Reasoning with Inconsistent Information

Usuzování s nekonzistentními informacemi

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V Praze dne 18. 5. 2018 Adam Přenosil

Dedicated in loving memory to my grandfather, Emil Ondruška.

Abstract

This thesis studies the extensions of the four-valued Belnap–Dunn logic, called super-Belnap logics, from the point of view of abstract algebraic logic. We describe the global structure of the lattice of super-Belnap logics and show that this lattice can be fully described in terms of classes of finite graphs satisfying some closure conditions. We also introduce a theory of so-called explosive extensions and use it to prove new completeness theorems for super-Belnap logics. A Gentzen-style proof theory for these logics is then developed and used to establish interpolation for many of them. Finally, we also study the expansion of the Belnap–Dunn logic by the truth operator Δ.

Keywords: abstract algebraic logic, Belnap–Dunn logic, paraconsistent logic, super-Belnap logics

Abstrakt

Tato dizertační práce studuje extenze čtyřhodnotové Belnapovy–Dunnovy logiky, tzv. superbelnapovské logiky, z pohledu abstraktní algebraické logiky. Popisujeme v ní globální strukturu svazu superbelnapovských logik a ukazujeme, že tento svaz lze zcela popsat pomocí tříd konečných grafů splňujících jisté uzávěrové podmínky. Také zde zavádíme teorii tzv. explozivních extenzí a používáme ji k důkazu nových vět o úplnosti pro superbelnapovské logiky. Poté rozvíjeme gentzenovskou teorii důkazů pro tyto logiky a použijeme ji k důkazu věty o interpolaci pro mnoho z těchto logik. Nakonec také studujeme rozšíření Belnapovy–Dunnovy logiky o operátor pravdivosti Δ.

Klíčová slova: abstraktní algebraická logika, Belnapova–Dunnova logika, parakonzistentní logika, superbelnapovské logiky

Preface

I believe that the reader of this thesis is owed a warning and an explanation, hence this preface.

The warning is simple: *you will learn very little about actually reasoning with inconsistent information in this thesis.* If you wish to read what the author of this thesis has to say about reasoning with inconsistent information, you should instead read the paper [54].

The original intention behind this thesis was to systematically investigate both the extensions and the expansions of the four-valued Belnap–Dunn logic and to study how these can be used to actually reason with inconsistent information. (The Belnap–Dunn logic itself does not tell us how to reason in the face of inconsistent information: we do not infer that $p \wedge -p$ is true given two sources of information supporting the truth of p and the truth of $-p$, even though the inference $p, -p \vdash p \wedge -p$ is valid in this logic.)

Only the first of these intentions is fully realized in the thesis that you are now reading. The discrepancy between the intentions and the actual thesis is simple to explain: the first part of the research project is by far the best developed one, and it makes for, I believe, a satisfactory thesis on its own (with a chapter about a particular expansion of the Belnap–Dunn logic thrown in for good measure). Again, if you wish to read what the author of this thesis has to say about expansions of the Belnap–Dunn logic which internalize the notion of inconsistency in the object language (in a different way than so-called logics of formal inconsistency), you should consult the papers [56, 57] or the manuscript [53] instead of this thesis.

In the end, the thesis lies firmly within the territory of algebraic logic. Still, it is my hope that even more philosophically inclined logicians may find the exploration of the logical space between the four-valued Belnap–Dunn logic and the two-valued classical logic to be of some interest. Although few of these logics will be of any use to the philosopher, knowing what, if any, logics are on offer may still have some value.

Conversely, I hope that this thesis demonstrates that paraconsistent logics, although often pursued in connection with some particular philosophical motivation, can serve as a source of problems which are interesting on their own even from a purely mathematical point of view.

Acknowledgments

Many people contributed to the making of this thesis and helped me take my first steps in academia. I am happy to have the opportunity to acknowledge them here.

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Although I spent little time at the Department of Logic of the Faculty of Arts of Charles University during my Ph.D. studies, it was my experience as a Bachelor's and Master's student there that motivated me to pursue a Ph.D. in logic. For this I also want to thank my teachers from the Department of Logic, namely Marta Bílková, Radek Honzík, Michal Peliš, Vítězslav Švejdar, and Jonathan Verner.

Thanks are also due to Jiří Velebil, who was my co-advisor when this thesis was still supposed to be about coalgebras, for patiently explaining enriched category theory to me. The effort was not lost, I believe.

I am grateful to have been able to visit several academic institutions abroad during my studies. These visits were all the more enjoyable thanks to my gracious hosts Nick Galatos (University of Denver), Hitoshi Omori and Kazushige Terui (Kyoto University), and Ofer Arieli (The Academic College of Tel Aviv-Yafo), Arnon Avron (Tel Aviv University), and Anna Zamansky (University of Haifa).

This thesis would have a completely different shape if it were not for Umberto Rivieccio's idea that the extensions of the Belnap–Dunn logic deserve a systematic investigation. I am indebted to Umberto for sharing his extensive research notes on the topic with me. Some of his unpublished results were incorporated (with proper credit) into this thesis.

I am also happy to have had the opportunity to collaborate with and in the process learn from Hugo Albuquerque, Nick Galatos, Tomáš Lávička, Hitoshi Omori, and Umberto Rivieccio.

Not only did I enjoy our discussions on logic (and other topics), but I am also grateful to Tomáš Lávička for agreeing to organize the tenth edition of the student conference PhDs in Logic with me. I appreciate how easy it was to collaborate with him.

Ending on a more personal level, I want to thank Dominika Černá for all her love and the laughs that we shared together, not to mention all the books. I am also grateful to her for volunteering to proof-read this thesis and catching many of my typos. Hopefully my last-minute edits did not introduce too many new ones.

Contents

Introduction

Aim of the thesis

The aim of this thesis is to systematically study extensions of the four-valued Belnap–Dunn logic using the methods of abstract algebraic logic.

The idea that extensions of the Belnap–Dunn logic, also known as the logic of first-degree entailment (FDE), form a family which deserves a systematic study in its own right is fairly recent. It was first suggested by Rivieccio [66], who coined the term *super-Belnap logics* for these extensions. The proposal to systematically explore this family of logics was motivated by the discovery of a previously unknown extension of the Belnap–Dunn logic, called the Exactly True Logic by Pietz and Rivieccio [50]. This discovery made it natural to ask what other unknown extensions there are. Rivieccio's paper [66] provided a first glimpse at a territory which was by and large previously uncharted. It is our goal in this thesis to pick up where this paper left off and explore the landscape of super-Belnap logics in more detail in order to provide future researchers with a reasonably comprehensive map. This investigation will be conducted within the framework of abstract algebraic logic [14, 24, 27], which views logics as structural single-conclusion consequence relations and studies them via their matrix semantics.

There are some exceptions to this paucity of information about super-Belnap logics. The *Belnap–Dunn logic* itself has attracted a good deal of attention from logicians, philosophers, and computer scientists since the seminal papers of Dunn [18] and Belnap [8, 9] published over 40 years ago, which argued that it is a natural logic for dealing with inconsistent and incomplete information. The *strong three-valued Kleene logic* [39, 40] and the *Logic of Paradox* [52] are also well-known non-classical logics, which have been used as bases for theories of truth [41] and for proposed solutions to semantic paradoxes such as the Liar paradox. Less attention was paid to what we call, following Rivieccio, *Kleene's logic order*, identified by Dunn [17] as the first-degree fragment of the relevance logic R-Mingle. Of course, *classical logic* belongs to the family of super-Belnap logics too.

Several factors may be responsible for this lack of previous research into super-Belnap logics. Firstly, each of the above logics was introduced with a fairly specific purpose in mind, which it generally serves well. Researchers employing these logics therefore have little need to look for alternative logics in their neighbourhood. (The lack of a systematic understanding of the *expansions* of the Belnap–Dunn logic is much more puzzling in this respect.) Moreover, according some definitions of logics, there indeed are no other extensions of the Belnap–Dunn logic. This is e.g. the case with Dunn's study [20] of these extensions, which essentially builds the proof by cases property, i.e. disjunction introduction in the antecedent, into the definition of a super-Belnap logic. Finally, the investigation of super-Belnap logics involves technical obstacles which do not come up in the study of, say, super-intuitionistic or normal modal logics. This is because super-Belnap logics are, in a precise sense, not algebraizable, therefore their study cannot be reduced to the study of some class of algebras.

There are, on the other hand, also several reasons for investigating this family of logics in more detail, in addition to the intrinsic mathematical interest of the task. Firstly, although most super-Belnap logics may have little use compared to the prominent logics mentioned above (just like most super-intuitionistic logics have little use compared to the prominent ones like the Gödel–Dummett logic), knowing precisely what gives the above logics special status among all super-Belnap logics gives us more insight into them. As we shall see, we may even gain more insight into classical logic by studying it in the context of other super-Belnap logics.

Secondly, studying super-Belnap logics contributes to our understanding of so-called non-protoalgebraic logics. In contrast to super-intuitionistic or normal modal logics, which can be studied using Heyting algebras or Boolean algebras with operators, these are logics where the link between logic and algebra is too weak to allow us to study them directly by studying the corresponding algebras. Moreover, many of the theorems of abstract algebraic logic relating syntactic and semantic properties of logics rely on the assumption of protoalgebraicity. Thus, although it is common nowadays to this study entire families of non-classical logics, as far as we know there has been no systematic investigation of a family of non-protoalgebraic logics comparable to the investigation of super-intuitionistic logics, substructural logics, or normal modal logics.

Thirdly, related to the previous point, the study of super-Belnap logics provides a motivation, as well as a testing ground, for new developments in abstract algebraic logic. One new direction which naturally suggests itself in connection with super-Belnap logics is the study of explosive or anti-axiomatic extensions of logics. Just like axiomatic extensions postulate that certain formulas are always true, explosive extensions postulate that certain sets of formulas are never true. In the case of super-Belnap logics, it is the lattice of explosive extensions rather than the lattice of axiomatic extensions that forms an interesting object of study. Remarkably, it turns out that this lattice is dually isomorphic, give or take an element at the top and bottom, to the lattice of classes of finite graphs closed under homomorphisms.

Finally, in their Gentzen-style formulation super-Belnap logics provide semantics for sequent calculi without the Cut rule and the Identity axiom. Just like substructural logics provide an algebraic semantics for calculi which keep these rules but relax the structural rules of Exchange, Weakening, and Contraction in the sequent calculi for classical logic, super-Belnap logics keep these rules while relaxing Cut and Identity. Elimination rules, i.e. the inverses of the introduction rules, are part of these calculi. Studying super-Belnap logics therefore amounts to studying cut-free and identity-free Gentzen calculi with elimination rules.

Outline

We now outline the structure of this thesis. A summary of the main results can be found in the following section. Let us note here that throughout the thesis we restrict our attention to propositional logics.

The preliminary part of the thesis consists of chapters 1–3. Here we review the general algebraic and logical preliminaries (Chapter 1), introduce the variety of De Morgan algebras (Chapter 2), and finally introduce the Belnap–Dunn logic and its best-known extensions (Chapter 3). The material presented in these chapters is, except for some parts of Chapter 3, not new.

The main arc of the thesis consists of chapters 4–8. These chapters build on the preceding ones and should therefore be read in linear order. We first prepare the ground for later chapters by introducing explosive extensions of logics as extensions by antiaxioms and investigating their basic properties (Chapter 4). Explosive parts of logics are also introduced and shown to be helpful when axiomatizing logics determined by products of matrices. This general theory is then applied to obtain a crop of new completeness results for super-Belnap logics (Chapter 5). Several completeness theorems for super-Belnap logics are also proved directly.

The global structure of the lattice of super-Belnap logics is investigated, using so-called splitting pairs of logics (Chapter 6). In particular, we split the lattice of super-Belnap logics into three main parts. The reader may wish to skip ahead and consult Figure 6.1 to get an overview of the super-Belnap landscape. We then describe the fine structure of the lattice of super-Belnap logics in terms of finite graphs (Chapter 7). This link between the realms of super-Belnap logics and graph theory is perhaps the most surprising and mathematically pleasing part of this thesis. Finally, metalogical properties of super-Belnap logics are studied, including their classification in the Leibniz and Frege hierarchies and their algebraic counterparts and strong versions (Chapter 8). We show that only very few super-Belnap logics enjoy the desirable properties of the Belnap–Dunn logic.

The final three chapters of the thesis deal with three separate topics, and full acquaintance with the main arc of the thesis is not required in most places. We first develop the rudiments of a Gentzen-style proof theory for super-Belnap logics, including an analogue of the cut elimination theorem, and use this theorem to prove interpolation theorems for super-Belnap logics (Chapter 9). We then consider what changes have to be made to the results of the thesis if we modify our framework by dropping the truth constants from the Belnap–Dunn logic, or moving to multiple-conclusion consequence, or adding an extra predicate to the Belnap–Dunn logic (Chapter 10). In the final chapter, we study the expansion of the Belnap–Dunn logic by the truth operator Δ and its algebraic counterpart, the variety of De Morgan algebras with Δ (Chapter 11).

The bulk of this thesis (Chapters 4–8 and Chapter 10) presents material from the unpublished manuscript [58]. Parts of Chapter 3 and Chapter 8, in particular the description of the truth-equational and assertional super-Belnap logics and most of the results on strong versions of super-Belnap logics and strong versions of explosive extensions, are based on joint work with Hugo Albuquerque and Umberto Rivieccio, published in [2]. Moreover, several results proved in this thesis were first obtained by Umberto Rivieccio in his unpublished notes [67]. Proper credit for these will be given at the appropriate places throughout the thesis. Finally, Chapter 9 is entirely based on the paper [55].

Main results

Let us now briefly summarize the main results or definitions of each chapter of the thesis, skipping the first two preliminary chapters.

Chapter 3 (The Belnap–Dunn logic and its cousins)

- The basic properties of the Belnap–Dunn logic $\mathcal{BD}$, the strong Kleene logic $\mathcal{K}$, the Logic of Paradox $\mathcal{LP}$, Kleene's logic of order $\mathcal{KO}$, the Exactly True Logic $\mathcal{ETL}$, and classical logic $\mathcal{CL}$ are reviewed.

- Completeness theorems are proved for these logics.

Chapter 4 (Explosive extensions)

- An explosive extension is defined as an extension by antiaxioms, which postulate that a certain set of formulas cannot be jointly designated.

- The explosive part of an extension $\mathcal{L}$ of a base logic $\mathcal{B}$ is defined as the largest explosive extension of $\mathcal{B}$ lying below $\mathcal{L}$.

- Computing the explosive parts of the logics determined by $\mathbb{M}$ and $\mathbb{N}$ is helpful when axiomatizing the logic determined by $\mathbb{M} \times \mathbb{N}$.

Chapter 5 (Completeness theorems)

- A completeness theorem for the logic $\mathcal{ECQ}$, which extends $\mathcal{BD}$ by the principle of *ex contradictione quodlibet* $p, -p \vdash q$.

- A completeness theorem for the logic $\mathcal{K}_{-}$, which is the strongest extension of $\mathcal{ETL}$ strictly below $\mathcal{K}$.

Chapter 6 (The lattice of super-Belnap logics)

- The lattice of non-trivial super-Belnap logics splits into the three disjoint intervals $[\mathcal{BD}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $[\mathcal{ETL}, \mathcal{CL}]$.

- The lattice of non-trivial super-Belnap logics also splits into the three disjoint intervals $[\mathcal{BD}, \mathcal{ETL}]$, $[\mathcal{LP} \cap \mathcal{ECQ}_2, \mathcal{K}_{-}]$, $[\mathcal{KO}, \mathcal{CL}]$.

Chapter 7 (Super-Belnap logics and finite graphs)

- Finite reduced models of $\mathcal{BD}$ correspond precisely to triples $\langle G, H, k \rangle$, where G and H are finite graphs and $k \in \omega$ (loops are allowed).

- Finitary super-Belnap logics in $[\mathcal{ETL}, \mathcal{ETL}_\omega]$ correspond precisely to classes of finite graphs closed under surjective homomorphisms, disjoint unions, and contracting isolated edges.

- Finitary explosive extensions of $\mathcal{BD}$ correspond precisely to classes of finite graphs closed under homomorphisms.

- There are continuum many finitary super-Belnap logics, as well as continuum many antivarieties of De Morgan algebras.

- There is a non-finitary explosive extension of $\mathcal{BD}$.

Chapter 8 (Metalogical properties of super-Belnap logics)

- The logics $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ are the only well-behaved super-Belnap logics from several points of view.

- With one exception, the algebraic counterpart of a super-Belnap logic $\mathcal{L}$ is a (quasi)variety if and only if $\mathcal{L} \in [\mathcal{BD}, \mathcal{ETL}]$ or $\mathcal{L} \in [\mathcal{KO}, \mathcal{CL}]$.

Chapter 9 (Sequent calculi for super-Belnap logics)

- Each super-Belnap logic has an equivalent Gentzen counterpart, which is axiomatized by adding elimination rules to a standard calculus for classical logic and relaxing Cut and Identity.

- A normal form for proofs in these calculi is defined and a normalization theorem is proved. For classical proofs from an empty set of premises this theorem essentially reduces to the cut elimination theorem.

- Extensions of $\mathcal{BD}$ by a set of so-called generalized cut rules, such as the logic $\mathcal{ETL}$, are shown enjoy the interpolation property.

- A new syntactic proof is provided of an interpolation theorem which splits consequence in $\mathcal{CL}$ between $\mathcal{K}$ and $\mathcal{LP}$.

Chapter 10 (Other frameworks)

- The lattice of super-Belnap logics remains essentially the same whether the truth constants are included in the signature or not.

- The multiple-conclusion versions of the logics $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ are the only extensions the multiple-conclusion version of $\mathcal{BD}$.

- Expansions of the Belnap–Dunn logic by an exact truth predicate and by a non-falsity predicate are axiomatized.

Chapter 11 (The truth operator Δ)

- A structure theory for De Morgan algebras with Δ, the (quasi)variety generated by the four-element De Morgan algebra expanded by the truth operator Δ, is developed.

- The expansion of the Belnap–Dunn logic by the truth operator Δ is studied and axiomatized.

Chapter 1

Preliminaries

1.1 Universal algebra

We are assuming familiarity on part of the reader with the basic notions of universal algebra, covered e.g. in the core chapters of the textbooks [12, 46]. In particular, we are assuming familiarity with the notion of an *algebra* and a *term* in a given *signature*, and with the notions of *subalgebras*, *products* and *ultraproducts* of algebras, *homomorphisms*, *embeddings*, and *homomorphic images* of algebras, and *congruences* on algebras.

Some further notions and facts will be recalled below. However, readers without a previous acquaintance with the basics of universal algebra may find some proofs difficult to follow. Conversely, readers acquainted with the basics of universal algebra may choose to skip this section. The only non-standard notion introduced here is the notion of an antivariety.

Algebras will be denoted $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc., while congruences will be denoted θ, ϕ, ψ, etc. The fact that an algebra $\mathbf{A}$ embeds into $\mathbf{B}$ will be denoted as $\mathbf{A} \leq \mathbf{B}$. The lattice of congruences on $\mathbf{A}$ will be denoted $\operatorname{Con} \mathbf{A}$. The congruence generated by a set of pairs $X \subseteq \mathbf{A}^2$, i.e. the smallest congruence $\theta \in \operatorname{Con} \mathbf{A}$ such that $X \subseteq \theta$, will be denoted $\operatorname{Cg}^{\mathbf{A}} X$. The congruence generated by a single pair is called *principal*. The *principal congruence* generated by a pair $\langle a, b \rangle \in \mathbf{A}^2$ will be denoted $\operatorname{Cg}^{\mathbf{A}} \langle a, b \rangle$. The largest congruence on $\mathbf{A}$ is called the *trivial congruence* and denoted $\nabla_{\mathbf{A}}$, while the smallest congruence on $\mathbf{A}$ is called the *identity congruence* and denoted $\Delta_{\mathbf{A}}$.

A *subdirect embedding* of $\mathbf{A}$ into a product $\Pi_{i \in I} \mathbf{B}_i$ is an embedding $\iota : \mathbf{A} \to \Pi_{i \in I} \mathbf{B}_i$ such that each of the maps $\pi_i \circ \iota : \mathbf{A} \to \mathbf{B}_i$ is surjective, where $\pi_i : \Pi_{i \in I} \mathbf{B}_i \to \mathbf{B}_i$ are the appropriate projection maps. If there is a subdirect embedding of $\mathbf{A}$ into $\Pi_{i \in I} \mathbf{B}_i$, we say that $\mathbf{A}$ is a *subdirect product* of the algebras $\mathbf{B}_i$. An algebra $\mathbf{A}$ is called *(finitely) subdirectly irreducible* if it cannot be represented as a subdirect product of a (finite) family of algebras in a non-trivial way. That is, in case $\mathbf{A}$ is isomorphic to some $\mathbf{B}_i$ whenever $\mathbf{A}$ is a subdirect product of the (finitely many) algebras $\mathbf{B}_i$.

If $\mathbf{A}$ is a subdirect product of the algebras $\mathbf{B}_i$ for $i \in I$, then there are surjective homomorphisms $h_i : \mathbf{A} \to \mathbf{B}_i$ for each $i \in I$, i.e. the algebras $\mathbf{B}_i$ are homomorphic images of $\mathbf{A}$. The family of algebras $\mathbf{B}_i$ thus corresponds to a family congruences θ_i on $\mathbf{A}$ such that $\Delta_{\mathbf{A}} = \bigcap_{i \in I} \theta_i$. Subdirectly irreducible algebras are then precisely those where the identity congruence cannot be represented non-trivial as a meet of congruences. In other words, an algebra is subdirectly irreducible if and only if it has a smallest non-identity congruence, which is then called the *monolith* of $\mathbf{A}$. Likewise, an algebra $\mathbf{A}$ is finitely subdirectly irreducible if and only if $\phi \cap \psi = \Delta_{\mathbf{A}}$ in $\operatorname{Con} \mathbf{A}$ implies that either $\phi = \Delta_{\mathbf{A}}$ or $\psi = \Delta_{\mathbf{A}}$.

The subalgebra of $\mathbf{A}$ generated by $X \subseteq \mathbf{A}$ is the smallest subalgebra of $\mathbf{A}$ which contains X. An algebra is *finitely generated* if it is generated by some finite set. An algebra is *locally finite* if its finitely generated subalgebras are finite. Each algebra embeds into an ultraproduct of its finitely generated subalgebras, and in particular each locally finite algebra embeds into an ultraproduct of its finite subalgebras.

Let us now review several important kinds of classes of algebras. Here we are implicitly assuming that we are talking about algebras in some fixed signature. We say that a class of algebras is *axiomatized* by a certain set of first-order formulas if it is precisely the class of all algebras which satisfy those formulas. If the formulas contain free variables, these are interpreted as if they were universally quantified.

We first recall several important class operators. Let K be a class of algebras in some given signature. Then:

- $\mathbb{H}(\mathsf{K})$ is the class of all homomorphic images of algebras in K,
- $\mathbb{S}(\mathsf{K})$ is the class of all algebras which embed into some $\mathbf{A} \in \mathsf{K}$,
- $\mathbb{P}(\mathsf{K})$ is the class of all products of algebras in K,
- $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ is the class of all ultraproducts of algebras in K.

A *variety* is a class of algebras axiomatized by a set of equations, i.e. formulas of the form $t \approx u$, where t and u are terms. Equivalently, it is a class of algebras closed under homomorphic images, subalgebras, and products. The smallest variety which contains K is the class $\mathbb{HSP}(\mathsf{K})$. Each algebra in a variety is a subdirect product of its subdirectly irreducible members.

A *quasivariety* is a class of algebras axiomatized by a set of quasi-equations, i.e. implications of the form

$$t_1 \approx u_1 \ \& \ \dots \ \& \ t_n \approx u_n \implies t \approx u.$$

Equivalently, it is a class of algebras closed under subalgebras, products, and ultraproducts. The smallest quasivariety which contains K is the class $\mathbb{SPP}_{\mathrm{U}}(\mathsf{K})$.

The notion of an antivariety, introduced and studied by Gorbunov and Kravchenko [30, 31], is less known. An *antivariety* is a class of algebras axiomatized by a set of negative universal classes, i.e. disjunctions of negated equalities

$$t_1 \not\approx u_1 \vee \cdots \vee t_n \not\approx u_n.$$

Equivalently, it is a class of algebras closed under homomorphic pre-images, subalgebras, and ultraproducts. Here, an algebra $\mathbf{A}$ is a *homomorphic pre-image* of $\mathbf{B}$ if there is a surjective homomorphism $h : \mathbf{A} \to \mathbf{B}$. The smallest antivariety which contains K is the class $\mathbb{H}^{-1}\mathbb{S}\mathbb{P}^*_{\mathrm{U}}(\mathsf{K})$, where $\mathbb{H}^{-1}(\mathsf{L})$ denotes the class of all homomorphic pre-images of algebras in L and $\mathbb{P}^*_{\mathrm{U}}(\mathsf{L})$ denotes the class of all ultraproducts of non-empty families of algebras in L. Each antivariety either contains all algebras in the signature or it is a quasivariety minus the trivial singleton algebra.

We will also need to recall some basic facts about distributive lattices. Firstly, let us recall that a *lattice*, from the point of view of universal algebra, is an algebra with two binary operations $x \wedge y$ and $x \vee y$ which satisfy the following equations:

$$\begin{array}{llll} x \wedge y \approx y \wedge x & x \wedge (y \wedge z) \approx (x \wedge y) \wedge z & x \wedge x \approx x & x \wedge (x \vee y) \approx x \\ x \vee y \approx y \vee x & x \vee (y \vee z) \approx (x \vee y) \vee z & x \vee x \approx x & x \vee (x \wedge y) \approx x \end{array}$$

It follows that $a \wedge b$ is the greatest lower bound and $a \vee b$ is the least upper bound of a and b in the order $x \leq y$ defined as

$$a \wedge b = a \iff a \leq b \iff a \vee b = b.$$

We will in fact only be concerned with distributive lattices in this thesis. Recall that a lattice is *distributive* if it moreover satisfies the following two equivalent equations:

$$\begin{aligned} x \wedge (y \vee z) &\approx (x \wedge y) \vee (x \wedge z), \\ x \vee (y \wedge z) &\approx (x \vee y) \wedge (x \vee z). \end{aligned}$$

A *bounded distributive lattice* is a distributive lattice equipped with two additional constants t and f which satisfy the following equations:

$$\begin{aligned} x \wedge \mathsf{t} &\approx x, \\ x \vee \mathsf{f} &\approx x. \end{aligned}$$

In other words, t and f are the upper and lower bounds, i.e. the largest and smallest elements, of the lattice. Note that because the constants t and f are part of the signature of bounded distributive lattices, homomorphisms of bounded distributive lattices are required to preserve these constants, and subalgebras of bounded lattices must contain t and f.

