JEM004 Macroeconomics
IES, Fall 2008
Lecture Notes
Eva Hromádková

2 The Solow-Swan Growth Model

2.1 Basic structure

General structure of growth models: how to build a model (decentralized version)

Households:

- owners of all inputs and assets in economy ⇒ their decisions determine outcomes
- # of children + work (how many hours)/leisure decision ⇒ L_t (work force)
- consumption (C_t)/savings (=investment) decision ⇒ K_t (capital)

Firms:

- hire people (L_t) and capital (K_t) to produce good (Y_t) by production technology ($F(\cdot)$)
- have access to knowledge (A_t) that makes the production more effective

Other institutions: introduction depends on what issue we want to analyze=

- government (taxation, social security)
- central bank (monetary transmission)

All agents meet at the markets for goods and inputs (labor and capital market), where equilibrium prices and quantities are determined.

Structure of Solow-Swan model: simplified approach

- savings are constant fraction $s \in [0, 1]$ of output
- labor force and knowledge grow at given (exogenous) rates (n and g, respectively)

⇒ all decisions of HHs and firms are already made

2.2 Neoclassical production function:

$Y(t) = F(K(t), A(t)L(t))$

- capital $K(t)$ - durable physical inputs (machines, buildings, computers, etc.)
- labor $L(t)$ - number of workers and the amount of time they work
- knowledge $A(t)$ - the effectiveness of production

Properties:

1. **Constant returns to scale** (CRS) in capital and effective labor:

$$F(cK, cAL) = cF(K, AL) \quad \text{for all}\, c \geq 0$$

   - no gains from further specialization
   - other imputs (e.g. land, natural resources) are not important
   - we can rewrite production function into intensive form (per effective worker): if we denote $k = \frac{K}{AL}, y = \frac{Y}{AL}, f(k) = F(k, 1)$ then we can rewrite production function as

$$y = \frac{Y}{AL} = \frac{F(K, AL)}{AL} = F(\frac{K}{AL}, 1) = F(k, 1) = f(k)$$

2. **Positive and diminishing returns to inputs**:

$$\frac{\partial F}{\partial K} > 0, \frac{\partial F}{\partial L} > 0; \quad \frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0$$

   This also translates into conditions on intensive-form production function:

$$\frac{\partial F}{\partial K} = AL\frac{\partial F(\frac{K}{AL}, 1)}{\partial K} = AL\frac{df(k)}{dK} = AL\frac{1}{AL}f'(k) = f'(k) > 0$$

$$\frac{\partial^2 F}{\partial K^2} = \frac{df'(k)}{dK} = \frac{f''(k)}{AL} \quad \Rightarrow \quad \text{if} \quad AL > 0 \quad \text{then} \quad f''(k) < 0$$

3. **Inada conditions and essentiality:** marg. products at extremes

$$\begin{aligned}
\lim_{K\to 0} \frac{\partial F}{\partial K} &= \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty \quad \Rightarrow \quad \lim_{k\to 0} f'(k) = \infty \\
\lim_{K\to \infty} \frac{\partial F}{\partial K} &= \lim_{L\to \infty} \frac{\partial F}{\partial L} = 0 \quad \Rightarrow \quad \lim_{k\to \infty} f'(k) = 0 \\
F(K, 0) &= F(0, L) = f(0) = 0
\end{aligned}$$

Example 1: Cobb-Douglas production function

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1$$

Prop 1: $\quad F(cK, cAL) = (cK)^{\alpha}(cAL)^{1-\alpha} = c^{\alpha}c^{1-\alpha}F(K, AL) = cF(K, AL)$

Intensive form: $f(k) = F(\frac{K}{AL}, 1) = \left(\frac{K}{AL}\right)^{\alpha} = k^{\alpha}$

Prop 2: $f'(k) = \alpha k^{\alpha-1} > 0; f''(k) = (\alpha - 1)\alpha k^{\alpha-1} < 0$

