, vol. 36 of Studies in Logic, College Publications, London, England, 2011, pp. 367–400.

[13] P. Hájek, Metamathematics of Fuzzy Logic, vol. 4 of Trends in Logic, Kluwer, Dordrecht, Netherlands, 1998.

[14] P. Hájek, S. Tulipani, Complexity of fuzzy probability logics, Fundam. Inform. 45 (2001), 207–213.

[15] J.Y. Halpern, Reasoning About Uncertainty, MIT Press, 2005.

[16] D.M. Gabbay, G. Metcalfe, N. Olivetti, Proof Theory for Fuzzy Logics, vol. 36 of Applied Logic Series, Springer, Dordrecht, Netherlands, 2008.

[17] A. Ciabattoni, C. Fermüller, G. Metcalfe, Uniform rules and dialogue games for fuzzy logics, in: F. Baader, A. Voronkov (Eds.), Logic for Programming, Artificial Intelligence, and Reasoning, Lecture Notes in Computer Science, vol. 3452, Springer, Berlin, Heidelberg, Germany, 2005, pp. 496–510.

[18] Z. Ognjanović, M. Rašković, Z. Marković, Probability Logics, Springer International Publishing, Cham, Switzerland, 2016.