A *(lattice) filter* on a lattice $\mathbf{L}$ is a subset $F \subseteq \mathbf{L}$ such that

$$x \in F \;\&\; x \leq y \implies y \in F,$$
$$x \in F \;\&\; y \in F \implies x \wedge y \in F.$$

A filter F on $\mathbf{L}$ is *trivial* if $F = \mathbf{L}$. It is *prime* if it is non-trivial and

$$x \vee y \in F \implies x \in F \text{ or } y \in F.$$

Prime filters on bounded distributive lattices correspond precisely to homomorphisms into the two-element bounded distributive lattice $\mathbf{L_2}$. For each homomorphism of bounded distributive lattices $h : \mathbf{L} \to \mathbf{L_2}$, the set $h^{-1}\{\mathsf{t}\}$ is a prime filter. Conversely, each prime filter F on a bounded distributive lattice defines a homomorphism of bounded distributive lattices $h : \mathbf{L} \to \mathbf{L_2}$ such that $h(a) = \mathsf{t} \iff a \in F$.

Elements of distributive lattices can be separated by prime filters.

Lemma 1.1 (Prime Filter Separation Lemma).
Let $\mathbf{L}$ be a distributive lattice. If $a \nleq b$ in $\mathbf{L}$, then there is a prime filter F on $\mathbf{L}$ such that $a \in F$ and $b \notin F$.

In fact, a somewhat stronger separation principle can be formulated.

Lemma 1.2 (Filter–Ideal Separation Lemma).
Let $\mathbf{L}$ be a distributive lattice. If F is a filter and I is an ideal on $\mathbf{L}$ with $F \cap I = \emptyset$, then there is a prime filter G and a prime ideal J on $\mathbf{L}$ with $G \cap J = \emptyset$ such that $F \subseteq G$ and $I \subseteq J$.

The first of these lemmas implies, thanks to the correspondence between prime filters and homomorphisms into $\mathbf{L_2}$, that $\mathbf{L_2}$ is the only subdirectly irreducible bounded distributive lattice. Each bounded distributive lattice is therefore a subdirect power of $\mathbf{L_2}$. In particular, each quasi-equation which holds in $\mathbf{L_2}$ must hold in every (bounded) distributive lattice. This yields the following lemma.

Lemma 1.3 (Equality Lemma for DLs).
In a distributive lattice, if $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$, then $x = y$.

1.2 Abstract algebraic logic

The framework of abstract algebraic logic will be used throughout the thesis, it is therefore crucial for the reader to have at least a passing acquaintance with it. Although we attempt to define all the necessary notions below, the reader unfamiliar with the abstract algebraic approach to logic is encouraged to consult the textbook of Font [24] for a thorough introduction to the field. Many of the results mentioned in this section are also proved in the monograph of Czelakowski [14].

Abstract algebraic logic studies logics primarily as single-conclusion consequence relations. Consequence in a logic obtains between a (possibly infinite) set of formulas and a single formula. Specifying a logic $\mathcal{L}$ therefore amounts to specifying a set of formulas and a consequence relation on this set. The formulas of a logic $\mathcal{L}$ are obtained by combining the propositional variables of the logic or the constants of the logic by means of the connectives of the logic. The set of variables of $\mathcal{L}$ will be denoted $\mathrm{Var}\,\mathcal{L}$ and assumed to be infinite. The set of all formulas will be denoted $\mathrm{Fm}\,\mathcal{L}$. Since the logic $\mathcal{L}$ will usually be clear from the context, we shall often write simply Fm.

The formulas of a logic in fact form an algebra (in the obvious way), denoted **Fm**. This algebra is sometimes called the *absolutely free algebra* generated in the given signature by $\mathrm{Var}\,\mathcal{L}$. Endomorphisms of this algebra, i.e. homomorphisms $\sigma : \mathbf{Fm} \to \mathbf{Fm}$, are called *substitutions*. The identity substitution will be denoted σ_{id}, i.e. $\sigma_{\mathrm{id}}(\varphi) = \varphi$.

Having specified the formulas of a logic, it remains to specify in which cases consequence obtains between a set of premises Γ and a conclusion φ. That is, to specify which rules $\Gamma \vdash \varphi$ are valid in the logic. In a logic defined semantically, the notion of validity in a matrix does the job. In a logic defined syntactically by means of a set of rules, it is instead the notion of provability which defines the consequence relation.

A *rule* is simply a pair $\Gamma \vdash \varphi$ consisting of a set of formulas Γ (called the *premises* or the *antecedent* of the rule) and a formula φ (called the *conclusion* of the rule). If $\mathcal{L}$ is a set of rules, we use $\Gamma \vdash_{\mathcal{L}} \varphi$ to denote that the rule $\Gamma \vdash \varphi$ belongs to (is *valid* in, *holds* in) $\mathcal{L}$. We use $\Gamma \vdash_{\mathcal{L}} \Phi$, where Φ is a non-empty set of formulas, as an abbreviation for the claim that $\Gamma \vdash_{\mathcal{L}} \varphi$ for each $\varphi \in \Phi$. (The notation $\Gamma \vdash_{\mathcal{L}} \emptyset$ will be used in a different sense later.) A rule is called an *axiomatic rule* or simply an *axiom* if it has the form $\emptyset \vdash \varphi$. A *(substitution) instance* of a rule $\Gamma \vdash \varphi$ is a rule of the form $\sigma[\Gamma] \vdash \sigma(\varphi)$ for some substitution σ.

A *matrix* is a pair $\langle \mathbf{A}, F \rangle$ consisting of an algebra $\mathbf{A}$ and a set $F \subseteq \mathbf{A}$ called the *filter* of the matrix. An element of $\mathbf{A}$ is called *designated* if it lies in the filter. A matrix $\langle \mathbf{A}, F \rangle$ is called *trivial* if $F = \mathbf{A}$. If a more compact notation than $\langle \mathbf{A}, F \rangle$ is called for, we shall use $\mathbb{M}$, $\mathbb{N}$, etc. to denote matrices.

A *valuation* on an algebra $\mathbf{A}$, and by extension on a matrix $\mathbb{M} = \langle \mathbf{A}, F \rangle$, is a homomorphism $h : \mathbf{Fm} \to \mathbf{A}$. It will also occasionally be convenient to use the notation $h : \mathbf{Fm} \to \mathbb{M}$. We say that a valuation on $\langle \mathbf{A}, F \rangle$ *designates* a formula φ if $v(\varphi) \in F$. It designates a set of formulas Γ if it designates each $\gamma \in \Gamma$. A rule $\Gamma \vdash \varphi$ *holds* (is *valid*) in a matrix $\langle \mathbf{A}, F \rangle$ if each valuation on $\langle \mathbf{A}, F \rangle$ which designates Γ also designates φ. That is, if

$$v[\Gamma] \subseteq F \implies v(\varphi) \in F \text{ for each } v : \mathbf{Fm} \to \mathbf{A}.$$

In that case the matrix $\langle \mathbf{A}, F \rangle$ is said to be a *model* of the rule $\Gamma \vdash \varphi$. It is a model of a set of rules $\mathcal{L}$ if it is a model of each rule in $\mathcal{L}$. A set $F \subseteq \mathbf{A}$ is called a filter of $\mathcal{L}$ or an $\mathcal{L}$*-filter* if $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{L}$.

The notion of validity in a matrix yields a Galois connection between sets of rules and classes of matrices. To each set of rules $\mathcal{L}$ we can assign the class of all models of $\mathcal{L}$, denoted $\operatorname{Mod}\mathcal{L}$. Conversely, to each class of matrices K we can assign the set of all rules which hold in all matrices of K, called the *logic* of K and denoted $\operatorname{Log}\mathsf{K}$. The logic of a single matrix $\mathbb{M}$ will be denoted $\operatorname{Log}\mathbb{M}$. A logic *simplicter* will be the logic of some class of matrices. Equivalently, a set of rules $\mathcal{L}$ is a *logic* if $\mathcal{L} = \operatorname{Log}\operatorname{Mod}\mathcal{L}$. We then say that a logic $\mathcal{L}$ is *complete* with respect to a class of matrices K if $\mathcal{L} = \operatorname{Log}\mathsf{K}$. Clearly each logic $\mathcal{L}$ is complete with respect to $\operatorname{Mod}\mathcal{L}$. A set of formulas closed under consequence in $\mathcal{L}$ will be called a *theory* of $\mathcal{L}$.

Logics can be characterized intrinsically as precisely those sets of rules $\mathcal{L}$ which satisfy the following conditions:

$$\varphi \vdash_{\mathcal{L}} \varphi \qquad \text{(reflexivity)}$$
$$\Gamma \vdash_{\mathcal{L}} \varphi \implies \Gamma, \Delta \vdash_{\mathcal{L}} \varphi \qquad \text{(monotonicity)}$$
$$\Gamma \vdash_{\mathcal{L}} \Phi \text{ and } \Phi, \Delta \vdash_{\mathcal{L}} \psi \implies \Gamma, \Delta \vdash_{\mathcal{L}} \psi \qquad \text{(cut)}$$
$$\Gamma \vdash_{\mathcal{L}} \varphi \implies \sigma[\Gamma] \vdash_{\mathcal{L}} \sigma(\varphi) \text{ for each substitution } \sigma \qquad \text{(structurality)}$$

This is in fact often taken to be the definition of a logic.

The largest logic in a given signature is the *trivial logic*, which validates every rule. This is the logic determined by any trivial matrix. The smallest logic is the *identity logic*, which validates $\Gamma \vdash \varphi$ only if $\varphi \in \Gamma$.

The notion of a finitary logic may be obtained by replacing arbitrary rules in the definition of a logic by *finitary rules*, i.e. rules $\Gamma \vdash \varphi$ with Γ finite. We may identify *finitary* logics with logics such that $\Gamma \vdash_{\mathcal{L}} \varphi$ implies that $\Gamma' \vdash_{\mathcal{L}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$. The finitary logic determined by a class of matrices K will be denoted $\operatorname{Log}_{\omega}\mathsf{K}$. That is, $\operatorname{Log}_{\omega}\mathsf{K}$ is finitary and a finitary rule $\Gamma \vdash \varphi$ holds in $\operatorname{Log}_{\omega}\mathsf{K}$ if and only if it holds in each matrix of K. We say that a finitary logic is *ω-complete* or *complete as a finitary logic* with respect to a class of matrices K if $\mathcal{L} = \operatorname{Log}_{\omega}\mathsf{K}$. It is important to note that each logic generated by a finite set of finite matrices is finitary.

Logics are ordered by inclusion:

$$\mathcal{L}_1 \leq \mathcal{L}_2 \iff \Gamma \vdash_{\mathcal{L}_1} \varphi \text{ implies } \Gamma \vdash_{\mathcal{L}_2} \varphi \text{ for all } \Gamma, \varphi.$$

In this case we say that $\mathcal{L}_2$ is an *extension* of $\mathcal{L}_1$. The extensions of $\mathcal{L}$ form a complete lattice under this ordering, denoted $\operatorname{Ext}\mathcal{L}$. The finitary extension of a finitary logic $\mathcal{L}$ form an algebraic lattice $\operatorname{Ext}_{\omega}\mathcal{L}$. Observe that $\operatorname{Ext}_{\omega}\mathcal{L}$ is a sublattice of $\operatorname{Ext}\mathcal{L}$, but not necessarily a complete sublattice.

Meets in the lattice $\operatorname{Ext}\mathcal{L}$ are intersections and will be denoted as such:

$$\Gamma \vdash_{\bigcap_{i \in I} \mathcal{L}_i} \varphi \iff \Gamma \vdash_{\mathcal{L}_i} \varphi \text{ for each } i \in I.$$

If $\mathcal{L}_1 = \operatorname{Log}\mathsf{K}$ and $\mathcal{L}_2 = \operatorname{Log}\mathsf{L}$, then clearly $\mathcal{L}_1 \cap \mathcal{L}_2 = \operatorname{Log}(\mathsf{K} \cup \mathsf{L})$.

To compute joins of logics in $\mathrm{Ext}\,\mathcal{L}$, it will be useful to see consequence in a logic as provability in a corresponding Hilbert-style system. Let us make this correspondence explicit. A *proof* of a formula φ from the premises Γ using a set of rules R is a well-founded tree (i.e. a tree without infinite branches) whose nodes are labelled by formulas such that:

(i) each terminal node is labelled by a formula from Γ or by a substitution instance of the conclusion of an axiomatic rule in R,

(ii) each non-determinal node is labelled by the conclusion and its parents by the premises of a substitution instance of a rule in R,

(iii) the root of the tree is labelled by φ.

A formula φ is said to be *provable* from Γ using a set of rules R if it has a proof from Γ using R. We say that a logic $\mathcal{L}$ is *axiomatized* by a set of rules R if consequence in $\mathcal{L}$ coincides with provability using R.

Observe that the logic axiomatized by R is the smallest logic which validates all rules of R. Each logic $\mathcal{L}$ is trivially axiomatized by the set of all rules which hold in $\mathcal{L}$. Moreover, if R_1 axiomatizes $\mathcal{L}_1$ and R_2 axiomatizes $\mathcal{L}_2$, then the join of these logics $\mathcal{L}_1 \vee \mathcal{L}_2$ is axiomatized by $R_1 \cup R_2$.

A logic $\mathcal{L}$ called *finitely axiomatizable* (relative to some logic $\mathcal{B} \leq \mathcal{L}$) if it is axiomatized by a finite set of rules (plus the rules valid in $\mathcal{B}$). For each finitary logic $\mathcal{B}$ the finitely axiomatizable finitary extensions of $\mathcal{B}$ are precisely the compact elements of $\mathrm{Ext}_\omega\,\mathcal{B}$. Therefore to prove that a logic $\mathcal{L}$ is not finitely axiomatizable, it suffices to show that $\mathcal{L} = \bigvee_{n\in\omega} \mathcal{L}_n$ for some strictly increasing chain of logics $\mathcal{L}_1 < \mathcal{L}_2 < \cdots < \mathcal{L}$.

We now introduce some important constructions on matrices. A matrix $\langle \mathbf{B}, G\rangle$ is a *submatrix* of $\langle \mathbf{A}, F\rangle$, symbolically $\langle \mathbf{B}, G\rangle \leq \langle \mathbf{A}, F\rangle$, if $\mathbf{B} \leq \mathbf{A}$ and $G = F \cap \mathbf{B}$. A *product* of matrices $\langle \mathbf{A}_i, F_i\rangle$ is the matrix $\langle \Pi_{i\in I}\mathbf{A}_i, \Pi_{i\in I}F_i\rangle$. A *homomorphism* of matrices $h : \langle \mathbf{A}, F\rangle \to \langle \mathbf{B}, G\rangle$ is a homomorphism of algebras $h : \mathbf{A} \to \mathbf{B}$ such that $a \in F \implies h(a) \in G$. It is a *strict* homomorphism if in fact $a \in F \iff h(a) \in G$, i.e. if $F = h^{-1}[G]$. An *embedding* of matrices is a strict embedding of algebras. If $h : \mathbb{M} \to \mathbb{N}$ is a (strict) surjective homomorphism, we say that $\mathbb{N}$ a *(strict) homomorphic image* of $\mathbb{M}$ and $\mathbb{M}$ is a *(strict) homomorphic preimage* of $\mathbb{N}$. Ultraproducts of matrices are defined as ultraproducts of structures.

The class operators for algebras will also apply in an analogous sense to matrices. If K be a class of matrices in some given signature, then:

- $\mathbb{H}_\mathrm{S}(\mathsf{K})$ is the class of all strict homomorphic images of matrices in K,
- $\mathbb{S}(\mathsf{K})$ is the class of all matrices which embed into some $\mathbf{A} \in \mathsf{K}$,
- $\mathbb{P}(\mathsf{K})$ is the class of all products of matrices in K,
- $\mathbb{P}_\mathrm{U}(\mathsf{K})$ is the class of all ultraproducts of matrices in K.

Kernels of a strict homomorphisms from $\langle \mathbf{A}, F \rangle$ may be identified with congruences θ on $\mathbf{A}$ which are *compatible* with F in the sense that

$$a \in F \text{ and } \langle a, b \rangle \in \theta \implies b \in F.$$

Such congruences on $\mathbf{A}$ will be called *strict congruences* on $\langle \mathbf{A}, F \rangle$.

Strict surjective homomorphisms form a particularly important class of homomorphisms because they do not change the logic determined by the matrix. That is, $\operatorname{Log} \mathbb{M} = \operatorname{Log} \mathbb{N}$ if $\mathbb{N}$ is a strict homomorphic image of $\mathbb{M}$.

Like algebras, matrices also admit subdirect decompositions. A matrix $\mathbb{M}$ is called a *subdirect product* of matrices $\mathbb{M}_i$ for $i \in I$ if there is a matrix embedding $\mathbb{M} \leq \Pi_{i \in I} \mathbb{M}_i$ which is subdirect as an embedding algebras. A model $\mathbb{M}$ of $\mathcal{L}$ is *subdirectly irreducible* (relative to $\mathcal{L}$) if for each family of models $\mathbb{M}_i$ of $\mathcal{L}$ for $i \in I$ such that $\mathbb{M}$ is a subdirect product of the family $\mathbb{M}_i$, there is some $i \in I$ such that $\mathbb{M}$ is isomorphic to $\mathbb{M}_i$. Equivalently, a model $\langle \mathbf{A}, F \rangle$ of $\mathcal{L}$ is subdirectly irreducible (relative to $\mathcal{L}$) if there is a smallest $\mathcal{L}$-filter on $\mathbf{A}$ which properly extends F. Each (reduced) model of a finitary logic is known to be a subdirect product of some family of subdirectly irreducible (reduced) models of $\mathcal{L}$.

We can also describe subdirect decompositions of matrices in intrinsic terms. Recall that a subdirect decomposition of an algebra $\mathbf{A}$ corresponds to a family of congruences θ_i for $i \in I$ with $\Delta_{\mathbf{A}} = \bigcap_{i \in I} \theta_i$. A subdirect decomposition of a matrix $\langle \mathbf{A}, F \rangle$ then corresponds to such a family of congruence equipped with a family of $\mathcal{L}$-filters F_i for $i \in I$ such that θ_i is compatible with F_i. The filter F is then recovered as $F = \bigcap_{i \in I} \pi_i^{-1}[F_i]$, where $\pi_i : \mathbf{A} \to \mathbf{A}/\theta_i$ are the projection maps.

There is always a largest strict congruence on each matrix $\langle \mathbf{A}, F \rangle$, called the *Leibniz congruence* of F and denoted $\boldsymbol{\Omega}^{\mathbf{A}} F$. This congruence may be explicitly described as follows: $\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$ if and only if

$$t(a, c_1, \ldots, c_n) \in F \iff t(b, c_1, \ldots, c_n) \in F \text{ for each term } t(x, y_1, \ldots, y_n)$$

and all $c_1, \ldots, c_n \in \mathbf{A}$. Clearly a homomorphism $h : \langle \mathbf{A}, F \rangle \to \langle \mathbf{B}, G \rangle$ is strict if and only if $\operatorname{Ker} h \subseteq \boldsymbol{\Omega}^{\mathbf{A}} F$.

A matrix $\langle \mathbf{A}, F \rangle$ is called *(Leibniz) reduced* if $\boldsymbol{\Omega}^{\mathbf{A}} F = \Delta_{\mathbf{A}}$. Factoring a matrix $\langle \mathbf{A}, F \rangle$ by the congruence $\boldsymbol{\Omega}^{\mathbf{A}} F$ yield the reduced matrix $\langle \mathbf{A}, F \rangle^* = \langle \mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}} F, F/\boldsymbol{\Omega}^{\mathbf{A}} F \rangle$, called the *(Leibniz) reduct* of $\langle \mathbf{A}, F \rangle$. The class of all reduced models of $\mathcal{L}$ will be denoted $\operatorname{Mod}^* \mathcal{L}$. Clearly $\mathcal{L} = \operatorname{Log} \operatorname{Mod}^* \mathcal{L}$. We also introduce a corresponding class operator:

- $\mathbb{R}(\mathsf{K})$ is the class of all Leibniz reducts of matrices in K.

In some cases, the quantification over all terms in the above description of the Leibniz congruence can be replaced by quantification over some finite set of terms. If this holds for all models of $\mathcal{L}$, we say that $\mathcal{L}$ has *finitizable Leibniz congruences*. The condition $\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$ can then be expressed by a first-order formula (in a language with a single unary predicate $\mathrm{True}(x)$ but without equality) if $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{L}$.

The class of all algebraic reducts of matrices in $\mathrm{Mod}^* \mathcal{L}$ will be denoted $\mathrm{Alg}^* \mathcal{L}$. That is, $\mathbf{A} \in \mathrm{Alg}^* \mathcal{L}$ if and only if there is an $\mathcal{L}$-filter F on $\mathbf{A}$ such that $\Delta_{\mathbf{A}} = \boldsymbol{\Omega}^{\mathbf{A}} F$. It is, however, often more natural to consider a wider class of algebras. The *algebraic counterpart* of $\mathcal{L}$, denoted $\mathrm{Alg}\,\mathcal{L}$, is defined as the class of all algebras $\mathbf{A}$ such that $\Delta_{\mathbf{A}} = \bigcap \{ \boldsymbol{\Omega}^{\mathbf{A}} F \mid F \text{ is an } \mathcal{L}\text{-filter on } \mathbf{A} \}$. It turns out that $\mathrm{Alg}\,\mathcal{L}$ is precisely the class of all subdirect products of algebras in $\mathrm{Alg}^* \mathcal{L}$. In the study of a logic $\mathcal{L}$ it often suffices to restrict to matrices over algebras in $\mathrm{Alg}\,\mathcal{L}$.

Finally, we review the relevant parts of the Leibniz and Frege hierarchies of abstract algebraic logic and the notion of the strong version of a logic. An understanding of these topics will not essential for most of the thesis, therefore the reader unfamiliar with them may choose to skip the rest of the section, provided that he is also willing to skip most of Chapter 8 (Metalogical properties of super-Belnap logics).

The *Leibniz hierarchy* tries to classify logics according to the behaviour of the Leibniz operator on their filters. The most important classes of the Leibniz hierarchy are the classes of protoalgebraic, equivalential, truth-equational, assertional, and algebraizable logics. The importance of these classes is witnessed by the fact that each of them admits several natural definitions. We only present some of these definitions and refer the interested reader to the literature for a proper introduction to each of these classes.

A logic $\mathcal{L}$ is called *protoalgebraic* if the Leibniz operator is monotone on $\mathcal{L}$-filters. That is, for all $\mathcal{L}$-filters F and G on each algebra $\mathbf{A}$

$$F \subseteq G \implies \boldsymbol{\Omega}^{\mathbf{A}} F \subseteq \boldsymbol{\Omega}^{\mathbf{A}} G.$$

Equivalently, $\mathcal{L}$ is protoalgebraic if there is a set of formulas in two variables $\Delta(p, q)$, called a *protoimplication set*, such that

$$\emptyset \vdash \Delta(p, p) \qquad \text{and} \qquad p, \Delta(p, q) \vdash q.$$

Protoalgebraic logics are thus logics equipped with an implication which some fairly minimal principles.

A logic $\mathcal{L}$ is called *equivalential* if it is protoalgebraic and the Leibniz operator on $\mathcal{L}$-filters commutes with homomorphisms. That is, for each $\mathcal{L}$-filter F on each algebra $\mathbf{B}$

$$\boldsymbol{\Omega}^{\mathbf{A}} h^{-1}[F] = h^{-1} \boldsymbol{\Omega}^{\mathbf{B}} F \text{ for each homomorphism } h : \mathbf{A} \to \mathbf{B}.$$

Equivalently, $\mathcal{L}$ is equivalential if it has a protoimplication set $\Delta(p, q)$ which moreover for each n-ary function symbol f of $\mathcal{L}$ satisfies

$$\bigcup_{i=1}^{n} \Delta(p_i, q_i) \vdash_{\mathcal{L}} \Delta(f(p_1, \ldots, p_n), f(q_1, \ldots, q_n)).$$

A logic $\mathcal{L}$ is called *truth-equational* if the Leibniz operator on $\mathcal{L}$-filters is completely order reflecting. That is, for each family of $\mathcal{L}$-filters F_i for $i \in I$ on $\mathbf{A}$ and each $\mathcal{L}$-filter G on $\mathbf{A}$ we have

$$\bigcap_{i\in I} \mathbf{\Omega}^{\mathbf{A}} F_i \subseteq \mathbf{\Omega}^{\mathbf{A}} G \implies \bigcap_{i\in I} F_i \subseteq G.$$

Equivalently, $\mathcal{L}$ is truth-equational if there is a set of equations $E(x)$ in one variable such that for each reduced model $\langle \mathbf{A}, F\rangle$ of $\mathcal{L}$

$$F = \{a \in \mathbf{A} \mid \mathbf{A} \vDash E(a)\}.$$

A logic $\mathcal{L}$ is *assertional* if it is the logic determined by a class of matrices of the form $\langle \mathbf{A}, \{\mathsf{t}\}\rangle$, where t is a constant in the signature of $\mathcal{L}$ (more precisely, a term which may contain variables but whose value in each of these matrices is independent of the values of these variables). Assertional logics are known to form a subclass of truth-equational logics.

An *algebraizable* logic is a logic which is both equivalential and truth-equational. Algebraizable logics are exactly those logics $\mathcal{L}$ whose consequence relation is equivalent to the equational consequence relation of some class of algebras K in the following sense. There is a translation $\boldsymbol{\tau}$ sending formulas to sets of equations and a translation $\boldsymbol{\rho}$ sending equations to sets of formulas, both commuting with substitutions, such that

$$\begin{aligned} \Gamma \vdash_{\mathcal{L}} \varphi &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi), \\ E \vDash_{\mathsf{K}} t \approx u &\iff \boldsymbol{\rho}[E] \vdash_{\mathcal{L}} \boldsymbol{\rho}(\varphi), \end{aligned}$$

$$\begin{aligned} \varphi \vdash_{\mathcal{L}} \boldsymbol{\rho}\boldsymbol{\tau}(\varphi) \vdash_{\mathcal{L}} \varphi, \\ t \approx u \vDash_{\mathsf{K}} \boldsymbol{\tau}\boldsymbol{\rho}(t \approx u) \vDash_{\mathsf{K}} t \approx u. \end{aligned}$$

For each algebraizable logic, there is always a largest class K satisfying these conditions, called the *equivalent algebraic semantics* of $\mathcal{L}$. If the logic $\mathcal{L}$ as well as both of the translations are finitary in a natural sense, then its equivalent algebraic semantics is a quasivariety. If K is a variety, then the axiomatic extensions of $\mathcal{L}$ correspond precisely to subvarieties of K.