2.3 Dynamics and solution of the model:

Labor and technology: grow at exogenous (given) constant rates over time, described by differential equations

$$\begin{aligned} \dot{L}(t) &= \boldsymbol{n}L(t) \\ \dot{A}(t) &= \boldsymbol{g}A(t) \end{aligned}$$

Growth rate of a variable equals the rate of change of its natural logarithm, i.e.

$$\frac{d\ln(X(t))}{dt} = \frac{d\ln(X(t))}{dX(t)}\frac{dX(t)}{dt} = \frac{1}{X(t)}\dot{X}(t)$$

Applied to our case

$$\begin{aligned} \frac{\dot{L}(t)}{L(t)} &= \frac{d\ln(L(t))}{dt} = n \;\Rightarrow\; \ln L(t) = \ln L(0) + nt \;\Rightarrow\; L(t) = L(0)e^{nt} \\ \frac{\dot{A}(t)}{A(t)} &= \frac{d\ln(A(t))}{dt} = g \;\Rightarrow\; \ln A(t) = \ln A(0) + gt \;\Rightarrow\; A(t) = A(0)e^{gt} \end{aligned}$$

We thus assume that both labor and knowledge grow exponentially over time.

Capital:

- output is divided between consumption $C(t)$ and savings $S(t)$ - in this model, savings are constant fraction s of output and they are immediately used as investment $I(t)$ into new capital, i.e. $S(t) = sY(t) = I(t)$
- existing capital depreciates over time at rate δ

Change in capital stock is therefore difference between new investment and depreciation

$$\dot{K}(t) = I(t) - \delta K(t) = sY(t) - \delta K(t)$$

Now, let us rewrite it intensive units (for simpler analysis). First, divide the equation by $A(t)L(t)$.

$$\frac{\dot{K}(t)}{A(t)L(t)} = sy(t) - \delta k(t) = sf(k(t)) - \delta k(t)$$

Now, the only problem is LHS. We would like to express it in terms of a change in intensive variable, i.e. $\dot{k}$. How does that look like?

$$\begin{aligned}\dot{k}(t) = \left(\frac{\dot{K(t)}}{A(t)L(t)}\right) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}[A(t)\dot{L}(t) + L(t)\dot{A}(t)] \\ &= \frac{\dot{K}(t)}{A(t)L(t)} - k(t)[n+g]\end{aligned}$$

Which implies the **key equation of the Solow-Swan model**

$$\underbrace{\dot{k}(t)}_{\text{change in capital per effective worker}} = \underbrace{sf(k(t))}_{\text{actual investment}} - \underbrace{[\delta + n + g]k(t)}_{\text{break-even investment}} .$$

- $sf(k(t))$ - savings from output per effective worker
- $[\delta + n + g]k(t)$ - investment needed to keep the level of capital per effective worker that decreases due to
  - depreciation (δ)
  - growing quantity of effective labor ($n + g$)

Figure 1a) depicts the two terms of the expression for $\dot{k}$ as a function of k; while actual investment $sf(k)$ is a **concave** function of k due to the properties of production function, break-even investment is **linear** in k. Therefore, there exist certain k^* at which actual investment equals break-even investment (i.e. the lines cross), which is called stationary equilibrium.

Let us analyze the dynamics of the model. If the economy starts with $k_0 < k^*$ (to the "left"), we save and invest more than is needed for the covering of depreciation and growth, i.e. the net investment $\dot{k}$, depicted on Figure 1b), is positive and the stock of capital per effective worker is increasing up to the point when it is equal k^*. However, if the economy starts with $k_0 > k^*$ (to the "right"), our savings do not cover the losses and the stock of capital per effective worker is decreasing. In any case, regardless where the level of capital per effective worker initially starts, in time it **converges** to k^*, which is then called **stable equilibrium**.

Balanced growth path: In the equilibrium, the level of capital per effective worker converges to k^* - it remains constant with no growth. However, how do the original variables behave?

