Seminar to Advanced Macroeconomics

Estimation of the Solow Growth Model

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Outline

- Linear regression: very very brief review
- The Solow model and its empirical verification
- Parameterization
- OLS estimation of the cross-country growth regression

The Solow Growth Model

$Y(t)=K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \quad 0<\alpha<1 \qquad (1)$

Labor L and technology A grow by exogenously given rates n and g:

$L(t)=L(0)e^{nt} \qquad A(t)=L(0)e^{gt}$

Growth of capital K depends on savings

Intensive form of (1): $k=\frac{K}{AL}$ implies $y=k^{\alpha}$

$\dot{k}=s\,y_t-(n+g+\delta)k_t=s\,k_t^{\alpha}-(n+g+\delta)k_t \qquad (2)$

In equilibrium the output y is given by steady state of k from (2)

$k^{*}=(s/(n+g+\delta))^{1/(1-\alpha)}. \qquad (3)$

The main message: Equilibrium growth is determined by s, n, g and δ.

Question: can this be verified empirically? Is income higher in countries with higher s and lower in countries with higher $(n+g+\delta)$?

Cross-Country Regression

Empirical verification of the Solow model has 3 steps.
1. Derivation of the equilibrium level of output per capita
2. Linearize it in order to allow using OLS
3. Estimation and testing of hypotheses

We plug $k^* = (s/(n+g+\delta))^{1/(1-\alpha)}$ into $y = k^\alpha$
thus $y^* = (s/(n+g+\delta))^{\alpha/(1-\alpha)}$ (4)

Data for Y/L but not Y/AL, so eq. (4) is multiplied by $A(t)$

$$Y/L = (s/(n+g+\delta))^{\alpha/(1-\alpha)} A(t) = (s/(n+g+\delta))^{\alpha/(1-\alpha)} A(0)e^{gt}$$

Upper equation is not linear, we need to transform it - using logarithms we get:

$$\log\left(\frac{Y}{L}\right) = \log A_0 + gt + \frac{\alpha}{1-\alpha}\log s - \frac{\alpha}{1-\alpha}\log(n+g+\delta) \qquad (4)$$

Cross-Country Regression

- Resulting equation...

$$\log\left(\frac{Y}{L}\right) = \log A_0 + g\,t + \frac{\alpha}{1-\alpha}\log s - \frac{\alpha}{1-\alpha}\log(n+g+\delta) \qquad (4)$$

...can be rewritten as:

$$\log y_i = \beta_0 + \beta_1 \log s_i + \beta_2 \log(n_i + g + \delta) + \varepsilon_i \qquad (5)$$

- Equation (5) will be estimated.

Parameterisation of Cross-Country Regression

Our regression equation:

$$\log y_i = \beta_0 + \beta_1 \log s_i + \beta_2 \log(n_i+g+\delta) + \varepsilon_i \qquad (5)$$

- Not all data observable and available – we need to set approximate values.
- *n*...........average rate of population growth in working age 15-64
- $y=ln(Y/L)$
- $g+d$......non-observable, supposed to be constant and equal to 0.05
- $s=I/Y$....from the identity $S = I$

Solow model - Data

- Data: 75 countries, 1960-1985 averages for data of rates, incomes for both years.
- Data in tables: for each country we have YL60; YL85 (income/labour; absolute), DY6085 (average depreciation rate), N6085 (population growth rate, average, %), IY (I/Y ~ s; average, %).

Solow model – Estimation results (Original M-R-W data)

```
                       Ordinary least squares
    Dependent variable: ln(MacroSolow[YL85])
    Number of observations: 75
   Variable                 Coefficient  St. Error   t-statistic  Sign.
1 Constant                    5.3676983   1.540081     3.4853332   [0.0008]
2 ln((MacroSolow[N6085]/100)+0.05)
                             -2.0133899   0.5328300  -3.7786717   [0.0003]
3 ln(MacroSolow[IY]/100)
                              1.3253532   0.1706108    7.7682812   [0.0000]
  R^2adj. = 59.063938603%   DW = 1.9816
  R^2 = 60.170318641%       S.E. = 0.6094559879
  Residual sum of squares:  26.7434352866204
  Maximum loglikelihood: -67.750424484184
  AIC =  1.9133446529        BIC =  2.036944019
     F(2,72) = 54.38486 [0.0000]
  Normality: Chi^2(2) = 5.81677  [0.0546]
  Heteroskedasticity: Chi^2(1) = 0.321696 [0.5706]
  Functional form: Chi^2(1) = 0.456655 [0.4992]
  AR(1) in the error: Chi^2(1) = 1.27E-04 [0.9910]
```