Note that a logic is algebraizable if and only if it it has a protoimplication set which in addition to the condition required by equivalentiality satisfies

$$p \dashv\vdash_{\mathcal{L}} \boldsymbol{\rho}\boldsymbol{\tau}(p)$$

for some pair of translations $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$.

Each of the above classes is in fact closed under extensions of logics. That is, if $\mathcal{L}_1 \leq \mathcal{L}_2$ and $\mathcal{L}_1$ belongs to the class, then so does $\mathcal{L}_2$

In contrast to the Leibniz hierarchy, the *Frege hierarchy* deals with the behaviour of the interderivability relation. We will only introduce two classes of this hierarchy. A logic $\mathcal{L}$ is *selfextensional* if substituting $\mathcal{L}$-interderivable formulas in any context yields $\mathcal{L}$-interderivable formulas. That is, if

$$\varphi \dashv\vdash_{\mathcal{L}} \psi \implies \chi(\varphi) \dashv\vdash_{\mathcal{L}} \chi(\psi)$$

for each formula $\chi(p)$, where p need not be the only variable of χ. A logic $\mathcal{L}$ is *Fregean* if this implication holds for interderivability modulo each $\mathcal{L}$-theory Γ. That is, if for each formula $\chi(p)$ and each $\mathcal{L}$-theory Γ

$$\Gamma, \varphi \dashv\vdash_{\mathcal{L}} \Gamma, \psi \implies \Gamma, \chi(\varphi) \dashv\vdash_{\mathcal{L}} \Gamma, \chi(\psi).$$

Finally, we introduce strong versions of logics. The *strong version* $\mathcal{L}^+$ of a logic $\mathcal{L}$ is defined as the logic $\mathcal{L}^+$ determined by the class of matrices

$$\{\langle \mathbf{A}, F \rangle \mid F \text{ is the smallest } \mathcal{L}\text{-filter on } \mathbf{A}\}.$$

Equivalently, strong versions can be defined in terms of Leibniz filters. An $\mathcal{L}$-filter F on $\mathbf{A}$ is called a *Leibniz filter* of $\mathcal{L}$ if for each $\mathcal{L}$-filter G on $\mathbf{A}$

$$\boldsymbol{\Omega}^{\mathbf{A}} F \subseteq \boldsymbol{\Omega}^{\mathbf{A}} G \implies F \subseteq G.$$

Importantly, $\mathcal{L}^+ = \mathcal{L}$ if $\mathcal{L}$ is a truth-equational logic.

Leibniz filters can also be introduced via Leibniz classes. The *Leibniz class* of an $\mathcal{L}$-filter F on $\mathbf{A}$ is defined as the set of all $\mathcal{L}$-filters G on $\mathbf{A}$ such that $\boldsymbol{\Omega}^{\mathbf{A}} F \subseteq \boldsymbol{\Omega}^{\mathbf{A}} G$. The *Leibniz filter* of F is defined as the smallest filter $F^*_{\mathcal{L}}$ in the Leibniz class of F. Leibniz filters *simpliciter* are then precisely the filters of the form $F^*_{\mathcal{L}}$.

Chapter 2

De Morgan algebras

We now turn from the general algebraic and logical preliminaries to the particular class of algebras which will be prominent throughout this thesis as the algebraic counterpart of the Belnap–Dunn logic, namely the class of De Morgan algebras. Basic acquaintance with this class of algebras will be essential in the algebraic study of super-Belnap logics.

The purpose of this chapter is to introduce our notation for De Morgan algebras and to collect some basic results on these algebras which will be used throughout the thesis. We shall see that De Morgan algebras share many of the properties of bounded distributive lattices, such as local finiteness, equationally definable principal congruences, and a correspondence between lattice filters and homomorphisms into subdirectly irreducible algebras.

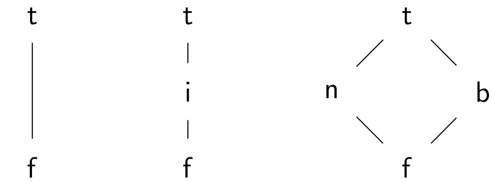
None of the results presented in this chapter are original or, indeed, particularly new. We merely introduce the reader to the classic results of Kalman [38], Sankappanavar [68], and Pynko [62] on De Morgan algebras. However, we choose to provide proofs of these results in the interest of providing a self-contained introduction to De Morgan algebras. We also aim to present these results in a way which will allow us to easily exploit them in our subsequent investigation of the extensions of the Belnap–Dunn logic.

2.1 The algebra $\mathbf{DM_4}$ and its subalgebras

The most important algebra in the study of the Belnap–Dunn logic is the four-element algebra $\mathbf{DM_4}$ (for "De Morgan"). The elements of this algebra are the four truth values of the Belnap–Dunn logic. They will be denoted t for "True", f for "False", n for "Neither (True nor False)", and b for "Both (True and False)". The signature of the algebra $\mathbf{DM_4}$ then consists of

- the constants t and f,
- the lattice connectives $\wedge$ and $\vee$, and
- the De Morgan negation $-$.

Figure 2.1: Subdirectly irreducible De Morgan algebras



$\mathbf{B_2}$ $\mathbf{K_3}$ $\mathbf{DM_4}$

Table 2.1: The primitive operations of $\mathbf{DM_4}$

$\wedge$	f	n	t	b
f	f	f	f	f
n	f	n	n	f
t	f	n	t	b
b	f	f	b	b

$\vee$	f	n	t	b
f	f	n	t	b
n	n	n	t	t
t	t	t	t	t
b	b	t	t	b

	$-$
f	t
n	n
t	f
b	b

The algebraic interpretation of the constants t and f is clear: they denote the elements of the same name. The lattice connectives $\wedge$ and $\vee$ are interpreted as meet and join in the lattice order on $\mathbf{DM_4}$ shown in Figure 2.1. Finally, the De Morgan negation behaves classically on the classical truth values, i.e. $-\mathsf{t} = \mathsf{f}$ and $-\mathsf{f} = \mathsf{t}$, while fixing the non-classical truth values, i.e. $-\mathsf{n} = \mathsf{n}$ and $-\mathsf{b} = \mathsf{b}$. $\mathbf{DM_4}$ thus differs from the four-element Boolean algebra only in the behaviour of De Morgan negation on the non-classical truth values. The (non-constant) operations of $\mathbf{DM_4}$ are also shown in Table 2.1.

The logical interpretation of the elements of $\mathbf{DM_4}$ and its primitive operations will be postponed until the following chapter.

The algebra $\mathbf{DM_4}$ has exactly three proper subalgebras, namely $\mathbf{B_2}$ (for "Boolean"), $\mathbf{K_3}$ (for "Kleene"), and $\mathbf{P_3}$ (for "Priest"). Their universes are

$$\mathbf{B_2} := \{\mathsf{f}, \mathsf{t}\}, \qquad \mathbf{K_3} := \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}, \qquad \mathbf{P_3} := \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}.$$

The algebras $\mathbf{K_3}$ and $\mathbf{P_3}$ are isomorphic. Indeed, there is up to isomorphism only one three-element De Morgan algebra (algebra in the variety generated by $\mathbf{DM_4}$). For the sake of simplicity, we use $\mathbf{K_3}$ to denote this algebra. Despite this notational convention, the three-element De Morgan algebra *per se* and the subalgebra $\mathbf{K_3}$ of $\mathbf{DM_4}$ should be thought of as slightly different objects. In situations where this algebra is not embedded into $\mathbf{DM_4}$, as in Figure 2.1, we shall therefore use i (for "Intermediate" or "Indeterminate") to denote the middle element of the algebra.

Note that the singleton subalgebras $\{\mathsf{n}\}$ and $\{\mathsf{b}\}$ are forbidden by the inclusion of the constants t and f in the signature of $\mathbf{DM_4}$. These constants are often excluded from the signature of the Belnap–Dunn logic by other researchers. For the most part, this choice will have few implications for our work. The few differences that result from including the truth constants in the signature will be discussed in Chapter 10 (Other frameworks).

There are exactly two prime filters on $\mathbf{DM_4}$, namely $\{\mathsf{t}, \mathsf{b}\}$ and $\{\mathsf{t}, \mathsf{n}\}$. Proving that $a \leq b$ holds in $\mathbf{DM_4}$ thus amounts precisely to proving that

$$a \in \{\mathsf{t}, \mathsf{b}\} \implies b \in \{\mathsf{t}, \mathsf{b}\} \qquad \text{and} \qquad a \in \{\mathsf{t}, \mathsf{n}\} \implies b \in \{\mathsf{t}, \mathsf{n}\}.$$

In addition to the lattice order shown in Figure 2.1, which will be referred to as the *truth order*, the truth values of $\mathbf{DM_4}$ also come equipped with a natural *information order*, denoted by $\sqsubseteq$. This is the order obtained by reading Figure 2.1 from left to right rather than from bottom to top. In other words, the smallest element in this order is n, the largest element is b, and the elements t and f are incomparable.

It is important to notice that although the subalgebras $\mathbf{K_3}$ and $\mathbf{P_3}$ of $\mathbf{DM_4}$ are isomorphic, the restriction of the information order on $\mathbf{DM_4}$ to these subagebras is different: the middle value of $\mathbf{K_3}$ is the minimum of the information order on $\mathbf{K_3}$, while the middle value of $\mathbf{P_3}$ is the maximum of the information order on $\mathbf{P_3}$.

The information order is again a distributive lattice with exactly two prime filters, namely $\{\mathsf{t}, \mathsf{b}\}$ and $\{\mathsf{f}, \mathsf{b}\}$. It will be useful for the reader to keep in mind that proving that $a = b$ holds in $\mathbf{DM_4}$ amounts to proving that $a \sqsubseteq b$ and $b \sqsubseteq a$, and proving that $a \sqsubseteq b$ amounts to proving that

$$a \in \{\mathsf{t}, \mathsf{b}\} \implies b \in \{\mathsf{t}, \mathsf{b}\} \qquad \text{and} \qquad a \in \{\mathsf{f}, \mathsf{b}\} \implies b \in \{\mathsf{f}, \mathsf{b}\}.$$

It will be equally important to observe that the primitive operations of $\mathbf{DM_4}$ are monotone with respect to the information order. This in particular means that the pointwise ordering of valuations on $\mathbf{DM_4}$ reduces to the pointwise ordering restricted to propositional atoms. The following fact will be crucial in many of our proofs.

Fact 2.1. *Let $v, w : \mathbf{Fm} \to \mathbf{DM_4}$. Then $v(\varphi) \sqsubseteq w(\varphi)$ for all formulas φ if and only if $v(p) \sqsubseteq w(p)$ for all atoms p. In that case we write $v \sqsubseteq w$.*

Finaly, observe that the algebra $\mathbf{DM_4}$ has exactly one non-trivial isomorphism, namely the map $\partial : \mathbf{DM_4} \to \mathbf{DM_4}$ such that

$$\partial\mathsf{t} = \mathsf{t} \qquad \partial\mathsf{n} = \mathsf{b}$$
$$\partial\mathsf{f} = \mathsf{f} \qquad \partial\mathsf{b} = \mathsf{n}$$

This map will be called *conflation*. It is monotone with respect to the truth order but antitone with respect to the information order (unlike the De Morgan negation, which is monotone with respect to the information order but antitone with respect to the truth order).

2.2 A primer on De Morgan algebras

We now turn our attention from the description of the algebra $\mathbf{DM_4}$ to the study of the variety generated by $\mathbf{DM_4}$.

Definition 2.2 (De Morgan algebras).
A *De Morgan lattice (De Morgan algebra)* is a (bounded) distributive lattice equipped with a unary operator $-x$ which satisfies the equations

$$x \approx --x \qquad \text{and} \qquad -(x \vee y) \approx -x \wedge -y,$$

or equivalently the equations

$$x \approx --x \qquad \text{and} \qquad -(x \wedge y) \approx -x \vee -y.$$

The operator $-x$ will be called *De Morgan negation*.

The distinction between De Morgan algebras and De Morgan lattices will be of little import throughout the thesis. We therefore generally choose to formulate our definitions in terms of De Morgan algebras. Unless we state otherwise, parallel definitions and results may be formulated for De Morgan lattices, although we generally will not care to do so explicitly.

Definition 2.3 (Boolean algebras and Kleene algebras).
A *Boolean algebra* is a De Morgan algebra which satisfies the inequality

$$x \wedge -x \leq y.$$

A *Kleene algebra* is a De Morgan algebra which satisfies the inequality

$$x \wedge -x \leq y \vee -y.$$

Boolean and Kleene lattices are defined similarly.

The variety of Boolean algebras will be denoted BA, the variety of Kleene algebras KA, and the variety of De Morgan algebras DMA.

We now work towards a proof that the algebras $\mathbf{B_2}$, $\mathbf{K_3}$, and $\mathbf{DM_4}$ are the only subdirectly irreducible De Morgan algebras, and that they generate, respectively, the variety of Boolean algebras, the variety of Kleene algebras, and the variety of De Morgan algebras. Unless we state otherwise, $\mathbf{A}$ will denote a De Morgan algebra throughout the rest of this section.

Although the full strength of the following lemma is not needed for this proof, it will be useful in the proof of the completeness theorem for the Belnap–Dunn logic in Section 3.3 (Completeness and normal forms).

Lemma 2.4 (Homomorphism Lemma for DMAs).
Let $\mathbf{B}$ *be an algebra in the signature of De Morgan algebras. Then a map* $f : \mathbf{B} \to \mathbf{DM_4}$ *is a homomorphism if and only if* $f(\mathsf{t}) = \mathsf{t}$ *and* $f(\mathsf{f}) = \mathsf{f}$ *and*

$$\begin{aligned}
f(a \wedge b) \in \{\mathsf{t}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{t}, \mathsf{b}\} \textit{ and } f(b) \in \{\mathsf{t}, \mathsf{b}\}, \\
f(a \wedge b) \in \{\mathsf{f}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{f}, \mathsf{b}\} \textit{ or } f(b) \in \{\mathsf{f}, \mathsf{b}\}, \\
f(a \vee b) \in \{\mathsf{t}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{t}, \mathsf{b}\} \textit{ or } f(b) \in \{\mathsf{t}, \mathsf{b}\}, \\
f(a \vee b) \in \{\mathsf{f}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{f}, \mathsf{b}\} \textit{ and } f(b) \in \{\mathsf{f}, \mathsf{b}\}, \\
f(-a) \in \{\mathsf{t}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{f}, \mathsf{b}\}, \\
f(-a) \in \{\mathsf{f}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{t}, \mathsf{b}\}.
\end{aligned}$$

If $\mathbf{B}$ *is a De Morgan algebra, then a map* $f : \mathbf{B} \to \mathbf{DM_4}$ *is a homomorphism if and only if* $f(\mathsf{t}) = \mathsf{t}$ *and* $f(\mathsf{f}) = \mathsf{f}$ *and*

$$\begin{aligned}
f(a \wedge b) \in \{\mathsf{t}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{t}, \mathsf{b}\} \textit{ and } f(b) \in \{\mathsf{t}, \mathsf{b}\}, \\
f(a \vee b) \in \{\mathsf{t}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{t}, \mathsf{b}\} \textit{ or } f(b) \in \{\mathsf{t}, \mathsf{b}\}, \\
f(-a) \in \{\mathsf{t}, \mathsf{b}\} &\iff f(a) \in \{\mathsf{f}, \mathsf{b}\}.
\end{aligned}$$

Proof. Each homomorphism clearly satisfies these equivalences. Conversely, case analysis shows that $f(-a) = -f(a)$. To prove that $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$ it suffices by the Prime Filter Separation Lemma (Lemma 1.1) to show for each prime filter F on $\mathbf{DM_4}$ that

$$\begin{aligned}
f(a \wedge b) \in F &\iff f(a) \in F \text{ and } f(b) \in F, \\
f(a \vee b) \in F &\iff f(a) \in F \text{ or } f(b) \in F.
\end{aligned}$$

But this is an immediate consequence of the assumptions, since $\{\mathsf{t}, \mathsf{b}\}$ and $\{\mathsf{t}, \mathsf{n}\}$ are the only prime filters on $\mathbf{DM_4}$.

If $\mathbf{B}$ is a De Morgan algebra which satisfies the stated equivalences, then

$$\begin{aligned}
f(-a) \in \{\mathsf{f}, \mathsf{b}\} &\iff -f(-a) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(--a) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(a) \in \{\mathsf{t}, \mathsf{b}\}
\end{aligned}$$

and

$$\begin{aligned}
f(a \wedge b) \in \{\mathsf{f}, \mathsf{b}\} &\iff -f(a \wedge b) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(-a \vee -b) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(-a) \in \{\mathsf{t}, \mathsf{b}\} \text{ or } f(-b) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(a) \in \{\mathsf{f}, \mathsf{b}\} \text{ or } f(b) \in \{\mathsf{f}, \mathsf{b}\}
\end{aligned}$$

and

$$\begin{aligned}
f(a \vee b) \in \{\mathsf{f}, \mathsf{b}\} &\iff -f(a \vee b) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(-a \wedge -b) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(-a) \in \{\mathsf{t}, \mathsf{b}\} \text{ and } f(-b) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff f(a) \in \{\mathsf{f}, \mathsf{b}\} \text{ and } f(b) \in \{\mathsf{f}, \mathsf{b}\}. \qquad \square
\end{aligned}$$

The following lemma was proved by Pynko in [62, Prop 3.2].

Lemma 2.5 [62] **(Filter–homomorphism correspondence in DMAs).** *The poset homomorphisms $h : \mathbf{A} \to \mathbf{DM_4}$ ordered by the information order is isomorphic to the poset of prime filters F on $\mathbf{A}$ ordered by inclusion via the maps $h \mapsto F_h$ and $F \mapsto h_F$, where*

$$F_h := h^{-1}\{\mathsf{t}, \mathsf{b}\}$$

and

$$h_F(a) := \begin{cases} \mathsf{t} \text{ if } a \in F \text{ and } -a \notin F, \\ \mathsf{b} \text{ if } a \in F \text{ and } -a \in F, \\ \mathsf{f} \text{ if } a \notin F \text{ and } -a \in F, \\ \mathsf{n} \text{ if } a \notin F \text{ and } -a \notin F. \end{cases}$$

Proof. The set F_h is a prime filter by virtue of being an inverse homomorphic image of a prime filter. The map h_F is a homomorphism of De Morgan algebras by the Homomorphism Lemma for DMAs (Lemma 2.4), using the assumption that F is a prime filter.

We show that the two maps are order-preserving. If $g \sqsubseteq h$, then

$$a \in F_g \implies g(a) \in \{\mathsf{t}, \mathsf{b}\} \implies h(a) \in \{\mathsf{t}, \mathsf{b}\} \implies a \in F_h.$$

On the other hand, if $F \subseteq G$, then

$$\begin{aligned} h_F(a) \in \{\mathsf{t}, \mathsf{b}\} &\implies \quad a \in F \implies \quad a \in G \implies h_G(a) \in \{\mathsf{t}, \mathsf{b}\}, \\ h_F(a) \in \{\mathsf{f}, \mathsf{b}\} &\implies -a \in F \implies -a \in G \implies h_G(a) \in \{\mathsf{f}, \mathsf{b}\}. \end{aligned}$$

It remains to show that the two maps are mutually inverse. But

$$a \in F_{h_F} \iff a \in h_F^{-1}\{\mathsf{t}, \mathsf{b}\} \iff h_F(a) \in \{\mathsf{t}, \mathsf{b}\} \iff a \in F.$$

On the other hand,

$$\begin{aligned} h_{F_h}(a) \in \{\mathsf{t}, \mathsf{b}\} &\iff a \in F_h \iff h(a) \in \{\mathsf{t}, \mathsf{b}\}, \\ h_{F_h}(a) \in \{\mathsf{f}, \mathsf{b}\} &\iff -a \in F_h \iff h(-a) \in \{\mathsf{t}, \mathsf{b}\} \iff h(a) \in \{\mathsf{f}, \mathsf{b}\}. \end{aligned}$$

Therefore $F_{h_F} = F$ and $h_{F_h} = h$. □

We are now ready to prove what may be called the algebraic completeness theorem for De Morgan algebras. It states that this variety is generated as a quasivariety by the algebra $\mathbf{DM_4}$ and the subvarieties of Kleene and Boolean algebras are generated as quasivarieties by $\mathbf{K_3}$ and $\mathbf{B_2}$.

Theorem 2.6 [38, 62, 68] (Algebraic completeness for DMAs).

(i) $\mathsf{DMA} = \mathbb{SP}(\mathbf{DM_4})$.

(ii) $\mathsf{KA} = \mathbb{SP}(\mathbf{K_3})$.

(iii) $\mathsf{BA} = \mathbb{SP}(\mathbf{B_2})$.

Proof. (i) Suppose that $a \nleq b$ in $\mathbf{A}$. Then by the Prime Filter Separation Lemma (Lemma 1.1) there is a prime filter F with $a \in F$ and $b \notin F$. Filter–homomorphism correspondence in DMAs (Lemma 2.5) then yields a homomorphism $h : \mathbf{A} \to \mathbf{DM_4}$ with $h(a) \in \{\mathsf{t}, \mathsf{b}\}$ and $h(b) \notin \{\mathsf{t}, \mathsf{b}\}$. Thus $h(a) \nleq h(b)$ in $\mathbf{DM_4}$. It follows that $\mathbf{A}$ embeds into a power of $\mathbf{DM_4}$.

(ii) Let $\mathbf{A}$ be a Kleene algebra and $h : \mathbf{A} \to \mathbf{DM_4}$ be a homomorphism. Kleene algebras form a variety, therefore the image of $\mathbf{A}$ is a Kleene subalgebra of $\mathbf{DM_4}$. But each Kleene subalgebra of $\mathbf{DM_4}$ is isomorphic to a subalgebra of $\mathbf{K_3}$.

(iii) Let $\mathbf{A}$ be a Boolean algebra and $h : \mathbf{A} \to \mathbf{DM_4}$ be a homomorphism. Boolean algebras form a variety, therefore the image of $\mathbf{A}$ is a Boolean subalgebra of $\mathbf{DM_4}$. But the only Boolean subalgebra of $\mathbf{DM_4}$ is $\mathbf{B_2}$. □

Theorem 2.7 [38, 68] (Subdirectly irreducible DMAs).
There are exactly three (finitely) subdirectly irreducible De Morgan algebras, namely $\mathbf{B_2}$, $\mathbf{K_3}$, $\mathbf{DM_4}$.

Proof. Each subdirectly irreducible De Morgan algebra embeds into a power of $\mathbf{DM_4}$, therefore it embeds into $\mathbf{DM_4}$. But each subalgebra of $\mathbf{DM_4}$ is isomorphic to $\mathbf{B_2}$, $\mathbf{K_3}$, or $\mathbf{DM_4}$, and these are all subdirectly irreducible.

The claim for finitely subdirectly irreducible De Morgan algebras was proved by Kalman [38, Lemma 2] as well as Sankappanavar [68, Thm 3.4]. We do not include its proof, as it will not be needed in the following. □

Corollary 2.8 (Varieties of DMAs).
There are exactly three non-trivial varieties of De Morgan algebras, namely Boolean algebra, Kleene algebras, and De Morgan algebras.

Corollary 2.9 (Local finiteness of DMAs).
De Morgan algebras are locally finite.

We end our review of De Morgan algebras by providing an equational description of their principal congruences which will be useful in our later investigations. In fact, this description holds even for De Morgan lattices.

Theorem 2.10 [68, Thm 2.2] (Principal congruences of DMLs).
Let a, b, x, y be elements of a De Morgan lattice $\mathbf{A}$ such that $a \leq b$. Then $\langle x, y \rangle \in \mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle$ if and only if the following four equalities hold:

$$x \wedge a \wedge -b = y \wedge a \wedge -b \qquad (x \wedge a) \vee -a = (y \wedge a) \vee -a$$
$$x \vee b \vee -a = y \vee b \vee -a \qquad (x \vee b) \wedge -b = (y \vee b) \wedge -b$$

Proof. The equalities define a congruence θ on $\mathbf{A}$ such that $\langle a, b \rangle \in \theta$, hence $\mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle \subseteq \theta$. Conversely, suppose that x and y satisfy the equalities. Let $\pi : \mathbf{A} \to \mathbf{A}/\mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle$ be the canonical projection. In particular, $\pi(a) = \pi(b)$ and $\pi(-a) = \pi(-b)$. But the equalities then imply that

$$\pi(x \wedge a) \wedge -\pi(a) = \pi(y \wedge a) \wedge -\pi(a),$$
$$\pi(x \wedge a) \vee -\pi(a) = \pi(y \wedge a) \vee -\pi(a),$$

therefore $\pi(x \wedge a) = \pi(y \wedge a)$ by the Equality Lemma for DLs (Lemma 1.3). Likewise

$$\pi(x \vee a) \wedge -\pi(a) = \pi(y \wedge a) \wedge -\pi(a),$$
$$\pi(x \vee a) \vee -\pi(a) = \pi(y \wedge a) \vee -\pi(a),$$

therefore $\pi(x \vee a) = \pi(y \vee a)$ by the Equality Lemma for DLs. But then

$$\pi(x) \wedge \pi(a) = \pi(y) \wedge \pi(a),$$
$$\pi(x) \vee \pi(a) = \pi(y) \vee \pi(a),$$

therefore $\pi(x) = \pi(y)$ by the same lemma and $\langle x, y \rangle \in \mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle$. □

In the statement of the following proposition, recall that for sets of pairs $X \subseteq \mathbf{A}^2$ we use the notation $\mathrm{Cg}^{\mathbf{A}} X = \bigvee_{\langle a,b \rangle \in X} \mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle$.[1]

Proposition 2.11 [68] **(Congruences generated by ideals on DMLs).** *Let I be an ideal on a De Morgan lattice $\mathbf{A}$. Then*

$$\langle x, y \rangle \in \mathrm{Cg}^{\mathbf{A}} I^2 \iff (x \vee a) \wedge -a = (y \vee a) \wedge -a \textit{ for some } a \in I.$$

Proof. The congruence $\mathrm{Cg}^{\mathbf{A}} I^2$ is in fact a directed join of the congruences $\mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle$ for $a \leq b \in I_{\mathbf{A}}$. Therefore $\langle x, y \rangle \in \mathrm{Cg}^{\mathbf{A}} I^2$ if and only if for some such a, b we have $\langle x, y \rangle \in \mathrm{Cg}^{\mathbf{A}}\langle a, b \rangle$. By the above description of Principal congruences of DMLs (Theorem 2.10) this holds if and only if

$$\begin{array}{ll} x \wedge a \wedge -b = y \wedge a \wedge -b & (x \wedge a) \vee -a = (y \wedge a) \vee -a \\ x \vee b \vee -a = y \vee b \vee -a & (x \vee b) \wedge -b = (y \vee b) \wedge -b \end{array}$$

In particular, $(x \vee b) \wedge -b) = (y \vee b) \wedge -b$ for some $b \in I_{\mathbf{A}}$. Conversely, if there is some such b, then taking $a := b \wedge x \wedge y \wedge -y$ validates all of these equations. □

[1] In fact [68, Cor 2.3] only deals with the special case where I is a principal ideal and $\mathbf{A}$ is a De Morgan algebra, but the generalization to arbitrary ideals on De Morgan lattices is straightforward.