- By assumption, labor L grows at rate n and technology A grows at rate g.
- Capital stock $K = ALk$; as k is constant K grows at rate $(n + g)$
- Output $Y = F(K, AL)$. As $F(\cdot)$ is CRS and K and AL both grow at rate $(n+g)$, also Y grows at rate $(n + g)$

- Output per worker Y/L as well as capital per worker K/L grow at rate g

Economy thus converges to a **balanced growth path**, where every variable of the model is **growing at a constant rate**.

Moreover, the above-stated results prove that very simple Solow model is able to mimic Kaldor's (1963) stylized facts:

- **Fact 1:** Output per worker Y/L grows over time and the growth rate does not tend to diminish. —> Y/L grows at constant rate g.
- **Fact 2:** Physical capital per worker K/L grows over time. —> K/L grows at constant rate g.
- **Fact 3:** The capital to output ratio K/Y is nearly constant. —> Yes, because capital K and output Y grow at the same rate $n+g$.

2.4 Effect of the change in the saving rate

- Saving rate can be affected by the government policies and decisions.
- In further analysis we assume Solow economy initially on a balanced growth path, which experiences a permanent increase in the saving rate at time t_0.

Qualitative effect on output:

Figure 2 shows the effect on investment. As s increases, line of actual investment shifts upward, which means that actual investment is now greater than break-even investment and the net investment $\dot{k}$ is positive. However, level of k does not jump at time t_0, but rise continuosly to its new equilibrium level k^*_{new}.

Figure 3 in its five panels summarizes the qualitative effect on capital and output in the economy. Panel 1 depicts the sudden permanent jump (increase) in saving rate s at time t_0. Panel 2 shows corresponding jump in net investment $\dot{k}$, which in turn gradually increases k towards the new balanced growth path level (Panel 3). Panel 4 depicts the evolution of the growth rate of output per capita Y/L. As $Y/L = Af(k)$, in steady state, when the capital per effective worker has zero growth rate, Y/L grows at rate g. However, in transition to new steady state, the growth rate of k is positive and therefore also growth rate of Y/K jumps up, afterwards gradually returning to g. This translates in permanently higher level of logarithm of output per capita lnY/K, which now settles on the higher growth path parallel to the original one[1].

[1] If a variable grows at constant rate, e.g. Y/L growing at rate g, it grows exponentially and its graph in time would be a parabola. However, the graph of its natural logarithm in time is a straight line.

Effect on consumption:

In the world of Solow model with no opportunity of storage everything that is not invested should be consumed, i.e. $c^* = (1-s)f(k^*)$. As at t_0 saving rate s jumps but k remains constant, consumption reacts by initial jump downward. Later, when k increases to its new equilibrium level k^*, also consumption increases. BUT, is new level c^*_{new} larger or smaller than original c^*_{old}?

That depends on the initial steady state of the economy. On the balanced growth path

$$c^* = f(k^*) - (n+g+\delta)k^*$$

Thus

$$\frac{\partial c^*}{\partial s} = f'(k^*)\frac{dk^*}{ds} - (n+g+\delta)\frac{dk^*}{ds} = [f'(k^*) - (n+g+\delta)]\frac{dk^*}{ds}.$$

As we know that $\frac{dk^*}{ds}$ is positive, therefore the effect on consumption is determined by the sign of the expression $[f'(k^*) - (n+g+\delta)]$, which can be both positive and negative. In other words, the question is whether the marginal product of additional capital created by increased saving rate is larger than its natural depreciation. If yes, there is output left for the increase in consumption. If not, we have to sacrifice consumption in order to keep the level of capital stable.

Figure 4 depicts both situations. In the panel 4a, the saving rate is relatively high and at the original equilibrium point, the slope of production function ($f'(k)$) is flatter than the slope of break-even investment (i.e. $f'(k) < (n+g+\delta)$). We see that further increase in s would lead to a lower level of consumption, measured as the gap between $f(k)$ and $(n+g+\delta)k$. On the other hand, in the panel 4b the saving rate is relatively small and marg. product is bigger than depreciation. Therefore, increase in saving rate implies also increase in consumption.