Solow model – Estimation results (Gretl sample data, „Oil“ excluded)

```
Dependent variable: l_gdp85
             coefficient   std. error   t-ratio   p-value
  --------------------------------------------------------
  const        5.42988      1.58389      3.428    0.0009   ***
  s            1.42401      0.143106     9.951    2.10e-16 ***
  ngd         -1.98977      0.563362    -3.532    0.0006   ***

R-squared            0.600865   Adjusted R-squared   0.592462

RESET test for specification (squares only) -
  Null hypothesis: specification is adequate
  Test statistic: F(1, 94) = 3.59855
  with p-value = P(F(1, 94) > 3.59855) = 0.0608999

Breusch-Pagan test for heteroskedasticity -
  Null hypothesis: heteroskedasticity not present
  Test statistic: LM = 4.5545
  with p-value = P(Chi-Square(2) > 4.5545) = 0.102566

White's test for heteroskedasticity -
  Null hypothesis: heteroskedasticity not present
  Test statistic: LM = 5.71153
  with p-value = P(Chi-Square(5) > 5.71153) = 0.335308

Test for normality of residual -
  Null hypothesis: error is normally distributed
  Test statistic: Chi-square(2) = 3.0022
  with p-value = 0.222885
```

Solow model – Comments

- 60% of differences explained with this model!
- Diagnostics OK (variables significant, heteroscedasticity OK, normality OK, DW-statistics is OK as well, Reset test gives good results, too).
- Signs – as expected.
- However – if the specification OK, $\beta_1 = -\beta_2$.
- Values? Different: -2.013 and 1.325 (and as MRW noted if $\alpha=1/3$, then $abs(\beta_i)=1/2$.
- But can we reject the hypothesis $\beta_1 = -\beta_2$?

Solow model

To be continued...

Key points

1. Assumptions of the OLS estimator
2. Interpreting the output of regression
3. Hypothesis testing – null hypothesis, p-value, tests of the OLS assumptions
4. Cross-country growth regression
5. Omitted variable bias (generally and with respect to the growth regression)
6. Testing linear restriction

Gretl script

```
open /.../mrw.gdt
logs gdp85
genr s = log(inv/100)
genr ngd = log(popgrow/100+0.05)
smpl nonoil --dummy
# model 1
ols l_gdp85 const s ngd
reset
modtest --breusch-pagan
modtest --white
modtest --normality
vif
restrict
b[2]+b[3]=0
end restrict
```

Appendix

- Production function

$$(1) \qquad Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \qquad 0 < \alpha < 1.$$

- Specification of L and A

$$(2) \qquad L(t) = L(0)e^{nt}$$

$$(3) \qquad A(t) = A(0)e^{gt}$$

- Note to (2) and (3): $dL(t)/dt = L_0 e^{nt} n = L(t)n$, which is just the form from the lecture (you can try it at home for $A(t)$).

Appendix

- The dynamics of the model: dynamics of capital

$$(4) \qquad \dot{k}(t) = sy(t) - (n + g + \delta)k(t)$$
$$= sk(t)^{\alpha} - (n + g + \delta)k(t)$$

- Assuming that $\dot{k}(t) =$ *0* we obtain equilibrium value $k*$

$$(5) \qquad k^* = [s/(n + g + \delta)]^{1/(1-\alpha)}$$

Appendix

- Derivation of **Cross-country regression equation**.
- Production function: $y = k^{\alpha}$ => substitute k with k^*. Then substiute k^* with (5):

$$y^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

- Rearranging from $y = Y/AL$ on the left side to Y/L, substituting $A(t) = A_0 e^{gt}$ and taking logs (this also makes (1) linear for OLS):

$$\text{(6)} \qquad \ln\left[\frac{Y(t)}{L(t)}\right] = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln(s) - \frac{\alpha}{1-\alpha}\ln(n+g+\delta).$$

Review of Linear Regression

$$y_i = \alpha + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \ldots \beta_k x_{k,i} + u_i$$

$i = 1 \ldots n$ are observations, $x_{i,k}$ are explanatory variables, y_i is a dependent variable , α an intercept , β slopes and u_i random disturbances.

$$y = X\beta + u$$

Assumptions:

1. The true model is $y = X\beta + u$ (functional form)
2. $E(u) = 0$
3. $var(u) = \sigma^2 I_n$ (homoscedasticity and no autocorrelation)
4. $k < n$
5. $u \in N(0, \sigma^2 I_n)$ & $cov(X, u) = 0$