Chapter 3

The Belnap–Dunn logic and its cousins

With the algebraic preliminaries out of the way, it is now time to introduce the Belnap–Dunn logic itself and bring the reader up to date with the current state of knowledge about this logic and its most important extensions. We take these to be the strong three-valued Kleene logic $\mathcal{K}$, Priest's three-valued Logic of Paradox $\mathcal{LP}$, Kleene's logic of order $\mathcal{KO}$, and (the newcomer among these basic extensions of $\mathcal{BD}$) the four-valued Exactly True Logic $\mathcal{ETL}$.

Familiarity with these logics is a necessary prerequisite for the study of extensions of the Belnap–Dunn logic $\mathcal{BD}$, called super-Belnap logics here. The current chapter therefore serves as a sort of springboard for a deeper understanding of the landscape of super-Belnap logics. The basic properties of these three- and four-valued logics will be invoked throughout the thesis to establish results about other extensions of $\mathcal{BD}$.

The chapter begins by defining these logics and outlining their history. We then describe the consequence relations of these logics from several points of view, including their relation to De Morgan algebras. Next, we prove completeness theorems for these logics and recall that each formula of $\mathcal{BD}$ has an equivalent conjunctive and disjunctive normal form. Finally, the matrix models of $\mathcal{BD}$ and its basic extensions are investigated, including their Leibniz congruences and subdirect decomposition.

The discussion of other important properties of these extensions of $\mathcal{BD}$ will be postponed until later chapters. Their classification in the Leibniz and Frege hierarchies of abstract algebraic logic will be discussed in Chapter 8 (Metalogical properties of super-Belnap logics). Gentzen-style proof theory for these logics will be developed in Chapter 9 (Metalogical properties of super-Belnap logics) and used to prove several interpolation theorems. Finally, multiple-conclusion versions of these logics as well as variants of $\mathcal{BD}$ with the exact truth predicate or the non-falsity predicate will be studied in Chapter 10 (Other frameworks).

Most of the material presented in this chapter is not new. In addition to the classic papers of Belnap and Dunn [8, 9, 18] and the relevant parts of Anderson and Belnap's monograph on the logic of entailment [3], the main sources that we draw on are the papers of Font [25] and Pynko [59, 63], as well as Rivieccio's unpublished notes [67], part of which was published as [66]. The reader may also wish to consult the relevant chapter of Priest's textbook [51], Dunn's paper [20], or the overview paper [2] for an alternative introduction to the Belnap–Dunn logic and its most important extensions.

Throughout most of the thesis we restrict our attention to propositional logics in the signature of $\mathcal{BD}$. The only exceptions are Chapters 10 (Other frameworks) and 11 (The truth operator Δ), where the constant-free fragment of $\mathcal{BD}$ and the expansion of $\mathcal{BD}$ by the truth operator Δ are studied. Predicate versions of $\mathcal{BD}$ are not discussed, neither are their expansions by modalities or connectives other than Δ.

3.1 History and definitions

We start by introducing the Belnap–Dunn logic $\mathcal{BD}$ and its two best-known extensions, the strong Kleene logic $\mathcal{K}$ and the Logic of Paradox $\mathcal{LP}$.

The Belnap–Dunn logic was first introduced as the logic of so-called tautological entailments by Anderson and Belnap [4] in connection with their project of identifying a logic of entailment. By a tautological entailment they meant a first-degree entailment — a formula of the form $\varphi \to \psi$ where φ and ψ are purely truth-functional, i.e. contain no logical connectives other than conjunction, disjunction, and negation — valid in the logic of entailment that was the object of Anderson and Belnap's interest [3]. For this reason, the logic is commonly called FDE for "first-degree entailment". However, we stick to the term "Belnap–Dunn logic" because we are studying the logic in its own right rather than as a first-degree fragment of some other logic.

The logic of first-degree entailment was subsequently studied by Dunn in his thesis [21] and in [18], where he provided a semantics for the logic in terms of what we call truth relations below. The key idea of his semantics is to allow for such truth relations to relate a proposition to neither of the values "True" and "False" or to both. In other words, propositions are evaluated by subsets of the set of the two classical truth values. This is clearly equivalent to a four-valued semantics where the values "Neither" and "Both" are added to the two classical truth values.

Around the same time, this four-valued semantics was also popularized by Belnap [8, 9], who proposed this system as a logic that a computer should use in order to process data which is potentially inconsistent and incomplete. The usefulness of the four-valued semantics in such situations was in fact already pointed out by Dunn in [19]. Note that Belnap and Dunn did not interpret the truth values "Both (True and False)" and "Neither (True nor

False)" ontologically. Rather, they were meant to describe the information available to us, which may of course be inconsistent as well as incomplete. (We may be in possession of information supporting a proposition p as well as information supporting its negation $-p$.) Indeed, in this context Belnap aptly called the two classical truth values "told True" and "told False".

Semantics for the strong Kleene logic $\mathcal{K}$ and the Logic of Paradox $\mathcal{LP}$ may be obtained by excluding the possibility of a proposition being both true and false (for $\mathcal{K}$) or the possibility of a proposition being neither (for $\mathcal{LP}$). However, this is not how the logics $\mathcal{K}$ and $\mathcal{LP}$ were introduced historically.

The three-element algebra of truth values of the strong Kleene logic $\mathcal{K}$ was first considered by Kleene [39, 40] in connection with partially defined functions. However, Kleene himself did not associate any consequence relation with this algebra. The three-valued semantics was later famously used by Kripke in his theory of truth [41].

The Logic of Paradox $\mathcal{LP}$ was introduced by Asenjo [5] in his calculus for antinomies and by Priest [52] as an attempt to handle paradoxes such as the Liar paradox. A semantics for $\mathcal{LP}$ is obtained by excluding the truth value "Neither" from the semantics of $\mathcal{BD}$. The Logic of Paradox and its expansions are often adopted by dialetheists, who claim that there are sentences (such as the Liar sentence) which are both true and false.

We now outline the semantics of $\mathcal{BD}$ due to Dunn [18] as well as its modifications for $\mathcal{K}$ and $\mathcal{LP}$. We take the signature of $\mathcal{BD}$ to consist of the binary distributive lattice connectives $\wedge$ and $\vee$, representing conjunction and disjunction, the unary De Morgan negation $-$, and the constants $\mathtt{t}$ and $\mathtt{f}$. Although the constants $\mathtt{t}$ and $\mathtt{f}$ are usually not included in the signature of the Belnap–Dunn logic (and their inclusion will make little difference for most of our results), we shall see that their inclusion will make our study of extensions of $\mathcal{BD}$ slightly more interesting. The implications of excluding the truth constants will be discussed in more detail in Chapter 10.

A *truth relation* u will be a pair of sets of formulas $\langle u_+, u_- \rangle$, where $\varphi \in u_+$ and $\varphi \in u_-$ will be written as $u \Vdash^+ \varphi$ and $u \Vdash^- \varphi$, such that

$$
\begin{aligned}
& u \Vdash^+ \mathtt{t} \text{ and } u \nVdash^- \mathtt{t}, \\
& u \nVdash^+ \mathtt{f} \text{ and } u \Vdash^- \mathtt{f}, \\
& u \Vdash^+ -\varphi \iff u \Vdash^- \varphi, \\
& u \Vdash^- -\varphi \iff u \Vdash^+ \varphi, \\
& u \Vdash^+ \varphi \wedge \psi \iff u \Vdash^+ \varphi \text{ and } u \Vdash^+ \psi, \\
& u \Vdash^- \varphi \wedge \psi \iff u \Vdash^- \varphi \text{ or } u \Vdash^- \psi, \\
& u \Vdash^+ \varphi \vee \psi \iff u \Vdash^+ \varphi \text{ or } u \Vdash^+ \psi, \\
& u \Vdash^- \varphi \vee \psi \iff u \Vdash^- \varphi \text{ and } u \Vdash^- \psi.
\end{aligned}
$$

A truth relation is *consistent* if for each atom p either $u \nVdash^+ p$ or $u \nVdash^- p$, and it is *complete* if for each atom p either $u \Vdash^+ p$ or $u \Vdash^- p$.

Fact 3.1. *Let u be a truth relation. If it is consistent, then for each formula φ either $u \nVdash^+ \varphi$ or $u \nVdash^- \varphi$. If it is complete, then for each formula φ either $u \Vdash^+ \varphi$ or $u \Vdash^- \varphi$. If it is both, then $u \Vdash^- \varphi \iff u \nVdash^+ \varphi$ for each φ.*

The consequence relation of the Belnap–Dunn logic $\mathcal{BD}$ is defined by truth-preservation on truth relations. That is,

$$\Gamma \vdash_{\mathcal{BD}} \varphi \iff u \Vdash^+ \Gamma \text{ implies } u \Vdash^+ \varphi \text{ for each truth relation } u,$$

where by $u \Vdash^+ \Gamma$ we mean $u \Vdash^+ \gamma$ for all $\gamma \in \Gamma$. The consequence relation of the Kleene logic $\mathcal{K}$ is defined by truth-preservation on consistent truth relations, while the consequence relation of the Logic of Paradox $\mathcal{LP}$ is defined by truth-preservation on complete truth relations. Of course, the consequence relation of classical logic $\mathcal{CL}$ is defined by truth-preservation on complete consistent truth relations.

Equivalently, $\mathcal{BD}$ may be defined by falsity-reflection (backward falsity-preservation) on truth relations. That is,

$$\Gamma \vdash_{\mathcal{BD}} \varphi \iff u \Vdash^- \varphi \text{ implies } u \Vdash^- \Gamma \text{ for each truth relation } u,$$

where by $u \Vdash^- \Gamma$ we mean $u \Vdash^- \gamma$ for some $\gamma \in \Gamma$. Consequence in $\mathcal{K}$ then amounts to falsity-reflection on complete truth relations, consequence in $\mathcal{LP}$ amounts to falsity-reflection on consistent truth relations, and consequence in $\mathcal{CL}$ amount to falsity-reflection on complete consistent truth relations.

Truth relations are in fact nothing but valuations on the algebra $\mathbf{DM_4}$, i.e. homomorphisms $v : \mathbf{Fm} \to \mathbf{DM_4}$, in a slightly different presentation. The truth valuation u corresponds to the valuation v on $\mathbf{DM_4}$ such that

$$\begin{aligned}
v(\varphi) = \mathsf{t} &\iff u \Vdash^+ \varphi \text{ and } u \nVdash^- \varphi, \\
v(\varphi) = \mathsf{f} &\iff u \nVdash^+ \varphi \text{ and } u \Vdash^- \varphi, \\
v(\varphi) = \mathsf{n} &\iff u \nVdash^+ \varphi \text{ and } u \nVdash^- \varphi, \\
v(\varphi) = \mathsf{b} &\iff u \Vdash^+ \varphi \text{ and } u \Vdash^- \varphi.
\end{aligned}$$

Conversely, the valuation v on $\mathbf{DM_4}$ corresponds to the truth relation u such that

$$\begin{aligned}
u \Vdash^+ \varphi &\iff v(\varphi) \in \{\mathsf{t}, \mathsf{b}\}, \\
u \Vdash^- \varphi &\iff v(\varphi) \in \{\mathsf{f}, \mathsf{b}\}.
\end{aligned}$$

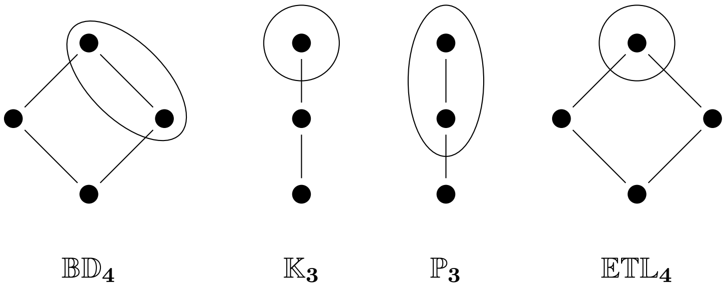
Under this correspondence, consistent truth relations correspond precisely to valuations into $\mathbf{K_3}$, and complete truth relations to valuations into $\mathbf{P_3}$.

The definition of consequence in $\mathcal{BD}$ then translates into

$$\Gamma \vdash_{\mathcal{BD}} \varphi \iff v[\Gamma] \subseteq \{\mathsf{t}, \mathsf{b}\} \text{ implies } v(\varphi) \in \{\mathsf{t}, \mathsf{b}\} \text{ for each } v : \mathbf{Fm} \to \mathbf{DM_4}.$$

That is, $\mathcal{BD}$ is the logic determined by the matrix $\mathbb{BD}_\mathbf{4} := \langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\} \rangle$. Likewise, $\mathcal{K}$ is the logic determined by the matrix $\mathbb{K}_\mathbf{3} := \langle \mathbf{K_3}, \{\mathsf{t}\} \rangle$, and $\mathcal{LP}$

Figure 3.1: Some logical matrices for super-Belnap logics



is the logic determined by the matrix $\mathbb{P}_3 := \langle \mathbf{P_3}, \{\mathsf{t}, \mathsf{b}\}\rangle$. These matrices are shown in Figure 3.1.

In addition to the well-known extensions of $\mathcal{BD}$ introduced above, two less well-known logics need to be introduced at this point. Firstly, *Kleene's logic of order* is defined as $\mathcal{KO} := \mathcal{K} \cap \mathcal{LP}$. In other words, it is the logic determined by the set of matrices $\{\mathbb{K}_3, \mathbb{P}_3\}$. The name was given to the logic by Rivieccio [66] due to its connection to the order on the three-element Kleene chain $\mathbf{K_3}$, which will be stated in the following section. It was also called Kalman implication by Makinson [43]. Just like $\mathcal{BD}$ is the first-degree fragment of the logic of entailment $\mathcal{E}$, $\mathcal{KO}$ was shown to be the first-degree fragment of the relevance logic R-Mingle by Dunn [17].

Secondly, the *Exactly True Logic* was defined as $\mathcal{ETL} := \mathrm{Log}\langle \mathbf{DM_4}, \{\mathsf{t}\}\rangle$ by Pietz and Rivieccio [50]. That is, $\mathcal{ETL}$ uses the same algebra of truth values of $\mathcal{BD}$, but its consequence relation is defined by the preservation of *exact* truth (i.e. truth and non-falsity) rather than preservation of mere truth. The matrix $\langle \mathbf{DM_4}, \{\mathsf{t}\}\rangle$ which defines this logic will be denoted $\mathbb{ETL}_4$. It is again shown in Figure 3.1. Although a relative newcomer compared to the other logics introduced in this section, the importance of the Exactly True Logic will be apparent throughout the thesis. Indeed, it was this logic which motivated Rivieccio [66] to initiate the systematic study of super-Belnap logics.[1]

3.2 Consequence relations

We now describe the consequence relations of the basic extensions of $\mathcal{BD}$ introduced in the previous section in more detail, starting with their relation to varieties of De Morgan algebras. We then relate these logics to each other by means of several translations.

[1]The idea of preserving exact truth was already suggested by Marcos [44], who in fact studied an expansion of the Exactly True Logic.

Theorem 3.2 (Semilattice-based completeness).
The following equivalences hold for finite Γ and Δ:

(i) $\Gamma \vdash_{\mathcal{BD}} \varphi$ *if and only if* $\bigwedge \Gamma \leq \varphi$ *holds in* DMA.

(ii) $\Gamma \vdash_{\mathcal{KO}} \varphi$ *if and only if* $\bigwedge \Gamma \leq \varphi$ *holds in* KA.

(iii) $\Gamma \vdash_{\mathcal{CL}} \varphi$ *if and only if* $\bigwedge \Gamma \leq \varphi$ *holds in* BA.

Proof. It suffices to prove the claim for $\Gamma := \{\varphi\}$. The right-to-left implications hold because the filters of $\mathbb{BD}_4$, $\mathbb{K}_3$, $\mathbb{P}_3$, and $\mathbb{B}_2$ are lattice filters.

(i) Suppose that $\psi \vdash_{\mathcal{BD}} \varphi$. But $\{\mathsf{t}, \mathsf{b}\}$ and $\{\mathsf{t}, \mathsf{n}\}$ are the only prime filters on $\mathbf{DM_4}$ and $\mathcal{BD} = \mathrm{Log}\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}\rangle = \mathrm{Log}\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{n}\}\rangle$. Therefore for each $v : \mathbf{Fm} \to \mathbf{DM_4}$ we have

$$v(\psi) \in \{\mathsf{t}, \mathsf{b}\} \implies v(\varphi) \in \{\mathsf{t}, \mathsf{b}\},$$
$$v(\psi) \in \{\mathsf{t}, \mathsf{b}\} \implies v(\varphi) \in \{\mathsf{t}, \mathsf{n}\},$$

hence $v(\psi) \leq v(\varphi)$.

(ii) Suppose that $\psi \vdash_{\mathcal{KO}} \varphi$. Then $\psi \vdash_{\mathcal{K}} \varphi$ and $\psi \vdash_{\mathcal{LP}} \varphi$. But $\{\mathsf{t}\}$ and $\{\mathsf{t}, \mathsf{i}\}$ are the only prime filters on $\mathbf{K_3}$ and $\mathcal{K} = \mathrm{Log}\langle \mathbf{K_3}, \{\mathsf{t}\}\rangle$ and $\mathcal{LP} = \mathrm{Log}\langle \mathbf{K_3}, \{\mathsf{t}, \mathsf{i}\}\rangle$. Therefore for each $v : \mathbf{Fm} \to \mathbf{K_3}$ we have

$$v(\psi) \in \{\mathsf{t}\} \implies v(\varphi) \in \{\mathsf{t}\}, \text{ since } \psi \vdash_{\mathcal{K}} \varphi,$$
$$v(\psi) \in \{\mathsf{t}, \mathsf{i}\} \implies v(\varphi) \in \{\mathsf{t}, \mathsf{i}\}, \text{ since } \psi \vdash_{\mathcal{LP}} \varphi,$$

hence $v(\psi) \leq v(\varphi)$.

(iii) The argument is entirely analogical to the previous cases. □

As a consequence, the logics $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$ enjoy the contraposition property. Moreover, as observed by Milne [47], the logic $\mathcal{LP}$ and $\mathcal{K}$ form what might be called a contrapositive pair.

Theorem 3.3 (Contraposition for the basic extensions of $\mathcal{BD}$).
Contraposition holds in the logics $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$ in the forms

$$\varphi \vdash_{\mathcal{BD}} \psi \iff -\psi \vdash_{\mathcal{BD}} -\varphi,$$
$$\varphi \vdash_{\mathcal{KO}} \psi \iff -\psi \vdash_{\mathcal{KO}} -\varphi,$$
$$\varphi \vdash_{\mathcal{CL}} \psi \iff -\psi \vdash_{\mathcal{CL}} -\varphi.$$

Moreover, the logics $\mathcal{LP}$ and $\mathcal{K}$ are related by the equivalences

$$\varphi \vdash_{\mathcal{LP}} \psi \iff -\psi \vdash_{\mathcal{K}} -\varphi \quad \text{and} \quad \varphi \vdash_{\mathcal{K}} \psi \iff -\psi \vdash_{\mathcal{LP}} -\varphi.$$

Proof. The equivalences for $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$ follow from the semilattice-based completeness theorems (Theorem 3.2), since $a \leq b \implies -b \leq -a$ in De Morgan algebras. To prove the penultimate equivalence, observe that

$$\begin{aligned}
\varphi \vdash_{\mathcal{LP}} \psi &\iff v(\varphi) \in \{\mathsf{t}, \mathsf{i}\} \text{ implies } v(\psi) \in \{\mathsf{t}, \mathsf{i}\} \text{ for each } v : \mathbf{Fm} \to \mathbf{K_3} \\
&\iff v(\psi) \notin \{\mathsf{t}, \mathsf{i}\} \text{ implies } v(\varphi) \notin \{\mathsf{t}, \mathsf{i}\} \text{ for each } v : \mathbf{Fm} \to \mathbf{K_3} \\
&\iff v(-\psi) \notin \{\mathsf{f}, \mathsf{i}\} \text{ implies } v(-\varphi) \notin \{\mathsf{f}, \mathsf{i}\} \text{ for each } v : \mathbf{Fm} \to \mathbf{K_3} \\
&\iff v(-\psi) \in \{\mathsf{t}\} \text{ implies } v(-\varphi) \notin \{\mathsf{t}\} \text{ for each } v : \mathbf{Fm} \to \mathbf{K_3} \\
&\iff -\psi \vdash_{\mathcal{K}} -\varphi.
\end{aligned}$$

The last equivalence now follows because

$$\varphi \vdash_{\mathcal{K}} \psi \iff --\varphi \vdash_{\mathcal{K}} --\psi \iff -\psi \vdash_{\mathcal{LP}} -\varphi. \qquad \square$$

We now turn to the consequence relations of $\mathcal{LP}$ and $\mathcal{K}$ and their relation to $\mathcal{BD}$ and $\mathcal{CL}$. The observation that $\mathcal{LP}$ has the same theorems as $\mathcal{CL}$ was already made by Priest when introducing the logic $\mathcal{LP}$ [52]. However, we include its proof for the sake of the reader. The observation that $\mathcal{K}$ and $\mathcal{BD}$ has not been made before, as far as we know. This is presumably because it is trivial in the framework without the constants t and f, where neither of the two logics has theorems.

Proposition 3.4 [**52**, **Thm III.8**] **(Theorems of $\mathcal{LP}$ and $\mathcal{K}$).**

(i) $\mathcal{LP}$ has the same theorems as $\mathcal{CL}$.

(ii) $\mathcal{K}$ has the same theorems as $\mathcal{BD}$.

Proof. Clearly each theorem of $\mathcal{LP}$ is a theorem of $\mathcal{CL}$, and each theorem of $\mathcal{BD}$ is a theorem of $\mathcal{K}$.

(i) Suppose that φ is not a theorem of $\mathcal{LP}$. Then there is a valuation $v : \mathbf{Fm} \to \mathbf{P_3}$ such that $v(\varphi) \notin \{\mathsf{t}, \mathsf{b}\}$, i.e. $v(\varphi) = \mathsf{f}$. Consider an arbitrary valuation $w : \mathbf{Fm} \to \mathbf{B_2}$ such that $w \sqsubseteq v$. That is, for each atom p with $v(p) = \mathsf{b}$ take $w(p) \in \{\mathsf{f}, \mathsf{t}\}$, otherwise take $w(p) := v(p)$. Then in particular $w(\varphi) \sqsubseteq v(\varphi) = \mathsf{f}$, therefore $w(\varphi) = \mathsf{f} \notin \{\mathsf{t}\}$.

(ii) Suppose that φ is not a theorem of $\mathcal{BD}$. Then there is a valuation $v : \mathbf{Fm} \to \mathbf{DM_4}$ such that $v(\varphi) \notin \{\mathsf{t}, \mathsf{b}\}$, i.e. $v(\varphi) \in \{\mathsf{n}, \mathsf{f}\}$. Consider an arbitrary valuation $w : \mathbf{Fm} \to \mathbf{K_3}$ such that $w \sqsubseteq v$, e.g. the valuation such that $w(p) = \mathsf{n}$ for each atom p. Then in particular $w(\varphi) \sqsubseteq v(\varphi) \in \{\mathsf{n}, \mathsf{f}\}$, therefore $w(\varphi) \in \{\mathsf{n}, \mathsf{f}\}$. $\square$

The following two propositions show that consequence in $\mathcal{LP}$ and $\mathcal{K}$ may be reduced to consequence in $\mathcal{BD}$ using classical tautologies and classical contradictions. In the statements of these propositions we use $\mathrm{At}(\Gamma)$ to denote the set of all atoms which occur in Γ, with $\mathrm{At}(\varphi) := \mathrm{At}(\{\varphi\})$.

Proposition 3.5 (Consequence in $\mathcal{LP}$).
The following are equivalent:

(i) $\Gamma \vdash_{\mathcal{LP}} \varphi$.

(ii) $\Gamma, \tau \vdash_{\mathcal{BD}} \varphi$ *for some classical tautology* τ.

(iii) $\Gamma, \{p \vee -p \mid p \in \mathrm{At}(\varphi)\} \vdash_{\mathcal{BD}} \varphi$.

Proof. The implication (iii) $\implies$ (ii) is trivial, and (ii) $\implies$ (i) holds by Proposition 3.4. It remains to prove (i) $\implies$ (iii). Suppose therefore that $\Gamma, \{p \vee -p \mid p \in \mathrm{At}(\varphi)\} \nvdash_{\mathcal{BD}} \varphi$. Then there is some valuation $v : \mathbf{Fm} \to \mathbf{DM_4}$ such that $v[\Gamma] \subseteq \{\mathsf{t}, \mathsf{b}\}$ and $v(p \vee -p) \in \{\mathsf{t}, \mathsf{b}\}$ for each $p \in \mathrm{At}(\varphi)$, but $v(\varphi) \notin \{\mathsf{t}, \mathsf{b}\}$. It follows that $v(p) \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ for each $p \in \mathrm{At}(\varphi)$. But then $v(\varphi) \in \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ because $\mathbf{P_3}$ is a subalgebra of $\mathbf{DM_4}$.

Now consider a valuation $w : \mathbf{Fm} \to \mathbb{P}_3$ such that $w(p) := v(p)$ if $p \in \mathrm{At}(\varphi)$ and $w(p) := \mathsf{b}$ otherwise. Then $v \sqsubseteq w$, therefore $w(\varphi) = v(\varphi) \notin \{\mathsf{t}, \mathsf{b}\}$ and $w(\gamma) \sqsupseteq v(\gamma) \in \{\mathsf{t}, \mathsf{b}\}$ for each $\gamma \in \Gamma$. Thus $\Gamma \nvdash_{\mathcal{LP}} \varphi$. □

Proposition 3.6 (Consequence in $\mathcal{K}$).
The following are equivalent for finite Γ*:*

(i) $\Gamma \vdash_{\mathcal{K}} \varphi$.