Finally, panel 4c present situation where $f'(k) = (n+g+\delta)$. In this point consumption is at its maximum possible level and saving rate is optimal. Therefore, steady state level of k^* that is consistent with this equilibrium is called **golden rule** level of capital.

2.5 Quantitative implications of the model:

Elasticity of output w.r.t. saving rate: How much does it adjust?

$$\frac{\partial y^*}{\partial s} \Big/ \frac{y^*}{s} = \frac{k^* f'(k^*)}{f(k^*)} \Big/ \Big(1 - \frac{k^* f'(k^*)}{f(k^*)}\Big)$$

In competitive economy with no externalities, production inputs earn their marginal products - 1 additional unit of capital earns $f'(k)$ units of output. Therefore, expression $\frac{k^* f'(k^*)}{f(k^*)}$ can be understood as the share of total income ($f(k^*)$) that goes to capital, denoted as $\alpha_k(k^*)$. Note that this share is constant, because k^* is constant (Fact 5 from Kaldor's stylized facts).

Ex.: If $\alpha_k(k^*) = \frac{1}{3}$ (as in most developed countries), then elasticity of output at steady state value of k^* is 0.5 - i.e. 10% increase in saving rate translates into 5% increase in output (little less, in fact, due to concavity of $f(k)$).

Significant changes in **saving rate** have only **moderate** effect on the level of **output** at the balanced growth path.

Speed of convergence of capital and output: How fast do they adjust?

$$\begin{aligned} \dot{k} &\cong -\lambda(k - k^*) \quad \text{or} \quad k(t) - k^* \cong e^{-\lambda t}[k(0) - k^*] \\ \dot{y} &= -\lambda(y - y^*) \quad \text{or} \quad y(t) - y^* \cong e^{-\lambda t}[y(0) - y^*] \\ & \quad \text{where} \quad \lambda = (n + g + \delta)[1 - \alpha_k(k^*)] \end{aligned}$$

That means that both output and capital move by a constant share of the remaining distance toward their new equilibrium values.

Ex.: If $\alpha_k(k^*) = \frac{1}{3}$ and $(n + g + \delta) = 0.06$ per year then $\lambda = 0.04$, i.e. output and capital move 4% of the remaining distance toward k^* and y^* each year - halftime of convergence is 17 years.

Not only is the long run effect of a change in saving rate modest, but **it does not occur very quickly**.