(ii) $\Gamma \vdash_{\mathcal{BD}} \varphi \vee \chi$ *for some classical contradiction* χ.

(iii) $\Gamma \vdash_{\mathcal{BD}} \varphi \vee \bigvee_{p \in \mathrm{At}(\Gamma)} (p \wedge -p)$.

Proof. This follows from the previous proposition by the contraposition relation between $\mathcal{K}$ and $\mathcal{LP}$ (Proposition 3.3). □

We now show that the logics determined by the singleton filter $\{\mathsf{t}\}$ on the algebras $\mathbf{B_2}$, $\mathbf{K_3}$, and $\mathbf{DM_4}$ may be translated into the logics $\mathcal{LP}$, $\mathcal{KO}$, and $\mathcal{BD}$ by means of a suitable translation. The equivalence for $\mathcal{ETL}$ was proved by Pietz and Rivieccio [50, Lemma 3.2], while the equivalence for $\mathcal{K}$ was proved by Rivieccio [67] but not published.

Theorem 3.7 [50, 67] (Translations between super-Belnap logics).
The following relations between logics hold:

$$\begin{aligned} \varphi \vdash_{\mathcal{ETL}} \psi &\iff \varphi \vdash_{\mathcal{BD}} -\varphi \vee \psi, \\ \varphi \vdash_{\mathcal{K}} \psi &\iff \varphi \vdash_{\mathcal{KO}} -\varphi \vee \psi, \\ \varphi \vdash_{\mathcal{CL}} \psi &\iff \varphi \vdash_{\mathcal{LP}} -\varphi \vee \psi. \end{aligned}$$

Proof. The right-to-left implications in all cases follow from the fact that $p, -p \vee q \vdash_{\mathcal{ETL}} q$ and $\mathcal{ETL} \leq \mathcal{K} \leq \mathcal{CL}$. The left-to-right implication for $\mathcal{ETL}$ was proved already by Pietz & Rivieccio [50, Lemma 3.2]. Its proof is in fact similar to the proof of the left-to-right implication for $\mathcal{K}$.

Table 3.1: Axiomatization of $\mathcal{BD}$

$p \wedge q \vdash p$	$p \vdash p \vee q$	$\emptyset \vdash \mathsf{t}$	$-\mathsf{t} \vee p \vdash p$
$p \wedge q \vdash q$	$q \vdash p \vee q$	$\emptyset \vdash -\mathsf{f}$	$\mathsf{f} \vee p \vdash p$

$$p \vee r \vdash --p \vee r \qquad p, q \vdash p \wedge q$$
$$--p \vee r \vdash p \vee r \qquad p \vee p \vdash p$$

$$p \vee (q \vee r) \vdash (p \vee q) \vee r$$
$$p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r)$$
$$(p \vee q) \wedge (p \vee r) \vdash p \vee (q \wedge q)$$

$$-(p \wedge q) \vee r \vdash (-p \vee -q) \vee r \qquad (-p \vee -q) \vee r \vdash -(p \wedge q) \vee r$$
$$-(p \vee q) \vee r \vdash (-p \wedge -q) \vee r \qquad (-p \wedge -q) \vee r \vdash -(p \vee r) \vee r$$

To prove the left-to-right implication for $\mathcal{K}$, suppose that $\varphi \nvdash_{\mathcal{KO}} -\varphi \vee \psi$. Then there is a valuation $v : \mathbf{Fm} \to \mathbf{K_3}$ such that $v(\varphi) \nleq -v(\varphi) \vee v(\psi)$. Thus $v(\varphi) = \mathsf{t}$ and $\mathsf{t} \nleq v(\psi)$, hence v witnesses that $\varphi \nvdash_{\mathcal{K}} \psi$.

To prove the left-to-right implication for $\mathcal{CL}$, let $\varphi \nvdash_{\mathcal{LP}} -\varphi \vee \psi$. Then there is a valuation $v : \mathbf{Fm} \to \mathbf{P_3}$ such that $v(\varphi) \in \{\mathsf{t}, \mathsf{b}\}$ and $v(-\varphi \vee \psi) \notin \{\mathsf{t}, \mathsf{b}\}$. Thus $v(\varphi) = \mathsf{t}$ and $v(\psi) = \mathsf{f}$. Now take a valuation $w : \mathbf{Fm} \to \mathbf{B_2}$ such that $w \sqsupseteq v$. Then $w(\varphi) \sqsupseteq v(\varphi) = \mathsf{t}$ and $w(\psi) \sqsupseteq v(\psi) = \mathsf{f}$, hence $w(\varphi) = \mathsf{t}$ and $w(\varphi) = \mathsf{f}$. The valuation w thus witnesses that $\varphi \nvdash_{\mathcal{CL}} \psi$. $\square$

3.3 Completeness and normal forms

We now recall the Hilbert-style axiomatization of $\mathcal{BD}$ due to Font [25] and Pynko [59] as well as axiomatizations of the super-Belnap logics introduced so far. Note that the Hilbert-style completeness theorem for $\mathcal{LP}$ was first proved by Pynko [59], for $\mathcal{ETL}$ by Pietz and Rivieccio [50], and for $\mathcal{K}$ and $\mathcal{KO}$ by Rivieccio [67] in his unpublished notes. The completeness theorems for $\mathcal{K}$ and $\mathcal{KO}$ were published in [2]. Note, however, that our completeness proofs for $\mathcal{K}$ and $\mathcal{KO}$ differ from Rivieccio's.

The results mentioned in the previous paragraph in fact have to be slightly modified to account for the presence of the truth constants in the signature. Apart from [2], all of these papers in fact study the constant-free fragment of $\mathcal{BD}$, although the difference turns out to be quite cosmetic.

Theorem 3.8 [25, 59] (Completeness for $\mathcal{BD}$).
The logic $\mathcal{BD}$ is axiomatized by the rules shown in Table 3.1.

Proof. All of these rules is valid in $\mathbb{BD}_4$. Conversely, let $\mathcal{L}$ be the logic axiomatized by these rules. By induction over the length of proofs we can show that $\mathcal{L}$ satisfies the proof by cases property:

$$\Gamma, \varphi \vee \psi \vdash_{\mathcal{L}} \chi \iff \Gamma, \varphi \vdash_{\mathcal{L}} \chi \text{ and } \Gamma, \psi \vdash_{\mathcal{L}} \chi.$$

The left-to-right direction is trivial because $p \vdash_{\mathcal{L}} p \vee q$ and $q \vdash_{\mathcal{L}} p \vee q$. To prove the right-to-left direction, it suffices to show that for each rule $\varphi \vdash \psi$ in the axiomatization the rule $\varphi \vee r \vdash \psi \vee r$ is derivable. Then by induction over the length of the proof $\Gamma, \varphi \vdash_{\mathcal{L}} \chi$ implies $\Gamma, \varphi \vee \psi \vdash_{\mathcal{L}} \chi \vee \psi$ and $\Gamma, \psi \vee \chi \vdash_{\mathcal{L}} \chi \vee \chi$. But $\chi \vee \psi \vdash_{\mathcal{L}} \psi \vee \chi$ and $\chi \vee \chi \vdash_{\mathcal{L}} \chi$.

The logic $\mathcal{L}$ is finitary by definition, therefore by Zorn's lemma each theory Γ of $\mathcal{L}$ such that $\Gamma \nvdash_{\mathcal{L}} \varphi$ extends to a maximal theory Δ of $\mathcal{L}$ such that $\Delta \nvdash_{\mathcal{L}} \varphi$. By the proof by cases property this theory is prime:

$$\Delta \vdash_{\mathcal{L}} \psi \vee \chi \iff \Delta \vdash_{\mathcal{L}} \psi \text{ or } \Delta \vdash_{\mathcal{L}} \chi.$$

But now by the Homomorphism Lemma for DMAs (Lemma 2.4) the map $v : \mathbf{Fm} \to \mathbf{DM_4}$ such that

$$\begin{aligned} v(\varphi) &= \mathsf{t} \text{ if } \Delta \vdash_{\mathcal{L}} \varphi \text{ and } \Delta \nvdash_{\mathcal{L}} -\varphi, \\ v(\varphi) &= \mathsf{f} \text{ if } \Delta \nvdash_{\mathcal{L}} \varphi \text{ and } \Delta \vdash_{\mathcal{L}} -\varphi, \\ v(\varphi) &= \mathsf{n} \text{ if } \Delta \nvdash_{\mathcal{L}} \varphi \text{ and } \Delta \nvdash_{\mathcal{L}} -\varphi, \\ v(\varphi) &= \mathsf{b} \text{ if } \Delta \vdash_{\mathcal{L}} \varphi \text{ and } \Delta \vdash_{\mathcal{L}} -\varphi, \end{aligned}$$

is a homomorphism of algebras. (We use the axioms of Table 3.1 to verify the assumptions of the lemma.) This homomorphism is strict by definition. It follows that the rule $\Delta \vdash \varphi$, hence also the rule $\Gamma \vdash \varphi$, fails in $\mathbb{BD}_4$. □

Theorem 3.9 (Completeness for the basic extensions of $\mathcal{BD}$). *$\mathcal{LP}$, $\mathcal{K}$, $\mathcal{KO}$, $\mathcal{ETL}$, and $\mathcal{CL}$ are the extensions of $\mathcal{BD}$ by the following rules:*

(i) $\mathcal{LP}$ by the rule $\emptyset \vdash p \vee -p$.

(ii) $\mathcal{K}$ by the rule $(p \wedge -p) \vee q \vdash q$.

(iii) $\mathcal{KO}$ by the rule $(p \wedge -p) \vee r \vdash (q \vee -q) \vee r$.

(iv) $\mathcal{ETL}$ by the rule $p, -p \vee q \vdash q$.

(v) $\mathcal{CL}$ by the rules $\emptyset \vdash p \vee -p$ and $p, -p \vee q \vdash q$.

Proof.

(i) holds by our description of consequence in $\mathcal{LP}$ (Proposition 3.5).

(ii) holds by our description of consequence in $\mathcal{K}$ (Proposition 3.6).

(iv) and (v) hold by virtue of the translations obtaining between $\mathcal{ETL}$ and $\mathcal{CL}$ on the one hand and $\mathcal{BD}$ and $\mathcal{LP}$ on the other (Theorem 3.7).

(iii) The rule $(p \wedge -p) \vee r \vdash (q \vee -q) \vee r$ is valid in both $\mathcal{LP}$ and $\mathcal{K}$. Conversely, suppose that $\varphi \vdash_{\mathcal{K}} \psi$. Then $\varphi \vdash_{\mathcal{BD}} \psi, \{p \wedge -p \mid p \in \mathrm{At}(\varphi)\}$ by Proposition 3.6, therefore $\varphi \vdash \psi \vee (\varphi \wedge \bigwedge\{q \vee -q \mid q \in \mathrm{At}(\psi)\})$ holds in the extension of $\mathcal{BD}$ by the rule $(p \wedge -p) \vee r \vdash (q \vee -q) \vee r$. But $\varphi \vdash_{\mathcal{LP}} \psi$ implies $\varphi \wedge \bigwedge\{q \vee -q \mid q \in \mathrm{At}(\psi)\} \vdash_{\mathcal{BD}} \psi$ by Proposition 3.5, therefore also $\psi \vee (\varphi \wedge \bigwedge\{q \vee -q \mid q \in \mathrm{At}(\psi)\}) \vdash_{\mathcal{BD}} \psi$. □

Observe that the rule $(p \wedge -p) \vee q \vdash q$, which axiomatized the logic $\mathcal{K}$, is equivalent in $\mathcal{BD}$ to the well-known rule of *resolution* $p \vee q, -q \vee r \vdash r$.

Fact 3.10. *$\mathcal{KO} \vee \mathcal{ETL} = \mathcal{K}$. $\mathcal{LP} \vee \mathcal{ETL} = \mathcal{CL}$.*

Proof. The second claim follows immediately from the completeness theorem for $\mathcal{CL}$ (Theorem 3.9). To prove the first claim, we need to derive the rule $(p \wedge -p) \vee q \vdash q$ from rules valid in $\mathcal{KO}$ and $\mathcal{ETL}$. Observe that

$$\begin{aligned}
(p \wedge -p) \vee q &\vdash_{\mathcal{KO}} (q \vee -q) \vee q, \\
(p \wedge -p) \vee q &\vdash_{\mathcal{KO}} -q \vee q, \\
(p \wedge -p) \vee q &\vdash_{\mathcal{KO}} (p \wedge -p \wedge -q) \vee q, \\
(p \wedge -p) \vee q &\vdash_{\mathcal{KO}} -(p \vee -p \vee q) \vee q.
\end{aligned}$$

But $(p \wedge -p) \vee q \vdash_{\mathcal{BD}} p \vee -p \vee q$ and $p \vee -p \vee q, -(p \vee -p \vee q) \vee q \vdash_{\mathcal{ETL}} q$. □

To describe the consequence relation of $\mathcal{BD}$ more explicitly, it will be useful to recall the Normal Form Theorem for $\mathcal{BD}$.

Definition 3.11 (Conjunctive and disjunctive normal forms). A *literal* is either an atom or a negated atom. A *conjunctive (disjunctive) clause* is a conjunction (disjunction) of literals. A formula in *conjunctive normal form* is a conjunction of disjunctive clauses, and a formula in *disjunctive normal form* is a disjunction of conjunctive clauses.

In the above definition we admit empty conjunctions and disjunctions, which are interpreted as $\mathtt{t}$ and $\mathtt{f}$, respectively.

Theorem 3.12 [25, Thm 3.9] (Normal Form Theorem for $\mathcal{BD}$). *Each formula is equivalent in $\mathcal{BD}$ to a formula in conjunctive normal form, as well as a formula in disjunctive normal form.*

The normal form theorem now allows us to fully describe the consequence relation of $\mathcal{BD}$. Observe that

$$\begin{aligned}
\Gamma \vdash_{\mathcal{BD}} \bigwedge_{i \in I} \varphi_i &\iff \Gamma \vdash_{\mathcal{BD}} \varphi_i \text{ for each } i \in I, \\
\Gamma, \bigvee_{i \in I} \varphi_i \vdash_{\mathcal{BD}} \psi &\iff \Gamma, \varphi_i \vdash_{\mathcal{BD}} \psi \text{ for each } i \in I,
\end{aligned}$$

therefore to fully describe consequence in $\mathcal{BD}$ it suffices to describe when consequence in $\mathcal{BD}$ obtains between a conjunctive clause on the left and a disjunctive clause on the right.

Proposition 3.13 (Consequence in $\mathcal{BD}$).
Let φ and ψ be a conjunctive and a disjunctive clause, respectively. Then:

(i) $\Gamma \vdash_{\mathcal{BD}} \psi$ if and only if $\gamma \vdash_{\mathcal{BD}} \psi$ for some $\gamma \in \Gamma$.

(ii) $\varphi \vdash_{\mathcal{BD}} \bigvee \Delta$ if and only if $\varphi \vdash_{\mathcal{BD}} \delta$ for some $\delta \in \Delta$ with Δ finite.

(iii) $\varphi \vdash_{\mathcal{BD}} \psi$ if and only if φ and ψ share a literal.

Proof.
(i) By the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) we may assume without loss of generality that every formula in Γ is a disjunctive clause. If $\gamma \nvdash_{\mathcal{BD}} \psi$ for each $\gamma \in \Gamma$, this means that each $\gamma \in \Gamma$ contains a literal which does not occur in ψ. The unique valuation on $\mathbb{BD}_\mathbf{4}$ which assigns an undesignated value to each literal which occurs in ψ and a designated value to every other literal then witnesses that $\Gamma \nvdash_{\mathcal{BD}} \psi$.
(ii) follows from (i) by contraposition (Theorem 3.3).
(iii) follows from (i) and (ii). $\square$

In fact, not only do normal forms exist in $\mathcal{BD}$, but they are in a suitable sense unique, unlike normal forms in classical logic, where we have e.g.

$$(-p \vee q) \wedge (-q \vee r) \wedge (-r \vee p) \dashv\vdash_{\mathcal{CL}} (-p \vee r) \wedge (-r \vee q) \wedge (-q \vee p).$$

Definition 3.14 (Irredundant normal forms).
A conjunctive (disjunctive) clause is *irredundant* if it contains each literal at most once. A formula in *irredundant conjunctive (disjunctive) normal form* is a conjunction (disjunction) of irredundant disjunctive (conjunctive) clauses φ_i for $i \in I$ and moreover $i \neq j$ for $i, j \in I$ implies that φ_i contains some literal not contained in φ_j.

Up to permutation of conjuncts and disjuncts, formulas in conjunctive (disjunctive) normal form may be identified with finite families of finite sets of literals. We now show that each formula of $\mathcal{BD}$ is equivalent to a unique conjunctively (disjunctively) interpreted irredundant family of sets of literals.

Theorem 3.15 (Irredundant Normal Form Theorem for $\mathcal{BD}$).
Each formula is equivalent in $\mathcal{BD}$ to a formula in irredundant conjunctive (disjunctive) normal form which is unique up to permutation of conjuncts and disjuncts.

Proof. We only prove the claim for the irredundant conjunctive normal form. We already know that each formula is equivalent to a formula in conjunctive normal form, which is then easily transformed into a formula in irredundant conjunctive normal form. It remains to prove uniqueness. Let therefore $\bigwedge_{i \in I} \varphi_i$ and $\bigwedge_{j \in J} \psi_j$ be two equivalent formulas in irredundant conjunctive normal form. Then $\{\varphi_i \mid i \in I\} \vdash_{\mathcal{BD}} \psi_j$ for each $j \in J$, therefore by the above description of consequence in $\mathcal{BD}$ (Proposition 3.13) there is some $I \in I$ such that $\varphi_i \vdash_{\mathcal{BD}} \psi_j$. By a symmetric argument there is then some $k \in J$ such that $\psi_k \vdash_{\mathcal{BD}} \varphi_i \vdash_{\mathcal{BD}} \psi_j$. But then $j = k$ by irredundancy, hence φ_i and ψ_j are equivalent. By irredundancy and Proposition 3.13 this means that φ_i and ψ_j are the same disjunctive clause up to permutation of disjuncts. Thus for each $j \in J$ there is some $i \in I$ (and by symmetry for each $i \in I$ there is some $j \in J$ such that ψ_j and φ_i are the same up to permutation of disjuncts. In other words, the formulas $\bigwedge_{i \in I} \varphi_i$ and $\bigwedge_{j \in J} \psi_j$ are the same up to permutation of conjuncts and disjuncts. □

3.4 Matrix models

We end our review of the basic extensions of $\mathcal{BD}$ by investigating their matrix models. In particular, we shall recall the description of the Leibniz congruences and the Leibniz reduced models of $\mathcal{BD}$ due to Font [25] and the description of the reduced models of the basic extensions of $\mathcal{BD}$ introduced due to Rivieccio [67]. We then discuss the subdirect decomposition of reduced models of $\mathcal{BD}$ and its basic extensions.

Proposition 3.16 [25, Prop 3.12] ($\mathcal{BD}$-filters on DMAs).
On a De Morgan algebra the $\mathcal{BD}$-filters are precisely the lattice filters.

The following lemma is an important consequence of the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12). The proof below is due to Font, we only include it because we shall rely on the lemma several times in this thesis.

Proposition 3.17 [25, Prop 3.13] (Leibniz congruences of $\mathcal{BD}$-filters).
Let $\langle \mathbf{A}, F \rangle$ be a model of $\mathcal{BD}$. Then $\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$ if and only if for all $c \in \mathbf{A}$

$$a \vee c \in F \iff b \vee c \in F \qquad \textit{and} \qquad -a \vee c \in F \iff -b \vee c \in F.$$

Proof. The left-to-right implication holds by the definition of the Leibniz Congruence. Conversely, suppose that $\langle a, b \rangle \notin \boldsymbol{\Omega}^{\mathbf{A}} F$. Then there are $c_1, \dots, c_n \in \mathbf{A}$ and a term $t(x, y_1, \dots, y_n)$ such that $t(a, c_1, \dots, c_n) \in F$ and $t(b, c_1, \dots, c_n) \notin F$. By the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) we can take t to be a conjunction of disjunctive clauses $t := \bigwedge_{i \in I} t_i$. It follows that $t_i(a, c_1, \dots, c_n) \in F$ and $t_i(a, c_1, \dots, c_n)$ for some $t_i := \bigvee_{j \in J} l_j$. But then the literals y_k and $-y_k$ evaluated to c_k and $-c_k$ may be replaced by a single atom y evaluated to the join of the appropriate elements c_k and $-c_k$. That is, we have $u(a, c) \in F$ and $u(b, c) \notin F$ for $u \in \{x \vee y, -x \vee y, x \vee -x \vee y\}$. If $u \in \{x \vee y, -x \vee y\}$, then the right-hand side of the equivalence fails. If $u = x \vee -x \vee y$, then $a \vee -a \vee c \in F$ and $b \vee -b \vee c \notin F$. But then either $a \vee -a \vee c \in F$ and $b \vee -a \vee c \notin F$ or $-a \vee b \vee c \in F$ and $-b \vee b \vee c \notin F$. □

The next proposition was in fact formulated by Font [25] with $b < a$ rather than $a \nleq b$. However, the formulation below will be more convenient when describing the reduced models of extensions of $\mathcal{BD}$.

Proposition 3.18 [**25, Thm 3.14**] **(Reduced models of $\mathcal{BD}$).**
A matrix $\langle \mathbf{A}, F \rangle$ is a Leibniz reduced model of $\mathcal{BD}$ if and only if

(i) $\mathbf{A}$ *is a De Morgan algebra,*

(ii) F *is a lattice filter on* $\mathbf{A}$*, and*

(iii) $a \nleq b$ *in* $\mathbf{A}$ *implies that for some* $c \in \mathbf{A}$ *either* $a \vee c \in F$ *and* $b \vee c \notin F$ *or* $-b \vee c \in F$ *and* $-a \vee c \notin F$*.*

Proof. If $\mathbf{A}$ is a De Morgan algebra and F is a lattice filter on $\mathbf{A}$, then $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{BD}$ (Proposition 3.16). If (iii) holds, then the matrix $\langle \mathbf{A}, F \rangle$ is reduced by the previous proposition.

Conversely, let $\langle \mathbf{A}, F \rangle$ be a reduced model of $\mathcal{BD}$. To prove that $\mathbf{A}$ is a De Morgan algebra, take an equational axiomatization of De Morgan algebras and use the previous proposition to check that for each equation $t(x_1, \dots, x_n) \approx u(x_1, \dots, x_n)$ in this axiomatization and each model $\langle \mathbf{A}, F \rangle$ of $\mathcal{BD}$ we have $\langle t(a_1, \dots, a_n), u(a_1, \dots, a_n) \rangle \in \mathbf{\Omega}^{\mathbf{A}} F$. The filter F is a lattice filter because $p, q \vdash_{\mathcal{BD}} p \wedge q$.

To prove (iii), suppose that $a \nleq b$. Then $a \wedge b < a$, therefore by (ii) $(a \wedge b) \vee c \in F$ implies $b \vee c \in F$, and $-a \vee c \in F$ implies $-(a \wedge b) \vee c \in F$. By the above description of Leibniz congruences of $\mathcal{BD}$-filters and the previous observation, there is some $c \in \mathbf{A}$ such that either $(a \wedge b) \vee c \notin F$ and $a \vee c \in F$ or $-a \vee c \notin F$ and $-a \vee -b \vee c = -(a \wedge b) \vee c \in F$. But in the former case $a \vee c \in F$ and $(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c \notin F$ implies that $b \vee c \notin F$, while in the latter case $-a \vee d \notin F$ and $-b \vee d \in F$ for $d := -a \vee c$. $\square$

The above proposition implies that we may in fact restrict to matrices consisting of a De Morgan algebra and a lattice filter when proving completeness theorems for super-Belnap logics. Such matrices will be called De Morgan matrices.

Definition 3.19 (De Morgan matrices).
A *De Morgan matrix* is a matrix $\langle \mathbf{A}, F \rangle$ such that $\mathbf{A}$ is a De Morgan algebra and F is a lattice filter on $\mathbf{A}$.

The reduced models of the basic extensions of $\mathcal{BD}$ were originally described by Rivieccio [67]. The proof for $\mathcal{ETL}$ was first published in [66, Prop 9] and the remaining proofs were published in [2, Thm 3.7]. Note, however, that our proof for $\mathcal{ETL}$ is rather different from Rivieccio's proof.

Proposition 3.20 [**2, 66, 67**] **(Reduced models of extensions of $\mathcal{BD}$).**
A Leibniz reduced model $\langle \mathbf{A}, F \rangle$ of $\mathcal{BD}$ is a model of:

(i) $\mathcal{ETL}$ *if and only if* $F = \{\mathtt{t}\}$*,*

(ii) $\mathcal{KO}$ *if and only if* $\mathbf{A}$ *is a Kleene algebra,*

(iii) $\mathcal{LP}$ *if and only if* $\mathbf{A}$ *is a Kleene algebra and* $F = \{a \in \mathbf{A} \mid -a \leq a\}$*,*

(iv) $\mathcal{K}$ *if and only if* $\mathbf{A}$ *is a Kleene algebra and* $F = \{\mathtt{t}\}$*,*

(v) $\mathcal{CL}$ *if and only if* $\mathbf{A}$ *is a Boolean algebra and* $F = \{\mathtt{t}\}$*.*

Proof. We omit the easy proofs of the right-to-left implications (for $\mathcal{K}$ we use the fact that $\mathcal{K} = \mathcal{KO} \vee \mathcal{ETL}$).

(i) To prove that $F = \{\mathtt{t}\}$, i.e. that F is a singleton, it suffices to show that $\theta := \mathrm{Cg}^{\mathbf{A}}\langle a, b\rangle$ is compatible with F for each $a, b \in F$: because the matrix $\langle \mathbf{A}, F\rangle$ is reduced, this implies that $\mathrm{Cg}^{\mathbf{A}}\langle a, b\rangle = \Delta_{\mathbf{A}}$ and $a = b$. Suppose therefore that $x \in F$ and $\langle x, y\rangle \in \theta$, assuming without loss of generality that $a \leq b$. Then $(x \wedge a) \vee -a = (y \wedge a) \vee -a$ by the equational description of principal congruences of De Morgan lattices (Theorem 2.10). But $x \in F$ and $a \in F$, therefore $(y \wedge a) \vee -a = (x \wedge a) \vee -a \in F$. Because $p, -p \vee \vdash_{\mathcal{ETL}} q$, it follows that $y \wedge a \in F$ and $y \in F$.