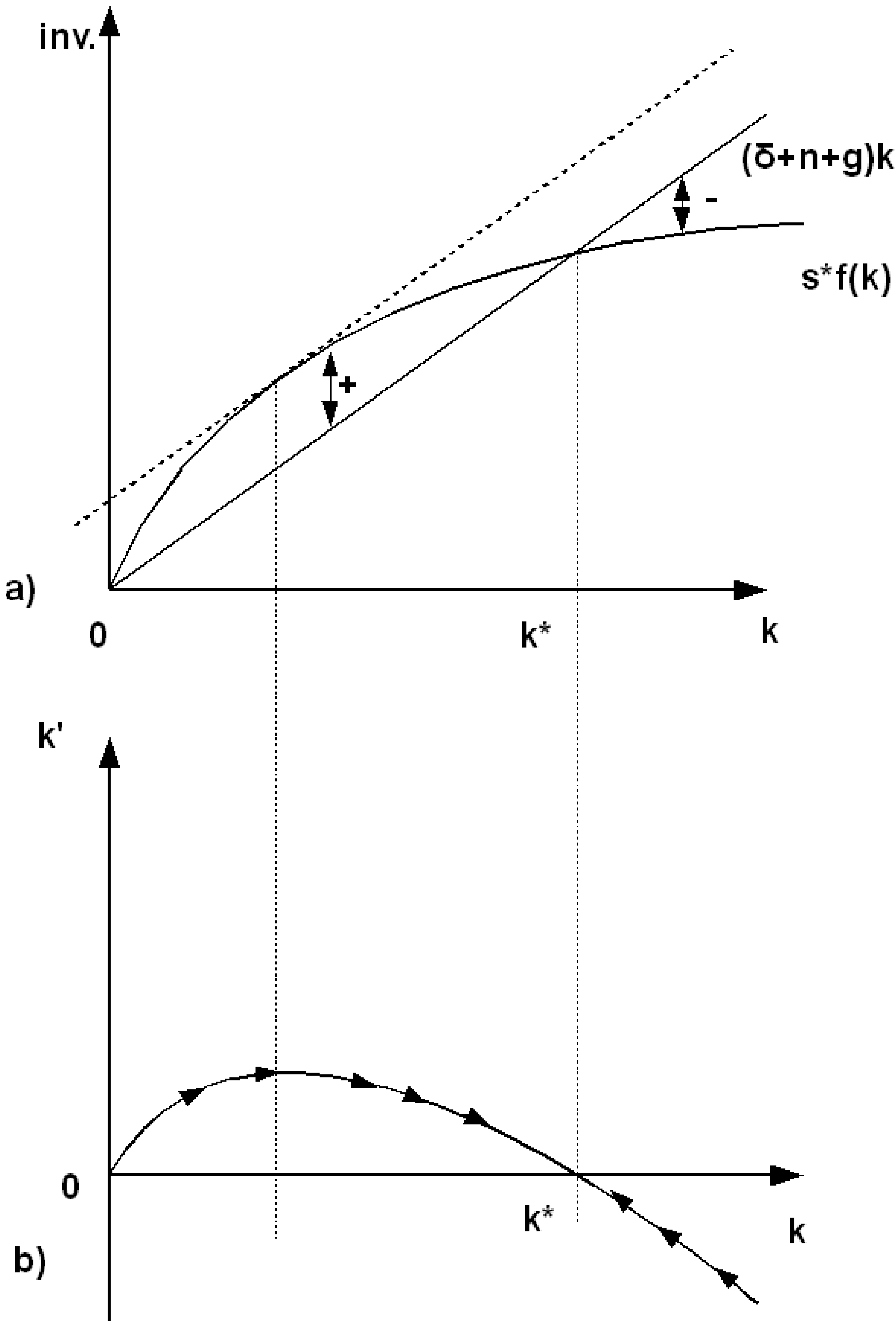

Figure 1: Dynamics and phase diagram for k in the Solow model.

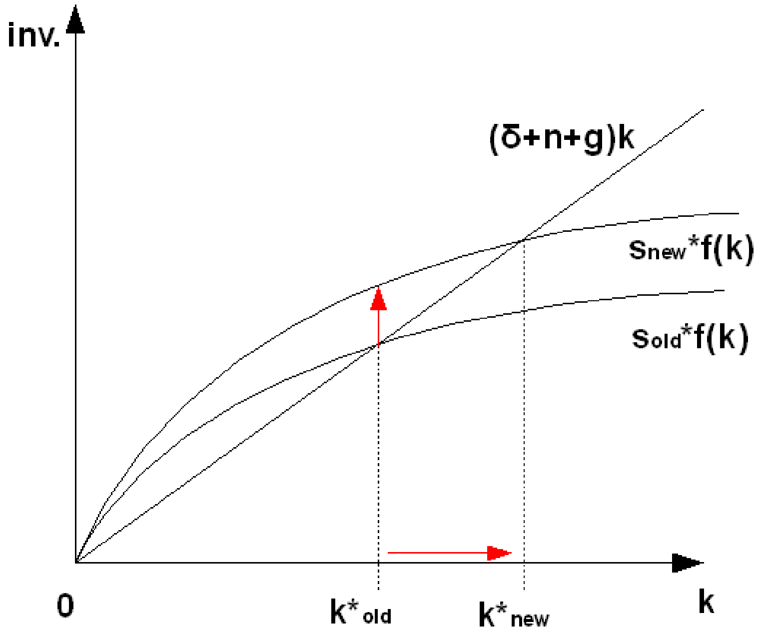

Figure 2: The effects of an increase in the saving rate on investment.