(ii) Suppose that $\mathbf{A}$ is not a Kleene algebra. Then $a \nleq b$ for some $a, b \in \mathbf{A}$ with $a \leq -a$ and $-b \leq b$. Therefore by the description of reduced models of $\mathcal{BD}$ (Proposition 3.18) there is some $c \in \mathbf{A}$ such that either $a \vee c \in F$ and $b \vee c \notin F$ or $-b \vee c \in F$ and $-a \vee c \notin F$. In the former case the valuation $v : \mathbf{Fm} \to \mathbf{A}$ with $v(p) := a$ and $v(q) := b$ and $v(r) := c$ witnesses the failure of the rule $(p \wedge -p) \vee r \vdash q \vee -q \vee r$. In the latter case the failure of this rule is witnessed by the valuation v with $v(p) := -b$ and $v(q) := -a$ and $v(r) := c$.

(iii) $\mathbf{A}$ is a Kleene algebra by (i). The inclusion $\{a \in \mathbf{A} \mid -a \leq a\} \subseteq F$ holds because $\emptyset \vdash_{\mathcal{LP}} a \vee -a$. Conversely, suppose that $a \in F$ but $-a \nleq a$. Then by Proposition 3.18 there is some $b \in \mathbf{A}$ such that $-a \vee b \in F$ and $a \vee b \notin F$, contradicting that $a \in F$.

(iv) follows from (i) and (ii), since $\mathcal{KO} \leq \mathcal{K}$ and $\mathcal{ETL} \leq \mathcal{K}$.

(v) By (i) we have $F = \{\mathtt{t}\}$. But $\emptyset \vdash_{\mathcal{CL}} p \vee -p$, therefore $\mathbf{A}$ satisfies the equation $x \vee -x \approx \mathtt{t}$ which defines the variety of Boolean algebras relative to De Morgan algebras. □

It is worth remarking that the claim for $\mathcal{ETL}$ in fact does *not* rely on the presence of the constants in the language (indeed, Rivieccio originally proved it for the constant-free fragment of $\mathcal{ETL}$). That is, although in general the reduced models of the constant-free fragment of $\mathcal{BD}$ need not be bounded, the reduced models of the constant-free fragment of $\mathcal{ETL}$ are bounded.

Finally, we discuss the subdirect decomposition of reduced models of $\mathcal{BD}$. Recall that each reduced model of a finitary logic is a subdirect product of subdirectly irreducible reduced models.

Proposition 3.21 (Subdirect decomposition of models of $\mathcal{BD}$). *The only relatively (finitely) subdirectly irreducible reduced models of*

(i) $\mathcal{BD}$ *are* $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, $\mathbb{P}_\mathbf{3}$, *and* $\mathbb{BD}_\mathbf{4}$.

(ii) $\mathcal{KO}$ *are* $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, *and* $\mathbb{P}_\mathbf{3}$.

(iii) $\mathcal{LP}$ *are* $\mathbb{B}_\mathbf{2}$ *and* $\mathbb{P}_\mathbf{3}$.

(iv) $\mathcal{ETL}$ *are* $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, *and* $\mathbb{ETL}_\mathbf{4}$.

(v) $\mathcal{K}$ *are* $\mathbb{B}_\mathbf{2}$ *and* $\mathbb{K}_\mathbf{3}$.

The only relatively (finitely) subdirectly irreducible model of

(vi) $\mathcal{CL}$ *is* $\mathbb{B}_\mathbf{2}$.

Proof.
(i) The matrices in question are relatively subdirectly irreducible reduced models of $\mathcal{BD}$. Conversely, let $\langle \mathbf{A}, F \rangle$ be a non-trivial relatively finitely subdirectly irreducible reduced models of $\mathcal{BD}$. Then $\langle \mathbf{A}, F \rangle$ is a De Morgan matrix (Proposition 3.18). We now show that F is prime as a lattice filter. Suppose that $a, b \notin F$. Then the intersection of the lattice filter generated by F, a and the lattice filter generated by F, b (both of which are $\mathcal{BD}$-filters) cannot be F. Thus there are $f_1, f_2 \in F$ such that $(a \wedge f_1) \vee (b \wedge f_2) \notin F$, hence $(a \vee b) \wedge f = (a \wedge f) \vee (b \wedge f) \notin F$ for $f := f_1 \wedge f_2 \in F$. It follows that $a \vee b \notin F$.

Because F is a prime lattice filter, the description of Leibniz congruences of $\mathcal{BD}$-filters (Proposition 3.17) yields that $\langle a, b \rangle \in \mathbf{\Omega}^{\mathbf{A}} F$ if and only if both $a \in F \iff b \in F$ and $-a \in F \iff -b \in F$. It follows that $\mathbf{A}$ has at most four elements. Since the four-element chain which is not the underlying algebra of any reduced model of $\mathcal{BD}$, the algebra $\mathbf{A}$ is either $\mathbf{B_2}$, $\mathbf{K_3}$, or $\mathbf{DM_4}$. Ruling out the matrix $\mathbb{ETL}_\mathbf{4}$, which is not finitely subdirectly irreducible because $\{\mathsf{t}\} = \{\mathsf{t}, \mathsf{b}\} \cap \{\mathsf{t}, \mathsf{n}\}$, leaves $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, $\mathbb{P}_\mathbf{3}$, and $\mathbb{BD}_\mathbf{4}$.

(ii) The same argument applies, and moreover we know that $\mathbf{A}$ is a Kleene algebra by Proposition 3.20.

(iii) Each reduced model of $\mathcal{LP}$ has the form $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a Kleene algebra and $F = \{a \in \mathbf{A} \mid -a \leq a\}$ (Proposition 3.18). But then $\mathbf{A}$ is a subdirect product of subdirectly irreducible Kleene algebras, i.e. $\mathbf{B_2}$ and $\mathbf{K_3}$ (Theorem 2.7), therefore the matrix $\langle \mathbf{A}, F \rangle$ is a subdirect product of the matrices $\mathbb{B}_\mathbf{2}$ and $\mathbb{P}_\mathbf{3}$, i.e. the extensions of $\mathbf{B_2}$ and $\mathbf{K_3}$ by the sets $\{a \in \mathbf{B_2} \mid -a \leq a\}$ and $\{a \in \mathbf{K_3} \mid -a \leq a\}$. If $\langle \mathbf{A}, F \rangle$ is subdirectly irreducible, then it is isomorphic to one of these matrices.

(iv) Each reduced model of $\mathcal{ETL}$ has the form $\langle \mathbf{A}, \{\mathsf{t}\} \rangle$ for some De Morgan algebra $\mathbf{A}$ (Proposition 3.20). But $\mathbf{A}$ is a subdirect product of the algebras $\mathbf{B_2}$, $\mathbf{K_3}$, $\mathbf{DM_4}$ (Theorem 2.7), therefore $\langle \mathbf{A}, \{\mathsf{t}\} \rangle$ is a subdirect product of the matrices $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, $\mathbb{ETL}_\mathbf{4}$. If $\langle \mathbf{A}, \{\mathsf{t}\}\rangle$ is subdirectly irreducible, then it is isomorphic to one of these matrices.

(v) The same argument applies, and moreover we know that $\mathbf{A}$ is a Kleene algebra by Proposition 3.20.

(vi) The same argument applies, and moreover we know that $\mathbf{A}$ is a Boolean algebra by Proposition 3.20. $\square$

It will also be useful to introduce what we call witnessed subdirect decomposition. A matrix $\mathbb{M}$ is a *witnessed subdirect product* of the matrices $\mathbb{M}_i$ for $i \in I$ if it is a subdirect product such that for each designated element $b \in \mathbb{M}_i$ there is some designated $a \in \mathbb{M}$ with $\pi_i(a) \leq b$, where $\pi_i : \Pi_{i \in I}\mathbb{M}_i \to \mathbb{M}_i$ is the appropriate projection map.

Proposition 3.22
(Witnessed subdirect decomposition of models of $\mathcal{BD}$).

(i) Each reduced model of $\mathcal{BD}$ is a witnessed subdirect product of the matrices $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, $\mathbb{P}_\mathbf{3}$, $\mathbb{BD}_\mathbf{4}$, and $\mathbb{ETL}_\mathbf{4}$.

(ii) Each reduced model of $\mathcal{KO}$ is a witnessed subdirect product of the matrices $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, and $\mathbb{P}_\mathbf{3}$.

(iii) Each reduced models of $\mathcal{CL}$ is a witnessed subdirect power of $\mathbb{B}_\mathbf{2}$.

Proof. The claim for $\mathcal{CL}$ follows trivially from the previous proposition, and the claim for $\mathcal{KO}$ is proved in exactly the same way as the claim for $\mathcal{BD}$. We thus only prove the claim for $\mathcal{BD}$.

By the previous proposition the matrix $\langle \mathbf{A}, F\rangle$ is a subdirect product of the matrices $\mathbb{B}_\mathbf{2}$, $\mathbb{K}_\mathbf{3}$, $\mathbb{P}_\mathbf{3}$, and $\mathbb{BD}_\mathbf{4}$. That is, $\langle \mathbf{A}, F\rangle \leq \Pi_{i\in I}\langle \mathbf{B}_i, F_i\rangle$ where $\langle \mathbf{B}_i, F_i\rangle \in \{\mathbb{B}_\mathbf{2}, \mathbb{K}_\mathbf{3}, \mathbb{P}_\mathbf{3}, \mathbb{BD}_\mathbf{4}\}$ and the embedding is subdirect. Now instead of the filters F_i consider the filters

$$G_i := \{b \in \mathbf{B}_i \mid b \geq \pi_i(a) \text{ for some } a \in F\},$$

where $\pi_i : \Pi_{i\in I}\mathbf{B}_i \to \mathbf{B}_i$ are the projection maps. In particular, G_i is a lattice filter on $\mathbf{B}_i$ such that $G_i \subseteq F_i$. Let $\mathbb{M}_i := \langle \mathbf{B}_i, G_i\rangle$. Then we have a subdirect embedding $\langle \mathbf{A}, F\rangle \leq \Pi_{i\in I}\mathbb{M}_i$ because for each $a \in \mathbf{A}$

$$\pi_i(a) \in F_i \text{ for each } i \in I \iff \pi_i(a) \in G_i \text{ for each } i \in I.$$

Moreover, the embedding $\langle \mathbf{A}, F\rangle \leq \Pi_{i\in I}\langle \mathbf{B}_i, G_i\rangle$ has the property that for each $b \in G_i$ there is some $a \in F$ such that $\pi_i(a) \leq b$. Finally, $\langle \mathbf{B}_i, G_i\rangle \in \{\mathbb{B}_\mathbf{2}, \mathbb{K}_\mathbf{3}, \mathbb{P}_\mathbf{3}, \mathbb{BD}_\mathbf{4}, \mathbb{ETL}_\mathbf{4}\}$ because $G_i \subseteq F_i$ and $\mathbf{B}_i \in \{\mathbf{B_2}, \mathbf{K_3}, \mathbf{DM_4}\}$. $\square$

Chapter 4

Explosive extensions

In this chapter, we prepare the ground for the investigation of the lattice of super-Belnap logics by introducing the notion of an *explosive extension* of a logic. Just like axiomatic extensions of a base logic $\mathcal{B}$ are extensions of $\mathcal{B}$ by a set of axioms (theorems), explosive extensions of $\mathcal{B}$ are extensions of $\mathcal{B}$ by a set of antiaxioms (antitheorems). Here an *antitheorem* of $\mathcal{L}$ is a set of formulas which cannot be jointly designated in a non-trivial model of $\mathcal{L}$.

Like axiomatic extensions, the explosive extensions of $\mathcal{B}$ form a lattice. Unlike the lattice of axiomatic extensions, this lattice is always distributive. Although axiomatizing and describing the models of the intersection of two logics are complicated tasks in general, we shall see that they simplify substantially for intersections of explosive extensions with arbitrary extensions.

In later chapters, we shall usually take the base logic to be $\mathcal{BD}$ or $\mathcal{ETL}$. In particular, the lattice of finitary explosive extensions of $\mathcal{BD}$ will be studied in Chapter 7 (Super-Belnap logics and finite graphs) and shown to admit a simple description in terms of finite graphs. In the current chapter, however, we only prove general results about explosive extensions which do not depend on any particular choice of the base logic.

The *explosive part* of an extension $\mathcal{L}$ of $\mathcal{B}$ is the second key notion introduced in this chapter. It is defined as the strongest explosive extension of $\mathcal{B}$ lying below $\mathcal{L}$. In other words, the explosive part of an extension $\mathcal{L}$ of $\mathcal{B}$ is the extension of $\mathcal{B}$ by the antitheorems of $\mathcal{L}$. Informally speaking, taking the explosive part of $\mathcal{L}$ amounts to forgetting all information about $\mathcal{L}$ apart from its set of antitheorems. The definition of the explosive part of $\mathcal{L}$ is, of course, relative to a choice of the base logic $\mathcal{B}$.

The importance of this notion for our investigation of super-Belnap logics lies in the fact that the logic determined by a product of matrices $\Pi_{i \in I} \mathbb{M}_i$ may be computed from the logics determined by the matrices $\mathbb{M}_i$ and their explosive parts. Being able to compute the explosive parts of the major super-Belnap logics will therefore enable us to provide easy proofs of a slew of completeness theorems in the next chapter.

4.1 Antitheorems

We start by defining antitheorems semantically. Just like theorems are the formulas designated by every valuation, antitheorems are the sets of formulas not designated by any valuation. However, we must take care to exclude valuations on trivial matrices, which of course validate all formulas.

Definition 4.1 (Antitheorems).
A set of formulas Γ is an *antitheorem* of a logic $\mathcal{L}$ if Γ is not jointly designated by any valuation $v : \mathbf{Fm} \to \mathbb{M}$ such that $\mathbb{M}$ is a non-trivial model of $\mathcal{L}$.

We shall abbreviate the claim that Γ is an antitheorem of $\mathcal{L}$ as $\Gamma \vdash_{\mathcal{L}} \emptyset$. We omit the easy proof of the following fact.

Fact 4.2 (Basic properties of antitheorems).
Monotonicity, structurality, and cut hold for antitheorems in the following forms:

(i) If $\Gamma \vdash_{\mathcal{L}} \emptyset$*, then* $\Gamma \vdash_{\mathcal{L}} \varphi$.

(ii) If $\Gamma \vdash_{\mathcal{L}} \emptyset$*, then* $\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$.

(iii) If $\Gamma \vdash_{\mathcal{L}} \emptyset$*, then* $\sigma[\Gamma] \vdash_{\mathcal{L}} \emptyset$ *for each substitution* σ.

(iv) If $\Gamma \vdash_{\mathcal{L}} \delta$ *for each* $\delta \in \Delta$ *and* $\Delta, \Pi \vdash_{\mathcal{L}} \emptyset$*, then* $\Gamma, \Pi \vdash_{\mathcal{L}} \emptyset$.

Just like not all logics have theorems, not all logics have antitheorems. For example, positive classical logic — the fragment of classical logic obtained by restricting to conjunction, disjunction, and implication — does not have any antitheorems. This is because all formulas in this fragment are designated in the two-element Boolean matrix by the valuation which designates all propositional atoms. The same holds for the constant-free fragment of the Belnap–Dunn logic. This is one of the reasons why we include the truth constants in the signature: questions regarding antitheorems which are non-trivial to answer for the Belnap–Dunn logic with the truth constants become trivial for the constant-free fragment.

Fact 4.3. *The constant-free fragment of* $\mathcal{BD}$ *has no antitheorems.*

Proof. The valuation on $\mathbb{BD}_\mathbf{4}$ which assigns b to every atom designates all constant-free formulas. □

It follows immediately from monotonicity for antitheorems that

$$\Gamma \vdash_{\mathcal{L}} \emptyset \implies \Gamma \vdash_{\mathcal{L}} \mathrm{Fm}\,\mathcal{L}.$$

If $\mathcal{L}$ has an antitheorem, then $\mathrm{Fm}\,\mathcal{L}$ is an antitheorem and

$$\Gamma \vdash_{\mathcal{L}} \emptyset \iff \Gamma \vdash_{\mathcal{L}} \mathrm{Fm}\,\mathcal{L}.$$

In order to determine whether $\mathrm{Fm}\,\mathcal{L}$ is indeed an antitheorem, we may use the following syntactic characterization of antitheorems.

Proposition 4.4 (Syntactic description of antitheorems).
The following are equivalent for a suitable invertible substitution $\sigma_{push\ p}$:

(i) Γ *is an antitheorem of* $\mathcal{L}$*.*

(ii) $\sigma[\Gamma] \vdash_{\mathcal{L}} \mathrm{Fm}\,\mathcal{L}$ *for each substitution* σ*.*

(iii) $\sigma_{push\ p}[\Gamma] \vdash_{\mathcal{L}} p$*.*

Proof.
(i) $\implies$ (ii): by monotonicity and structurality for antitheorems we have $\Gamma \vdash_{\mathcal{L}} \emptyset \implies \sigma[\Gamma] \vdash_{\mathcal{L}} \emptyset \implies \sigma[\Gamma] \vdash_{\mathcal{L}} \varphi$ for each φ.

(ii) $\implies$ (iii): trivial.

(iii) $\implies$ (i): we pick $\sigma_{\text{push } p}$ to be an invertible substitution such that no formula in the image of $\sigma_{\text{push } p}$ contains the atom p. For example, pick a sequence of variables p_i for $i \in \omega$ with $p_0 := p$ and define $\sigma_{\text{push } p}(p_i) := p_{i+1}$ and $\sigma_{\text{push } p}(q) := q$ otherwise. For the inverse subsitution $\sigma_{\text{pop } p}$ we take $\sigma_{\text{pop } p}(p_{i+1}) := p_i$ and $\sigma_{\text{pop } p}(q) := q$ otherwise.

Now suppose that Γ is not an antitheorem of $\mathcal{L}$. Then there is a non-trivial model $\mathbb{M}$ of $\mathcal{L}$ and a valuation $v : \mathbf{Fm} \to \mathbb{M}$ which designates Γ. Consider the valuation $w : \mathbf{Fm} \to \mathbb{M}$ such that $w(p)$ is a non-designated element of $\mathbb{M}$ and $w(q) := v(\sigma_{\text{pop } p}(q))$ for each atom q distinct from p. The valuation w then witnesses that $\sigma_{\text{push } p}[\Gamma] \nvdash_{\mathcal{L}} p$, since $w[\sigma_{\text{push } p}[\Gamma]] = v[(\sigma_{\text{pop } p} \circ \sigma_{\text{push } p})[\Gamma]] = v[\Gamma]$ is designated in $\mathbb{M}$ and $w(p)$ is not. $\square$

In view of the above proposition, we shall talk about the rule $\Gamma \vdash \emptyset$ as a shorthand for the rule $\sigma_{\text{push } p}[\Gamma] \vdash_{\mathcal{L}} p$ (given some choice of p and $\sigma_{\text{push } p}$). In practice, this technical complication can usually be avoided: if there is some atom p which does not occur in Γ, e.g. whenever Γ is finite, we may simply identify $\Gamma \vdash \emptyset$ with $\Gamma \vdash p$.

4.2 Explosive extensions and explosive parts

We shall not only be interested in the antitheorems of a given logic, but also in the extensions of that logic by antitheorems. Throughout the rest of the chapter, we pick some base logic $\mathcal{B}$ and assume that all logics that we talk about are extensions of $\mathcal{B}$ and that all matrices that we talk about are models of $\mathcal{B}$. This involves no loss of generality, as we may always choose the base logic $\mathcal{B}$ to be the smallest logic in the given language.

We now introduce the notion of an explosive (or antiaxiomatic) extension as the dual of the notion of an axiomatic extension, and show that the explosive extensions of a given base logic form a distributive lattice. If we take the base logic to be $\mathcal{BD}$, this lattice turns out to be an interesting object study. The lattice of axiomatic extensions of $\mathcal{BD}$, on the other hand, is rather uninteresting: as we shall see, $\mathcal{BD}$ only has one non-trivial proper axiomatic extension, namely $\mathcal{LP}$. The exact opposite is the case for intuitionistic logic: the lattice of axiomatic extensions is the interesting object there, while (as the reader may wish to verify) the lattice of explosive extensions of intuitionistic logic is trivial.

Definition 4.5 (Explosive extensions).
An *explosive extension* of $\mathcal{B}$ is an extension of $\mathcal{B}$ by a set of antitheorems.

The consequence relation of extension of $\mathcal{B}$ by a set of antitheorems may be described quite explicitly. (It may be helpful for the reader to try to formulate precisely the dual lemma for axiomatic extensions.)

Proposition 4.6 (Consequence in explosive extensions).
Let $\mathcal{L}_{exp}$ be the extension of $\mathcal{B}$ by the rules $\Delta_i \vdash \emptyset$ for $i \in I$. Then

$$\Gamma \vdash_{\mathcal{L}_{exp}} \varphi \iff \Gamma \vdash_{\mathcal{B}} \varphi \text{ or } \Gamma \vdash_{\mathcal{B}} \sigma[\Delta_i] \text{ for some } i \in I \text{ and some } \sigma.$$

Proof. The right-to-left direction is trivial. Conversely, it suffices to verify that the right-hand side of the equivalence defines a logic and that Δ_i for $i \in I$ are antitheorems of this logic. The implication then follows by virtue of the fact that $\mathcal{L}_{\text{exp}}$ is by definition the smallest such logic.

We only verify that the relation on the right-hand side of the equivalence satisfies cut. Suppose thefore that $\Gamma \vdash \varphi$ holds in this relation for all $\varphi \in \Phi$, as does $\Phi, \Delta \vdash \psi$. If it is the case that $\Gamma \vdash_{\mathcal{B}} \sigma[\Delta_i]$ for some σ and some $i \in I$, then $\Gamma, \Delta \vdash \psi$ holds in this relation. Otherwise, we have $\Gamma \vdash_{\mathcal{B}} \varphi$ for each $\varphi \in \Phi$. Cut then yields either $\Gamma, \Delta \vdash_{\mathcal{B}} \psi$ or $\Gamma, \Delta \vdash_{\mathcal{B}} \sigma[\Delta_i]$ for some σ and some $i \in I$. In either case $\Gamma, \Delta \vdash \psi$ holds in this relation. □

Just like the class of all models of an axiomatic extension of $\mathcal{B}$ is closed under homomorphic images (relativized to models of $\mathcal{B}$), the class of all models of an explosive extension of $\mathcal{B}$ is closed under homomorphic preimages (relativized to models of $\mathcal{B}$).

Fact 4.7 (Models of explosive extensions).
Let $\mathcal{L}_{exp}$ be an explosive extension of $\mathcal{B}$ and $h : \mathbb{M} \to \mathbb{N}$ be a homomorphism of non-trivial models of $\mathcal{B}$. Then $\mathbb{M}$ is a model of $\mathcal{L}_{exp}$ whenever $\mathbb{N}$ is.

Proof. Suppose that the matrix $\mathbb{M}$ is not a model of $\mathcal{L}_{\text{exp}}$. Then by the previous proposition there is some antitheorem Γ of $\mathcal{L}_{\text{exp}}$ such that $\Gamma \vdash \emptyset$ fails in $\mathbb{M}$. It follows that there is a valuation $v : \mathbf{Fm} \to \mathbb{M}$ which designates Γ. But then the valuation $w : \mathbf{Fm} \to \mathbb{N}$ such that $w(\varphi) := h(v(\varphi))$ designates Γ in $\mathbb{N}$, hence $\mathbb{N}$ is not a model of $\mathcal{L}_{\text{exp}}$. □

Fact 4.8 (Filters of explosive extensions).
Let $\mathcal{L}_{exp}$ be an explosive extension of $\mathcal{B}$ and let $F \subseteq G$ be $\mathcal{B}$-filters on $\mathbf{A}$. Then $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{L}_{exp}$ whenever $\langle \mathbf{A}, G \rangle$ is.

Proof. The identity map is a homomorphism from $\langle \mathbf{A}, F\rangle$ to $\langle \mathbf{A}, G\rangle$. $\square$

In fact, the converse assertion was proved by Stronkowski [72, Thm 3.7].[1]

Theorem 4.9 [72] (Semantic description of explosive extensions). *Let $\mathcal{L}$ be an extension of $\mathcal{B}$. Then $\mathcal{L}$ is an explosive extension of $\mathcal{B}$ if and only if for each homomorphism $h : \mathbb{M} \to \mathbb{N}$ of non-trivial models of $\mathcal{B}$ we have $\mathbb{M} \in \operatorname{Mod} \mathcal{L}$ whenever $\mathbb{N} \in \operatorname{Mod} \mathcal{L}$.*

Explosive extensions are in fact the (infinitary) equality-free counterpart of antivarieties of algebras studied by Gorbunov and Kravchenko [30, 31]. These are classes of algebras axiomatized by negative universal clauses (finite disjunctions of negated equalities), or equivalently quasivarieties which are closed under homomorphic pre-images.

Definition 4.10 (Explosive parts).
The *explosive part* of a logic $\mathcal{L}$ relative to $\mathcal{B}$, denoted $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}$, is the largest explosive extension of $\mathcal{B}$ below $\mathcal{L}$.

Note that an arbitrary join of explosive extensions is by definition an explosive extension itself, therefore it is legitimate to talk about the largest explosive extension lying below a given logic.

Clearly, $\mathcal{L}$ is a explosive extension of $\mathcal{B}$ if and only if $\mathcal{L} = \operatorname{Exp}_{\mathcal{B}} \mathcal{L}$.

Proposition 4.11 (Consequence in explosive parts).
$\Gamma \vdash_{\operatorname{Exp}_{\mathcal{B}} \mathcal{L}} \varphi$ if and only if either $\Gamma \vdash_{\mathcal{B}} \varphi$ or $\Gamma \vdash_{\mathcal{L}} \emptyset$.

Proof. The right-hand side of the equivalence defines a logic by the basic properties of antitheorems (Fact 4.2). Moreover, this logic is the explosive extension of $\mathcal{B}$ by the rules $\Gamma \vdash \emptyset$, where Γ ranges over the antitheorems of $\mathcal{L}$. It remains to show that each antitheorem of $\mathcal{L}$ is in fact an antitheorem of this logic. If Γ is not an antitheorem of this logic, then $\sigma_{\text{push } p}[\Gamma] \nvdash p$ in this logic by the syntactic description of antitheorems (Proposition 4.4), hence $\sigma_{\text{push } p}[\Gamma] \nvdash_{\mathcal{L}} \emptyset$. But then $\Gamma \nvdash_{\mathcal{L}} \emptyset$ by structurality for antitheorems. $\square$

Let us now state some basic observations about the explosive part map $\operatorname{Exp}_{\mathcal{B}} : \operatorname{Ext} \mathcal{B} \to \operatorname{Ext} \mathcal{B}$.

Fact 4.12. *$\operatorname{Exp}_{\mathcal{B}}$ is an interior operator on $\operatorname{Ext} \mathcal{B}$. That is:*

(i) $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_1 \leq \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$ *if* $\mathcal{B} \leq \mathcal{L}_1 \leq \mathcal{L}_2$,

(ii) $\operatorname{Exp}_{\mathcal{B}} \operatorname{Exp}_{\mathcal{B}} \mathcal{L} = \operatorname{Exp}_{\mathcal{B}} \mathcal{L} \leq \mathcal{L}$ *if* $\mathcal{B} \leq \mathcal{L}$.

Fact 4.13. *If $\mathcal{L}_1 \vee \mathcal{L}_2 = \mathcal{L}_1 \cup \mathcal{L}_2$, then $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_1 \vee \mathcal{L}_2 = \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_1 \cup \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$.*

[1] To be more precise, it is an immediate corollary of his characterization of classes of matrices axiomatized by explosive rules as precisely those classes which are closed under submatrices and homomorphic pre-images of matrices.

The following proposition shows that $\operatorname{Exp}_{\mathcal{B}}$ is in fact a *topological* interior operator on $\operatorname{Ext}\mathcal{B}$. That is, $\operatorname{Exp}_{\mathcal{B}}(\mathcal{L}_1 \cap \mathcal{L}_2) = \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_1 \cap \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$.

Proposition 4.14 (Meets of explosive extensions).
$\operatorname{Exp}_{\mathcal{B}} \bigcap_{i \in I} \mathcal{L}_i = \bigcap_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$ *for each family* $\mathcal{L}_i \in \operatorname{Ext}\mathcal{B}$ *with* $i \in I$.

Proof. The operator $\operatorname{Exp}_{\mathcal{B}}$ is monotone, thus $\operatorname{Exp}_{\mathcal{B}} \bigcap_{i \in I} \mathcal{L}_i \subseteq \bigcap_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$. Conversely, suppose that $\Gamma \vdash \varphi$ holds in $\bigcap_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$. If $\Gamma \vdash_{\mathcal{B}} \varphi$, then $\Gamma \vdash \varphi$ holds in $\operatorname{Exp}_{\mathcal{B}} \bigcap_{i \in I} \mathcal{L}_i$. But if $\Gamma \nvdash_{\mathcal{B}} \varphi$, then $\Gamma \vdash \emptyset$ holds in each $\mathcal{L}_i$ by the description of consequence in explosive parts (Proposition 4.11), hence $\Gamma \vdash \emptyset$ and therefore also $\Gamma \vdash \varphi$ holds in $\operatorname{Exp}_{\mathcal{B}} \bigcap_{i \in I} \mathcal{L}_i$. □

Joins in the lattice of explosive extensions reduce to unions.

Proposition 4.15 (Joins of explosive extensions).
$\bigvee_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i = \bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$.

Proof. Clearly $\bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i \subseteq \bigvee_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$. To establish the converse inclusion, it suffices to show that $\bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$ is a logic. Since reflexivity, monotonicity, and structurality hold for any union of logics, it suffices to show that it satisfies cut.

Suppose therefore that for each $\delta \in \Delta$ there is some $\mathcal{L}_\delta \in \{\mathcal{L}_i \mid i \in I\}$ such that $\Gamma \vdash_{\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_\delta} \delta$ for each δ, and moreover $\Delta, \Pi \vdash_{\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_\varphi} \varphi$ for some $\mathcal{L}_\varphi \in \{\mathcal{L}_i \mid i \in I\}$. If $\Gamma \vdash_{\mathcal{B}} \delta$ for each $\delta \in \Delta$, then $\Gamma, \Pi \vdash_{\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_\varphi} \varphi$ by cut for $\mathcal{L}_\varphi$. On the other hand, if $\Gamma \nvdash_{\mathcal{B}} \delta$ for some $\delta \in \Delta$, then by the description of consequence in explosive parts (Proposition 4.11) $\Gamma \vdash_{\mathcal{L}_\delta} \emptyset$ for some $\delta \in \Delta$. But then $\Gamma, \Pi \vdash \varphi$ holds in $\mathcal{L}_\delta$ by the basic properties of antitheorems (Fact 4.2), therefore it also holds in $\bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$.

Alternatively, the claim may be derived immediately from the description of consequence in explosive extensions (Proposition 4.6). □

Proposition 4.16 (Lattices of explosive extensions).
The explosive extensions of $\mathcal{B}$ *form a completely distributive complete sublattice of* $\operatorname{Ext}\mathcal{B}$*, denoted* $\operatorname{Exp}\operatorname{Ext}\mathcal{B}$.

Proof. Meets of explosive extensions of $\mathcal{B}$ are explosive extensions of $\mathcal{B}$ by Proposition 4.14. Joins of explosive extensions of $\mathcal{B}$ are explosive extensions of $\mathcal{B}$ by definition, and they coincide with unions by Proposition 4.15. Therefore the explosive extensions of $\mathcal{B}$ form a complete sublattice of $\operatorname{Ext}\mathcal{B}$, and this sublattice is completely distributive by virtue of meets and joins being intersections and unions. □

In addition to investigating the lattice of all explosive extensions of an arbitrary base logic, we can also investigate the finitary explosive extensions of a finitary base logic, i.e. extensions of the finitary base logic by sets of finite antitheorems. These will again form a distributive lattice.

Fact 4.17 (Finitarity for antitheorems).
Let $\mathcal{L}$ be a finitary logic. Then $\Gamma \vdash_{\mathcal{L}} \emptyset$ implies $\Gamma' \vdash_{\mathcal{L}} \emptyset$ for some finite $\Gamma' \subseteq \Gamma$.

Proof. If $\Gamma \vdash_{\mathcal{L}} \emptyset$, then $\sigma_{\text{push } p}[\Gamma] \vdash_{\mathcal{L}} p$ by the syntactic description of antitheorems (Proposition 4.4), therefore by finitarity $\sigma_{\text{push } p}[\Gamma'] \vdash_{\mathcal{L}} p$ for some finite $\Gamma' \subseteq \Gamma$, and thus $\Gamma' \vdash_{\mathcal{L}} \emptyset$. $\square$

Proposition 4.18 (Lattices of finitary explosive extensions).
The finitary explosive extensions of a finitary logic $\mathcal{B}$ form a distributive complete (hence algebraic) sublattice of $\mathrm{Ext}_\omega \mathcal{B}$, denoted $\mathrm{Exp}\,\mathrm{Ext}_\omega \mathcal{B}$.

Proof. Given a family $\mathcal{L}_i$ with $i \in I$ of finitary explosive extensions of $\mathcal{B}$, its join in $\mathrm{Ext}_\omega \mathcal{B}$ is again a finitary explosive extension of $\mathcal{B}$. Its meet in $\mathrm{Ext}_\omega \mathcal{B}$ is the logic $\mathcal{L}$ such that $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if there is some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{L}_i} \varphi$ for each $i \in I$, i.e. if and only if either $\Gamma' \vdash_{\mathcal{B}} \varphi$ or $\Gamma' \vdash_{\mathcal{L}_i} \emptyset$ for each $i \in I$ by the description of consequence in explosive parts (Proposition 4.11). But this is precisely the explosive extension of $\mathcal{B}$ by all the finite antitheorems Γ' such that $\Gamma' \vdash_{\mathcal{L}_i} \emptyset$ for each $i \in I$. Arbitrary meets and joins of finitary explosive extensions in $\mathrm{Ext}_\omega \mathcal{B}$ are thus finitary explosive extensions of $\mathcal{B}$, therefore the finitary explosive extensions of $\mathcal{B}$ form a complete sublattice of $\mathrm{Ext}_\omega \mathcal{B}$. Moreover, this lattice is distributive by virtue of being a sublattice of $\mathrm{Exp}\,\mathrm{Ext}\,\mathcal{B}$. Finally, a complete sublattice of an algebraic lattice is known to be algebraic. $\square$

Let us also consider how the strong version $\mathcal{L}^+_{\text{exp}}$ of an explosive extension $\mathcal{L}_{\text{exp}}$ of $\mathcal{B}$ may be determined. We show that under a suitable assumption on $\mathcal{B}$, which will in particular be satisfied by the Belnap–Dunn logic, the strong version of $\mathcal{L}_{\text{exp}}$ is simply $\mathcal{B}^+ \vee \mathcal{L}_{\text{exp}}$. (Readers unfamiliar with strong versions of logics may safely skip the following proposition.)

Proposition 4.19 (Strong versions of explosive extensions).
Let $\mathcal{L}_{exp}$ be an explosive extension of $\mathcal{B}$. Then $\mathcal{L}^+_{exp} = (\mathcal{B}^+ \vee \mathcal{L}_{exp})^+$. If the reduced filters of $\mathcal{B}^+$ are minimal $\mathcal{B}$-filters, then in fact $\mathcal{L}^+_{exp} = \mathcal{B}^+ \vee \mathcal{L}_{exp}$.

Proof. If $\mathcal{L}_{\text{exp}}$ is the trivial logic, then both claims hold trivially. Suppose therefore that $\mathcal{L}_{\text{exp}}$ is non-trivial. To prove that $\mathcal{L}^+_{\text{exp}} = (\mathcal{B}^+ \vee \mathcal{L}_{\text{exp}})^+$, it suffices to show that for each algebra $\mathbf{A}$ the smallest $\mathcal{L}_{\text{exp}}$-filter on $\mathbf{A}$ coincides with the smallest filter of $\mathcal{B}^+ \vee \mathcal{L}_{\text{exp}}$ on $\mathbf{A}$. To do so, it suffices to show that the smallest $\mathcal{L}_{\text{exp}}$-filter on $\mathbf{A}$ is in fact the smallest filter of $\mathcal{B}$ on $\mathbf{A}$ whenever there is some non-trivial $\mathcal{L}_{\text{exp}}$-filter on $\mathbf{A}$. But this is an immediate consequence of the fact that $\mathcal{L}$-subfilters of $\mathcal{L}_{\text{exp}}$-filters on $\mathbf{A}$ are again $\mathcal{L}_{\text{exp}}$-filters on $\mathbf{A}$ (Fact 4.8).

It follows that $\mathcal{B}^+ \vee \mathcal{L}_{\text{exp}} \leq \mathcal{L}^+_{\text{exp}}$, since $\mathcal{B}^+ \leq \mathcal{B}^+ \vee \mathcal{L}_{\text{exp}} \leq (\mathcal{B}^+ \vee \mathcal{L}_{\text{exp}})^+ = \mathcal{L}^+_{\text{exp}}$. Conversely, let $\langle \mathbf{A}, F \rangle$ be a reduced model of $\mathcal{B}^+ \vee \mathcal{L}_{\text{exp}}$. Then by assumption F is the minimal $\mathcal{B}$-filter on $\mathbf{A}$. But it is again an immediate consequence of Fact 4.8 that F is the minimal filter of $\mathcal{L}_{\text{exp}}$ on $\mathbf{A}$, hence also a filter of $\mathcal{L}^+_{\text{exp}}$. Thus $\mathcal{L}^+_{\text{exp}} \leq \mathcal{B}^+ \vee \mathcal{L}_{\text{exp}}$. $\square$

4.3 Intersections with explosive extensions

We now show that certain tasks which are complicated to do for intersections of arbitrary extensions of $\mathcal{B}$ simplify substantially in case one of the logics is in fact an explosive extension. In particular, we show that it is easy to axiomatize the intersection of an explosive extension with an arbitrary extensions (given axiomatizations of the two logics) and to describe its models (given a description of the models of the two logics).

Proposition 4.20 (Axiomatization of $\mathcal{L} \cap \mathcal{L}_{\mathbf{exp}}$).
Let $\mathcal{L}$ be the extension of $\mathcal{B}$ by the rules $\Gamma_i \vdash \varphi_i$ for $i \in I$. Let $\mathcal{L}_{exp}$ be the extension of $\mathcal{B}$ by the explosive rules $\Delta_j \vdash \emptyset$ for $j \in J$. Suppose without loss of generality that no atom occurs in both $\Gamma_i \cup \{\varphi_i\}$ and Δ_j. Then the logic $\mathcal{L} \cap \mathcal{L}_{exp}$ is the extension of $\mathcal{B}$ by the rules $\Gamma_i, \Delta_j \vdash \varphi_i$.

Proof. Clearly $\Gamma_i, \Delta_j \vdash \varphi_i$ holds in $\mathcal{L} \cap \mathcal{L}_{\mathrm{exp}}$ for each $i \in I$ and $j \in J$. Conversely, suppose that $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \vdash_{\mathcal{L}_{\mathrm{exp}}} \varphi$. If $\Gamma \vdash_{\mathcal{B}} \varphi$, we are done. Otherwise, $\Gamma \vdash_{\mathcal{L}_{\mathrm{exp}}} \emptyset$, therefore by the description of consequence in explosive extensions (Proposition 4.6) there is some substitution σ and some $j \in J$ such that $\Gamma \vdash_{\mathcal{B}} \sigma[\Delta_j]$. Because $\Gamma \vdash_{\mathcal{L}} \varphi$, there is a proof of φ from Γ using the rules $\Gamma_i \vdash \varphi_i$ and the rules of $\mathcal{B}$. It now suffices to transform it into a proof of φ from Γ using the rules $\Gamma_i, \Delta_j \vdash \varphi$ and the rules of $\mathcal{B}$.

To do so, we first prove $\sigma[\Delta_j]$ from Γ using the rules of $\mathcal{B}$. Then we take the proof of φ from Γ and replace each instance of a rule $\Gamma_i \vdash \varphi_i$ by an instance of the rule $\Gamma_i, \Delta_j \vdash \varphi_i$. This yields a proof of φ from Γ because we have already proved a substitution instance of Δ_j and Δ_j has by assumption no variables in common with $\Gamma_i \vdash \varphi_i$. □

Intersections of infinite families of explosive extensions may in fact be axiomatized in an analogous manner. However, the requirement that the logics be axiomatized in disjoint variables may involve some loss of generality when taking intersections of large families of logics. We omit the proof of the proposition below because it will not be needed in the following and the proof is entirely analogous to the previous one.

Proposition 4.21 (Axiomatization of $\bigcap_{i \in I} \mathrm{Exp}_{\mathcal{B}} \mathcal{L}_i$).
Let $\mathcal{L}_i$ for $i \in I$ be a family of explosive extensions of $\mathcal{B}$ by the antitheorems Γ_{ij} for $i \in I$ and $j \in J_i$. Suppose moreover that no atom occurs in both Γ_{ij} and Γ_{kl} for distinct i and k. Then $\bigcap_{i \in I} \mathcal{L}_i$ is the extension of $\mathcal{B}$ by the antitheorems $\bigcup_{i \in I} \Gamma_{if(i)}$ where f ranges over functions $f : I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$.

It is also easy to describe the class of all models of the intersections.

Proposition 4.22 (Models of $\mathcal{L} \cap \mathcal{L}_{\mathbf{exp}}$).
Let $\mathcal{L}_{exp}$ and $\mathcal{L}$ be an explosive and an arbitrary extension of $\mathcal{B}$. Then $\mathrm{Mod}\, \mathcal{L} \cap \mathcal{L}_{exp} = \mathrm{Mod}\, \mathcal{L} \cup \mathrm{Mod}\, \mathcal{L}_{exp}$.

Proof. The left-to-right inclusion is trivial. Conversely, suppose that $\mathbb{M}$ is a model of $\mathcal{B}$ which is a model of neither $\mathcal{L}$ nor $\mathcal{L}_{\mathrm{exp}}$. It suffices to show that $\mathcal{B}$ is not a model of $\mathcal{L} \cap \mathcal{L}_{\mathrm{exp}}$.

There are $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Delta \vdash_{\mathcal{L}_{\mathrm{exp}}} \emptyset$ such that $\Gamma \nvdash \varphi$ and $\Delta \nvdash \emptyset$ in $\operatorname{Log} \mathbb{M}$, as witnessed by the valuations v and w. We may moreover assume without loss of generality that no atom occurs in both of these rules. The valuation which agrees with v on the variables occurring in $\Gamma \vdash \varphi$ and with w on the variables occurring in Δ then witnesses the failure in $\mathbb{M}$ of the rule $\Gamma, \Delta \vdash \varphi$, which is valid in $\mathcal{L} \cap \mathcal{L}_{\mathrm{exp}}$. □

The above proposition again extends to intersections of infinite families of explosive extensions, provided that we restrict to families of logics which may be axiomatized in disjoint sets of variables. In particular, this can be ensured whenever the family has cardinality at most equal to the cardinality of the set of variables. We again omit the proof of the following proposition.

Proposition 4.23 (Models of $\bigcap_{i \in I} \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_i$).
Let $\mathcal{L}_i$ for $i \in I$ be a family of at most κ explosive extensions of $\mathcal{B}$ for $\kappa = |\operatorname{Var} \mathcal{L}|$. Then $\operatorname{Mod}\left(\bigcap_{i \in I} \mathcal{L}_i\right) = \bigcup_{i \in I} \operatorname{Mod} \mathcal{L}_i$.

The hypothesis on the cardinality of the family is needed to ensure that the logics may be axiomatized in disjoint variables. Without this technical assumption the proposition may fail, as the following example shows. (Here by the identity logic we mean the weakest logic in a given signature, i.e. the logic $\mathcal{L}$ such that $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\varphi \in \Gamma$.)

Example 4.24. *For each infinite cardinal κ, there is a family $\mathcal{L}_i$ for $i \in I$ of explosive extensions of the identity logic in κ variables such that $|I| = \kappa^+$ and $\operatorname{Mod} \bigcap_{i \in I} \mathcal{L}_i \subsetneq \bigcup_{i \in I} \operatorname{Mod} \mathcal{L}_i$.*

Proof. Consider the family of logics $\mathcal{L}_i$ for $i \in \kappa^+$ in a language with a set of variables p_j for $j \in \kappa$ and with the unary connectives P_k for $k \in \kappa^+$ such that $\mathcal{L}_i$ is axiomatized by the rule $P_i p_0 \vdash \emptyset$.

Now consider a (non-trivial) matrix $\langle \mathbf{A}, F \rangle$ such that (i) for each $i \in I$ there is some $a_i \in \mathbf{A}$ with $P_i a_i \in F$, (ii) each $a \in \mathbf{A}$ lies in at most one of the sets $P_i^{-1}[F]$ for $i \in I$, and (iii) $P_i P_k a \notin F$ for all $a \in \mathbf{A}$ and all $i, k \in \kappa^+$. This matrix is not a model of any of the logics $\mathcal{L}_i$ by (i). We show that it is a model of $\mathcal{L} := \bigcap_{i \in I} \mathcal{L}_i$ by (ii) and (iii).

The logic $\mathcal{L}$ is an explosive extension of the identity logic by virtue of being an intersection of a family of explosive extensions (Proposition 4.16). To show that $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{L}$, it suffices to show for each antitheorem Γ of $\mathcal{L}$ that no valuation $v : \mathbf{Fm} \to \langle \mathbf{A}, F \rangle$ designates Γ.

But clearly $\Gamma \vdash_{\mathcal{L}} \emptyset$ if and only if there is for each $i \in \kappa^+$ some formula φ_i such that $P_i \varphi_i \in \Gamma$. Since $\mathcal{L}$ has only κ variables, either $\varphi_i = \varphi_k$ for some distinct $i, k \in \kappa^+$ or $\varphi_i = P_k \psi$ for some $i, k \in \kappa^+$ and ψ. The requirements (ii) and (iii) on $\langle \mathbf{A}, F \rangle$ then imply that no valuation can designate Γ. □

4.4 Logics of products

We now show that the explosive part operator turns out to be very useful when computing logics determined by products of matrices.

Theorem 4.25 (Logics of products).
$\operatorname{Log} \Pi_{i \in I} \mathbb{M}_i = \bigcap_{i \in I} \operatorname{Log} \mathbb{M}_i \cup \bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \operatorname{Log} \mathbb{M}_i$ *for non-trivial matrices* $\mathbb{M}_i$.

Proof. Clearly if $\Gamma \vdash \varphi$ is valid in $\operatorname{Log} \mathbb{M}_i$ for each $i \in I$, then it is valid in $\Pi_{i \in I} \mathbb{M}_i$. Likewise, if $\Gamma \vdash \varphi$ is valid in $\operatorname{Exp}_{\mathcal{B}} \operatorname{Log} \mathbb{M}_i$, then by Proposition 4.6 either $\Gamma \vdash_{\mathcal{B}} \varphi$ or for some $i \in I$ there is no valuation $v_i : \mathbf{Fm} \to \mathbb{M}_i$ designating Γ, and thus no valuation $v : \mathbf{Fm} \to \Pi_{i \in I} \mathbb{M}_i$ designating Γ. In either case $\Gamma \vdash \varphi$ is valid in $\operatorname{Log} \mathbb{M}_i$.

Conversely, suppose that $\Gamma \nvdash \varphi$ in $\operatorname{Exp}_{\mathcal{B}} \operatorname{Log} \mathbb{M}_i$ for each $i \in I$, as witnessed by the valuations $v_i : \mathbf{Fm} \to \mathbb{M}_i$, and $\Gamma \nvdash \varphi$ in $\operatorname{Log} \mathbb{M}_j$ for some $j \in I$, as witnessed by the valuation $w_j : \mathbf{Fm} \to \mathbb{M}_j$. Then product of the valuations w_j and v_i for i other than j yields a valuation $w : \mathbf{Fm} \to \Pi_{i \in I} \mathbb{M}_i$ which witnesses the failure of the rule $\Gamma \vdash \varphi$. □

Corollary 4.26 (Explosive parts of logics of products).
$\operatorname{Exp}_{\mathcal{B}} \operatorname{Log} \Pi_{i \in I} \mathbb{M}_i = \bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \operatorname{Log} \mathbb{M}_i$ *for non-trivial matrices* $\mathbb{M}_i$.

The above description of the logic determined by a product of matrices allows us to derive a completeness theorem for the logic $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}$ whenever we have completeness theorems for $\mathcal{B}$ and $\mathcal{L}$. In the simplest case where each of the two logics $\mathcal{B}$ and $\mathcal{L}$ is determined by a single matrix, we get the following completeness theorem for $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}$.

Corollary 4.27 (Completeness for explosive parts).
$\operatorname{Exp}_{\mathcal{B}} \mathcal{L} = \operatorname{Log} \mathbb{M} \times \mathbb{N}$ *whenever* $\mathcal{L} = \operatorname{Log} \mathbb{M}$ *and* $\mathcal{B} = \operatorname{Log} \mathbb{N}$ *for* $\mathcal{L}$ *non-trivial.*

Proof. $\operatorname{Log} \mathbb{M} \times \mathbb{N} = (\mathcal{L} \cap \mathcal{B}) \cup \operatorname{Exp}_{\mathcal{B}} \mathcal{B} \cup \operatorname{Exp}_{\mathcal{B}} \mathcal{L} = \mathcal{B} \cup \operatorname{Exp}_{\mathcal{B}} \mathcal{L} = \operatorname{Exp}_{\mathcal{B}} \mathcal{L}$. □

It will also be useful to observe that joins with explosive extensions may under certain conditions be replaced by unions.

Proposition 4.28 (Joins with explosive parts).
If $\mathcal{L}_1 \leq \mathcal{L}_2$, *then* $\mathcal{L}_1 \vee \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2 = \mathcal{L}_1 \cup \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$.

Proof. Suppose that $\Gamma \nvdash \varphi$ in both $\mathcal{L}_1$ and $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$. It suffices to show that $\Gamma \nvdash \varphi$ in $\mathcal{L}_1 \vee \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$. Let therefore $\mathbb{M}_1$ be a model of $\mathcal{L}_1$ where the valuation v witnesses the failure of $\Gamma \vdash \varphi$, and let $\mathbb{M}_2$ be a model of $\mathcal{L}_2$ where the valuation w witnesses the failure of $\Gamma \vdash \varphi$. Then the product valuation $v \times w$ witnesses the failure of $\Gamma \vdash \varphi$ in $\mathbb{M}_1 \times \mathbb{M}_2$. It remains to show that $\mathbb{M}_1 \times \mathbb{M}_2$ is a model of $\mathcal{L}_1 \vee \operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$, i.e. a model of both $\mathcal{L}_1$ and $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$. But it is indeed a model of $\mathcal{L}_1$ because $\mathbb{M}_2$ is a model of $\mathcal{L}_1 \leq \mathcal{L}_2$, and it is a model of $\operatorname{Exp}_{\mathcal{B}} \mathcal{L}_2$ by Theorem 4.25. □

Chapter 5

Completeness theorems

This chapter is devoted to proving several new completeness theorems for super-Belnap logics. Our strategy will be to first compute the explosive parts of the basic extensions of $\mathcal{BD}$ and then exploit the results of the previous chapter to establish completeness theorems for these explosive parts. In addition, we prove completeness theorems for three more logics directly.

The current chapter will introduce two increasing chains of logics

$$\mathcal{ECQ} = \mathcal{ECQ}_1 < \mathcal{ECQ}_2 < \cdots < \mathcal{ECQ}_\omega,$$
$$\mathcal{ETL} = \mathcal{ETL}_1 < \mathcal{ETL}_2 < \cdots < \mathcal{ETL}_\omega,$$

and prove completeness theorems for the smallest and largest logics in these chains, as well as for $\mathcal{KO} \vee \mathcal{ECQ}$ and $\mathcal{LP} \vee \mathcal{ECQ}$. The failure to prove completeness theorems for the other logics is not an accident: we shall see in Chapter 7 (Super-Belnap logics and finite graphs) that the intermediate logics in these chains are in fact not complete with respect to any finite set of finite matrices. Nevertheless, non-trivial completeness theorems for these logics will be formulated in Chapter 7.

The logic $\mathcal{ECQ}$ in fact occupies an important place in the structure of the lattice of super-Belnap logics $\operatorname{Ext} \mathcal{BD}$, as will become clear in the following chapter. It turns out to be the smallest explosive extension of $\mathcal{BD}$, while $\mathcal{ECQ}_\omega$ is the largest non-trivial explosive extension of $\mathcal{BD}$.