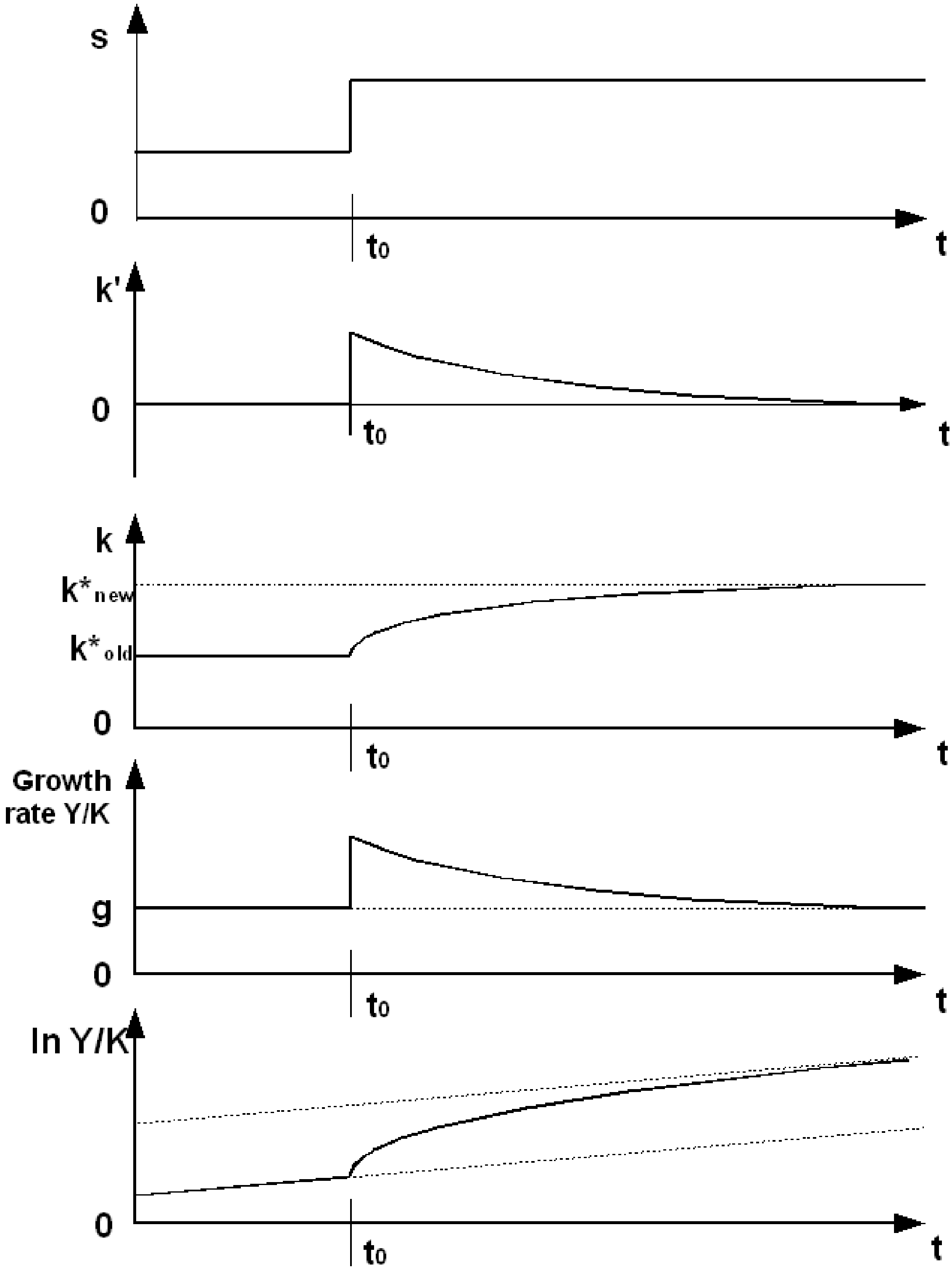

Figure 3: The effects of an increase in the saving rate.

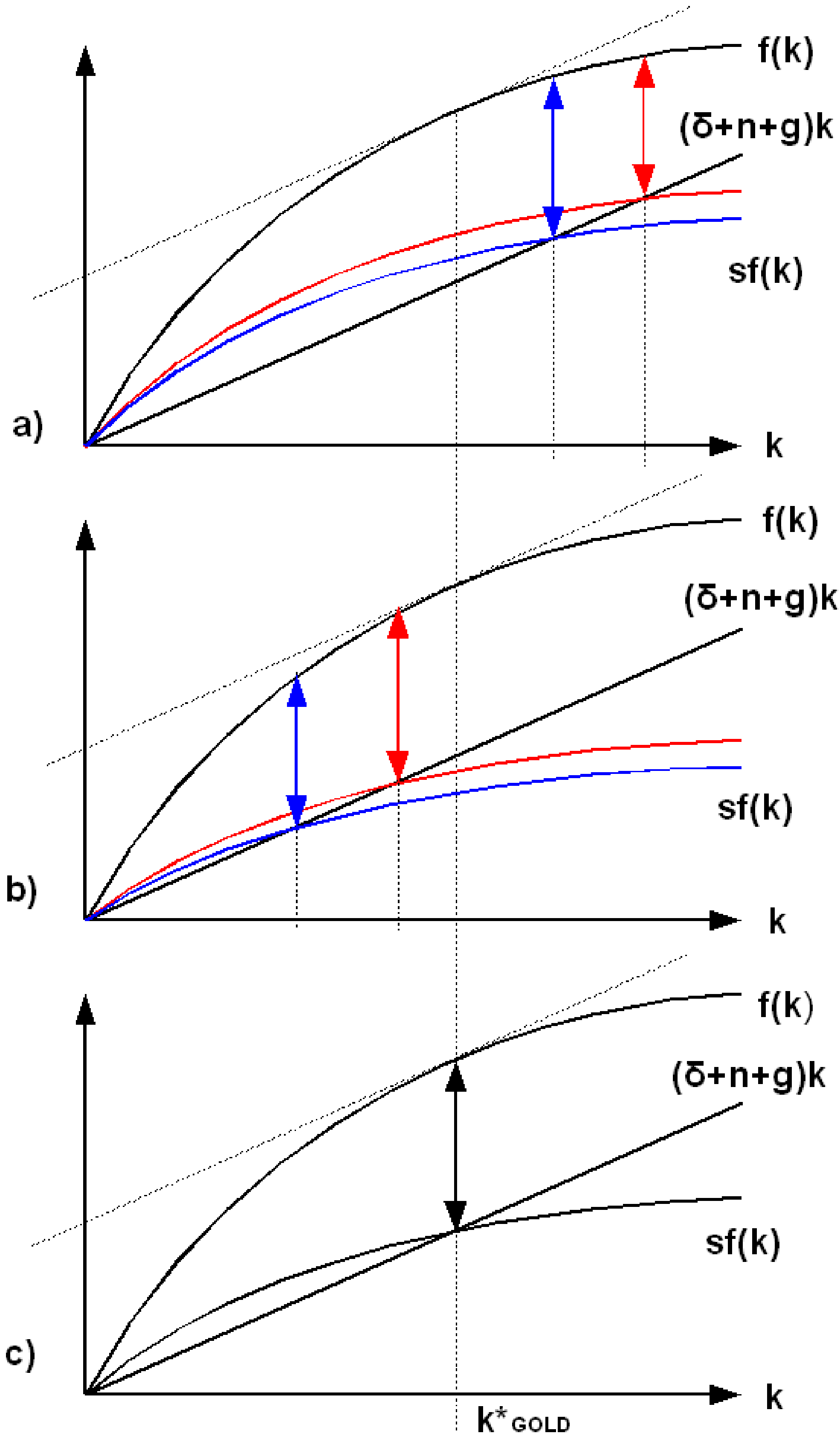

Figure 4: Consumption and changes in saving rate.