The current chapter will also introduce another chain of logics, namely

$$\mathcal{SDS}_1 < \mathcal{SDS}_2 < \cdots < \mathcal{SDS}_\omega.$$

We provide a completeness theorem for the logics $\mathcal{SDS}_\omega$ and $\mathcal{LP} \cap \mathcal{SDS}_\omega$. Although the significance of these logics may not be immediately apparent, we shall see in Chapter 6 (The lattice of super-Belnap logics) that $\mathcal{SDS}_\omega$ is the only lower cover of $\mathcal{K}$ in $\operatorname{Ext} \mathcal{ETL}$ and $\mathcal{LP} \cap \mathcal{SDS}_\omega$ is the only lower cover of $\mathcal{KO}$ in $\operatorname{Ext} \mathcal{BD}$. For this reason these two logics will also be denoted $\mathcal{K}_-$ and $\mathcal{KO}_-$.

5.1 Explosive parts of super-Belnap logics

This section will be devoted to computing the explosive parts of the basic super-Belnap logics relative to other basic super-Belnap logics.

We first introduce a chain of logics $\mathcal{ECQ}_n$ axiomatized by increasingly stronger forms of the rule of *ex contradictione quodlibet* $p, -p \vdash q$. Throughout the present chapter we shall use the notation

$$\chi_n := (p_1 \wedge -p_1) \vee \cdots \vee (p_n \wedge -p_n).$$

The logic $\mathcal{ECQ}_n$ for $n \geq 1$ is defined as the extension of $\mathcal{BD}$ by the rule $\chi_n \vdash \emptyset$. Clearly $\mathcal{ECQ}_n \leq \mathcal{ECQ}_{n+1}$. The logic $\mathcal{ECQ}_1$, i.e. the extension of $\mathcal{BD}$ by the rule $p, -p \vdash \emptyset$, will be called simply $\mathcal{ECQ}$.

The logics $\mathcal{ETL}_n$ are analogously defined as the extensions of $\mathcal{ETL}$ by the rule $\chi_n \vdash \emptyset$. In particular, $\mathcal{ETL}_1 = \mathcal{ETL}$. Clearly $\mathcal{ETL}_n \leq \mathcal{ETL}_{n+1}$. These inequalities (hence also those for $\mathcal{ECQ}_n$) are in fact strict, but to prove this we have to wait until Chapter 7 (Super-Belnap logics and finite graphs). The strictness of these inequalities immediately implies that the logics $\mathcal{ECQ}_\omega$ and $\mathcal{ETL}_\omega$ are not finitely axiomatizable.

The joins of these chains will be denoted $\mathcal{ECQ}_\omega$ and $\mathcal{ETL}_\omega$. Observe that they are in fact unions of the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$. That is,

$$\Gamma \vdash_{\mathcal{ECQ}_\omega} \varphi \iff \Gamma \vdash_{\mathcal{ECQ}_n} \varphi \text{ for some } n,$$
$$\Gamma \vdash_{\mathcal{ETL}_\omega} \varphi \iff \Gamma \vdash_{\mathcal{ETL}_n} \varphi \text{ for some } n.$$

The logics $\mathcal{ETL}_n$ and $\mathcal{ETL}_\omega$ were first introduced under the names $\mathcal{B}_n$ and $\mathcal{B}_\infty$ and shown to be distinct by Rivieccio [66, Thm 11].

Proposition 5.1.
$\mathrm{Exp}_{\mathcal{BD}}\, \mathcal{CL} = \mathcal{ECQ}_\omega$.

Proof. The inclusion $\mathcal{ECQ}_\omega \leq \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{CL}$ is clear. Conversely, suppose that $\Gamma \vdash_{\mathcal{CL}} \emptyset$. By finitarity for antitheorems (Fact 4.17) we may assume without loss of generality that Γ is finite. By the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) there is a formula $\varphi = \bigvee\{\varphi_i \mid i \in I\}$ in disjunctive normal form equivalent in $\mathcal{BD}$ to the conjunction $\bigwedge \Gamma$.

In the trivial case where $\varphi = \mathsf{f}$ clearly $\varphi \vdash_{\mathcal{ECQ}_\omega} \emptyset$, hence $\Gamma \vdash_{\mathcal{ECQ}_\omega} \emptyset$. Otherwise $\varphi_i \vdash_{\mathcal{CL}} \emptyset$ for each $i \in I$, therefore $\varphi_i = p_i \wedge -p_i \wedge \psi_i$ for some atom p_i and some formula ψ_i. Thus $\varphi \vdash_{\mathcal{BD}} (p_1 \wedge -p_1) \vee \cdots \vee (p_n \wedge -p_n)$ for some n and $\varphi \vdash_{\mathcal{ECQ}_n} \emptyset$. But $\Gamma \vdash_{\mathcal{BD}} \varphi$, therefore $\Gamma \vdash_{\mathcal{ECQ}_\omega} \emptyset$. □

Corollary 5.2 (Classical contradictions).
A formula χ is a classical contradiction if and only if $\chi \vdash_{\mathcal{BD}} \sigma(\chi_n)$ for some substitution σ.

Proof. This is an immediate consequence of our description of consequence in explosive extensions (Proposition 4.6). □

Proposition 5.3.
$\mathrm{Exp}_{\mathcal{ETL}}\,\mathcal{CL} = \mathcal{ETL}_\omega$.

Proof. $\mathrm{Exp}_{\mathcal{ETL}}\,\mathcal{CL} = \mathcal{ETL} \vee \mathrm{Exp}_{\mathcal{BD}}\,\mathcal{CL} = \mathcal{ETL} \vee \mathcal{ECQ}_\omega = \mathcal{ETL}_\omega$. □

Proposition 5.4.
$\mathrm{Exp}_{\mathcal{BD}}\,\mathcal{ETL}_n = \mathcal{ECQ}_n$. $\mathrm{Exp}_{\mathcal{BD}}\,\mathcal{ETL}_\omega = \mathcal{ECQ}_\omega$.

Proof. The inclusion $\mathcal{ECQ}_n \leq \mathrm{Exp}_{\mathcal{BD}}\,\mathcal{ETL}_n$ is clear. Conversely, suppose that $\Gamma \vdash_{\mathcal{ETL}_n} \emptyset$. We can without loss of generality assume that $\Gamma = \{\gamma\}$. Then by Corollary 5.2

$$\gamma \vdash_{\mathcal{ETL}} (\varphi_1 \wedge -\varphi_1) \vee \cdots \vee (\varphi_n \wedge -\varphi_n)$$

for some $\varphi_1, \dots, \varphi_n$. The translation between $\mathcal{ETL}$ and $\mathcal{BD}$ (Theorem 3.7) yields that

$$\gamma \vdash_{\mathcal{BD}} -\gamma \vee (\varphi_1 \wedge -\varphi_1) \vee \cdots \vee (\varphi_n \wedge -\varphi_n).$$

It now suffices to show that for some φ

$$p, -p \vee (q \wedge -q) \vee r \vdash_{\mathcal{BD}} (\varphi \wedge -\varphi) \vee r.$$

Substituting γ for p, p_1 for q, and $(p_2 \wedge -p_2) \vee \cdots \vee (p_n \wedge -p_n)$ for r then yields that $\gamma \vdash_{\mathcal{ECQ}_n} \emptyset$.

We take $\varphi := (q \wedge p) \vee -p$. Then $(\varphi \wedge -\varphi) \vee r$ is equivalent to the conjunction of $q \vee -p \vee r$, $p \vee -p \vee r$, $-q \vee -p \vee r$, and $p \vee r$. But all of these formulas are consequences of $p, -p \vee (q \wedge -q) \vee r$ in $\mathcal{BD}$.

The claim for $\mathcal{ETL}_\omega$ follows immediately from the fact that $\Gamma \vdash_{\mathcal{ETL}_\omega} \emptyset$ if and only if $\Gamma \vdash_{\mathcal{ETL}_n} \emptyset$ for some n. □

We now describe consequence in the logics $\mathcal{LP} \vee \mathcal{ECQ}$ and $\mathcal{KO} \vee \mathcal{ECQ}$ in more detail in order to show that they are precisely the explosive parts of $\mathcal{CL}$ with respect to $\mathcal{LP}$ and $\mathcal{KO}$.

Proposition 5.5.
$\mathcal{LP} \vee \mathcal{ECQ} = \mathcal{LP} \cup \mathcal{ECQ}_\omega$. $\mathcal{KO} \vee \mathcal{ECQ} = \mathcal{KO} \cup \mathcal{ECQ}_\omega$.

Proof. We first show that $\mathcal{ECQ}_\omega \subseteq \mathcal{KO} \vee \mathcal{ECQ}$. To do so, it suffices to show that $(p \wedge -p) \vee (q \wedge -q) \vee r \vdash_{\mathcal{KO}} (\varphi \wedge -\varphi) \vee r$ for some formula φ. Repeated applications of this rule then yield $(p_1 \wedge -p_1) \vee \cdots \vee (p_n \wedge -p_n) \vdash_{\mathcal{KO}} \psi \wedge -\psi$ for some formula ψ, and of course $\psi \wedge -\psi \vdash_{\mathcal{ECQ}} \emptyset$.

In particular, we take $\varphi := (p \vee q) \wedge (-p \vee -q)$. Then $(\varphi \wedge -\varphi) \vee r$ is equivalent to the conjunction of the formulas $p \vee q \vee r$, $p \vee -q \vee r$, $-p \vee q \vee r$, $-p \vee -q \vee r$, $p \vee -p \vee r$, $q \vee -q \vee r$. But the last two formulas are consequence of $(p \wedge -p) \vee (q \wedge -q) \vee r$ in $\mathcal{KO}$ and the rest are consequences of $(p \wedge -p) \vee (q \wedge -q)$ in $\mathcal{BD}$. Thus $\mathcal{ECQ}_\omega \leq \mathcal{KO} \vee \mathcal{ECQ}$.

It follows that $\mathcal{KO} \vee \mathcal{ECQ} = \mathcal{KO} \vee \mathcal{ECQ}_\omega$ and $\mathcal{LP} \vee \mathcal{ECQ} = \mathcal{LP} \vee \mathcal{ECQ}_\omega$. Since $\mathcal{KO} \leq \mathcal{LP} \leq \mathcal{CL}$ and $\mathcal{ECQ}_\omega = \mathrm{Exp}_{\mathcal{BD}}\,\mathcal{CL}$, we have $\mathcal{LP} \vee \mathcal{ECQ}_\omega = \mathcal{LP} \cup \mathcal{ECQ}_\omega$ and $\mathcal{KO} \vee \mathcal{ECQ}_\omega = \mathcal{KO} \cup \mathcal{ECQ}_\omega$ by Proposition 4.28. □

Proposition 5.6 (Joins with $\mathcal{ECQ}$).
$(\mathcal{LP} \cap \mathcal{L}) \vee \mathcal{ECQ} = (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{L} = (\mathcal{LP} \cap \mathcal{L}) \cup (\mathcal{ECQ}_\omega \cap \mathcal{L})$ *for* $\mathcal{L} \geq \mathcal{ECQ}_\omega$.

Proof. Clearly $(\mathcal{LP} \cap \mathcal{L}) \vee \mathcal{ECQ} \leq (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{L}$ for $\mathcal{L} \geq \mathcal{ECQ}_\omega \geq \mathcal{ECQ}$. Conversely, $(\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{L} = (\mathcal{LP} \cup \mathcal{ECQ}_\omega) \cap \mathcal{L} = (\mathcal{LP} \cap \mathcal{L}) \cup (\mathcal{ECQ}_\omega \cap \mathcal{L}) \leq (\mathcal{LP} \cap \mathcal{L}) \vee \mathcal{ECQ}$ holds if and only if $\mathcal{ECQ}_\omega \cap \mathcal{L} \leq (\mathcal{LP} \cap \mathcal{L}) \vee \mathcal{ECQ}$. To prove this for $\mathcal{L} \geq \mathcal{ECQ}_\omega$, it suffices to show that $\mathcal{ECQ}_\omega \leq (\mathcal{LP} \cap \mathcal{ECQ}_\omega) \vee \mathcal{ECQ}$.

Let $\mathbb{M}$ therefore be a non-trivial model of $\mathcal{LP} \cap \mathcal{ECQ}_\omega$ and $\mathcal{ECQ}$. By Proposition 4.22 the matrix $\mathbb{M}$ is either a model of $\mathcal{LP}$ or a model of $\mathcal{ECQ}_\omega$. But if it is a model of $\mathcal{LP}$ and $\mathcal{ECQ}$, then it is in fact a model of $\mathcal{ECQ}_\omega \leq \mathcal{LP} \cup \mathcal{ECQ}_\omega = \mathcal{LP} \vee \mathcal{ECQ}$. Thus in either case $\mathbb{M}$ is a model of $\mathcal{ECQ}_\omega$ and $\mathcal{ECQ}_\omega \leq (\mathcal{LP} \cap \mathcal{ECQ}_\omega) \vee \mathcal{ECQ}$. □

Proposition 5.7.
$\mathrm{Exp}_{\mathcal{LP}}\, \mathcal{CL} = \mathcal{LP} \vee \mathcal{ECQ}$. $\mathrm{Exp}_{\mathcal{KO}}\, \mathcal{CL} = \mathcal{KO} \vee \mathcal{ECQ}$.

Proof. $\mathcal{LP} \vee \mathcal{ECQ} = \mathcal{LP} \vee \mathcal{ECQ}_\omega = \mathcal{LP} \vee \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{CL} = \mathrm{Exp}_{\mathcal{LP}}\, \mathcal{CL}$. Likewise, $\mathcal{KO} \vee \mathcal{ECQ} = \mathcal{KO} \vee \mathcal{ECQ}_\omega = \mathcal{KO} \vee \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{CL} = \mathrm{Exp}_{\mathcal{KO}}\, \mathcal{CL}$, where the equality $\mathcal{KO} \vee \mathcal{ECQ} = \mathcal{KO} \vee \mathcal{ECQ}_\omega$ holds because by the previous proposition $\mathcal{KO} \vee \mathcal{ECQ} = (\mathcal{LP} \cap \mathcal{K}) \vee \mathcal{ECQ} = (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{K} = (\mathcal{LP} \cup \mathcal{ECQ}_\omega) \cap \mathcal{K} \geq \mathcal{ECQ}_\omega$. □

Finally, observe that $\mathcal{LP}$ and $\mathcal{BD}$ have the same antitheorems.

Proposition 5.8.
$\mathrm{Exp}_{\mathcal{BD}}\, \mathcal{LP} = \mathcal{BD}$.

Proof. Suppose that Γ is not an antitheorem of $\mathcal{BD}$. Then there is some valuation $v : \mathbf{Fm} \to \mathbb{BD}_\mathbf{4}$ such that $v[\Gamma] \subseteq \{\mathsf{t}, \mathsf{b}\}$. Now consider the valuation $w : \mathbf{Fm} \to \mathbb{BD}_\mathbf{4}$ such that $w(p) = \mathsf{b}$ for each atom p. Recall that $\mathbb{P}_\mathbf{3}$ is a submatrix of $\mathbb{BD}_\mathbf{4}$ and b is the largest element of $\mathbb{BD}_\mathbf{4}$ with respect to the information order. Therefore w is in fact a valuation $w : \mathbf{Fm} \to \mathbb{P}_\mathbf{3}$. But $v \sqsubseteq w$, thus $w(\gamma) \sqsupseteq v(\gamma) \in \{\mathsf{t}, \mathsf{b}\}$ for $\gamma \in \Gamma$ and w designates Γ in $\mathbb{P}_\mathbf{3}$. □

5.2 Completeness for super-Belnap logics

In the previous section, we identified several super-Belnap logics as the explosive parts of the basic extensions of $\mathcal{BD}$ for which we already have completeness theorems. But recall from the previous chapter (Corollary 4.27) that $\mathrm{Exp}_{\mathcal{B}}\, \mathcal{L} = \mathrm{Log}\, \mathbb{M} \times \mathbb{N}$ whenever $\mathcal{L} = \mathrm{Log}\, \mathbb{M}$ and $\mathcal{B} = \mathrm{Log}\, \mathbb{N}$ for $\mathcal{L}$ non-trivial. The following completeness theorems are thus immediate corollaries to the results of the previous section.

Proposition 5.9 [64, Thm 4.13] (Completeness for $\mathcal{LP} \vee \mathcal{ECQ}$).
$\mathcal{LP} \vee \mathcal{ECQ} = \mathrm{Log}\, \mathbb{P}_\mathbf{3} \times \mathbb{B}_\mathbf{2}$.

Proposition 5.10 (Completeness for $\mathcal{KO} \vee \mathcal{ECQ}$).
$\mathcal{KO} \vee \mathcal{ECQ} = \mathrm{Log}\{\mathbb{P}_\mathbf{3} \times \mathbb{B}_\mathbf{2}, \mathbb{K}_\mathbf{3} \times \mathbb{B}_\mathbf{2}\} = \mathrm{Log}\{\mathbb{P}_\mathbf{3} \times \mathbb{B}_\mathbf{2}, \mathbb{K}_\mathbf{3}\}$.

Proposition 5.11 (Completeness for $\mathcal{ECQ}_\omega$).
$\mathcal{ECQ}_\omega = \mathrm{Log}\, \mathbb{BD}_\mathbf{4} \times \mathbb{B}_\mathbf{2}$.

Proposition 5.12 (Completeness for $\mathcal{ETL}_\omega$).
$\mathcal{ETL}_\omega = \mathrm{Log}\, \mathbb{ETL}_\mathbf{4} \times \mathbb{B}_\mathbf{2}$.

Proposition 5.13 (Completeness for $\mathcal{ECQ}$).
$\mathcal{ECQ} = \mathrm{Log}\, \mathbb{BD}_\mathbf{4} \times \mathbb{ETL}_\mathbf{4}$.

Note that the completeness theorem for $\mathcal{LP} \vee \mathcal{ECQ}$ was already proved by Pynko [64, Thm 4.13]. A completeness theorem for the logic $\mathcal{ETL}_\omega$ with respect to the more complicated matrix $\mathbb{ETL}_\mathbf{4} \times \mathbb{K}_\mathbf{3}$ was proved by Rivieccio [66, Thm 14] using the results of Gaitán and Perea [28].

In the rest of the section, we prove three more completeness theorems, namely for $\mathcal{LP} \cap \mathcal{ETL}$ and for the logics $\mathcal{K}_-$ and $\mathcal{KO}_-$ which we introduce below. We shall rely on the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) and on our knowledge of consequence in $\mathcal{BD}$ (Proposition 3.13) as well as consequence in $\mathcal{LP}$ (Proposition 3.5).

Proposition 5.14 (Completeness for $\mathcal{LP} \cap \mathcal{ETL}$).
$\mathcal{LP} \cap \mathcal{ETL}$ is axiomatized by the rule $p, -p \vee q \vee -q \vdash q \vee -q$.

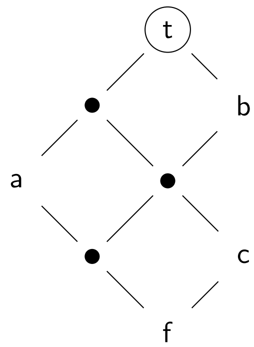
Proof. The rule is clearly valid in both $\mathcal{LP}$ and $\mathcal{ETL}$. Conversely, suppose that $\varphi \vdash_{\mathcal{LP} \cap \mathcal{ETL}} \psi$. By the finitarity of $\mathcal{LP} \cap \mathcal{ETL}$ it suffices to show that ψ is provable from φ in the extension of $\mathcal{BD}$ by the rule $p, -p \vee q \vee - \vdash q \vee -q$. If $\varphi \vdash_{\mathcal{BD}} \psi$, this holds trivially, we therefore assume that $\varphi \nvdash_{\mathcal{BD}} \psi$. By the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) we may also assume without loss of generality that ψ is a disjunctive clause.

But then $\varphi, \{p \vee -p \mid p \in \mathrm{At}(\psi)\} \vdash_{\mathcal{BD}} \psi$ by Proposition 3.5, and therefore $p \vee -p \vdash_{\mathcal{BD}} \psi$ for some $p \in \mathrm{At}(\psi)$ by Proposition 3.13, since $\varphi \nvdash_{\mathcal{BD}} \psi$. It follows that $\psi = p \vee -p \vee \chi$ for some χ. Now recall that $\varphi \vdash_{\mathcal{ETL}} \psi$ implies that $\varphi \vdash_{\mathcal{BD}} -\varphi \vee \psi$, i.e. $\varphi \vdash_{\mathcal{BD}} -\varphi \vee p \vee -p \vee \chi$ (Theorem 3.7). It follows that a single application of the rule $p, -p \vee q \vee -q \vee r \vdash q \vee -q \vee r$ is sufficient to derive ψ from φ.

But this rule is derivable from the simpler rule $p, -p \vee q \vee -q \vdash q \vee -q$. To see this, let $\varphi := q \vee -q \vee r$. Then $-p \vee q \vee -q \vee r \vdash_{\mathcal{BD}} -p \vee \varphi \vee -\varphi$, hence $\varphi \vee -\varphi$ is provable from p and $-p \vee q \vee -q \vee r$ using the simpler rule. But $\varphi \vee -\varphi \vdash_{\mathcal{BD}} q \vee -q \vee r$. □

We now introduce the logic $\mathcal{K}_-$, which we define semantically as the logic of the matrix $\mathbb{M}_\mathbf{8} = \langle \mathbf{DM_8}, \{\mathsf{t}\} \rangle$ depicted in Figure 5.1. The De Morgan negation of $\mathbf{DM_8}$ is uniquely determined by the lattice structure as the reflection across the horizontal axis of symmetry, i.e. $-\mathsf{a} = \mathsf{a}$ and $-\mathsf{b} = \mathsf{c}$.

Figure 5.1: The matrix $\mathbb{M}_{\mathbf{8}} = \langle \mathbf{DM_8}, \{\mathsf{t}\} \rangle$



This may seem like a very *ad hoc* logic to study at first sight, but we shall see in the next chapter that $\mathcal{K}_{-}$ is in fact a lower cover of $\mathcal{K}$ in $\operatorname{Ext} \mathcal{BD}$, and in particular it is the strongest extension of $\mathcal{ETL}$ strictly below $\mathcal{K}$.

To describe the consequence relation of $\mathcal{K}_{-}$, it will be useful to introduce the notion of the consistent part of a formula γ. Recall that by the Irredundant Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.15) the formula γ is equivalent to an essentially unique irredundant disjunction of irredundant conjunctive clauses $\bigvee_{i \in I} \gamma_i$. By the *consistent part* of γ we mean the disjunction of those conjuctive clauses γ_i which are classically consistent. The *inconsistent part* of γ is, of course, the disjunction of the remaining conjunctive clauses, empty disjunctions being interpreted as f in both cases. For example, the consistent part of $p \wedge (-p \vee q)$ is $p \wedge q$, while the inconsistent part is $p \wedge -p$ (up to equivalence in $\mathcal{BD}$).

Proposition 5.15 (Consequence in $\mathcal{K}_{-}$).
The following are equivalent:

(i) $\gamma \vdash_{\mathcal{K}_{-}} \varphi$.

(ii) $\gamma \vdash_{\mathcal{BD}} \chi \vee \psi$ *and* $\gamma \vdash_{\mathcal{BD}} -\psi \vee \varphi$ *for some formula* ψ *and some classical contradiction* χ.

(iii) $\gamma \vdash_{\mathcal{BD}} -\psi \vee \varphi$, *where* ψ *is the consistent part of* γ.

Proof. If γ is a classical contradiction, the equivalence holds trivially. Suppose therefore that γ is classically consistent.

(ii) $\implies$ (iii): let $\alpha = \bigvee_{i \in I} \alpha_i$ be the consistent part of γ. Then $\gamma \vdash_{\mathcal{BD}} \chi \vee \psi$ implies that $\alpha_i \vdash_{\mathcal{BD}} \chi \vee \psi$ for each $i \in I$. It follows that either $\alpha_i \vdash_{\mathcal{BD}} \chi$ or $\alpha_i \vdash_{\mathcal{BD}} \psi$ by our description of consequence in $\mathcal{BD}$ (Proposition 3.13). Since χ is a classical contradiction and α is not, $\alpha_i \vdash_{\mathcal{BD}} \psi$ for each $i \in I$. But then $\alpha \vdash_{\mathcal{BD}} \psi$, hence $-\psi \vdash_{\mathcal{BD}} -\alpha$. It follows that $\gamma \vdash_{\mathcal{BD}} -\psi \vee \varphi$ implies $\gamma \vdash_{\mathcal{BD}} -\alpha \vee \psi$.

(iii) $\implies$ (ii): let ψ and χ be respectively the consistent and the inconsistent of γ. Then $\gamma \vdash_{\mathcal{BD}} \chi \vee \psi$ by definition.

(ii) $\implies$ (i): it suffices to verify that $\chi \vee \psi, -\psi \vee \varphi \vdash_{\mathcal{K}_-} \varphi$. For each $v : \mathbf{Fm} \to \mathbf{DM_8}$ we have $v(\chi) \leq \mathsf{a} \vee \mathsf{c}$, therefore $v(\chi \vee \psi) = \mathsf{t}$ only if $v(\psi) \geq \mathsf{b}$. But $v(-\psi) \leq \mathsf{c}$, thus $v(-\psi \vee \varphi) = \mathsf{t}$ only if $v(\varphi) = \mathsf{t}$.

(i) $\implies$ (ii): we first prove two auxiliary claims. Firstly, we show that if the implication holds for each $\gamma \in \{\delta_1, \delta_2, \delta_3\}$, where

$$\begin{aligned} \delta_1 &:= \gamma_2 \vee \gamma_3 \\ \delta_2 &:= \gamma_3 \vee \gamma_1 \\ \delta_3 &:= \gamma_1 \vee \gamma_2 \end{aligned}$$

then holds for $\gamma := \gamma_1 \vee \gamma_2 \vee \gamma_3$. The assumption that the implication holds in these three cases yields formulas ψ_i and classical contradictions χ_i for $1 \leq i \leq 3$ such that

$$\delta_i \vdash_{\mathcal{BD}} \chi_i \vee \psi_i \qquad \text{and} \qquad \delta_i \vdash_{\mathcal{BD}} -\psi_i \vee \varphi.$$

Observe that

$$\gamma_1 \vee \gamma_2 \vee \gamma_3 \vdash_{\mathcal{BD}} (\delta_2 \vee \delta_3) \wedge (\delta_3 \vee \delta_1) \wedge (\delta_1 \vee \delta_2).$$

Now take

$$\begin{aligned} \psi &:= (\psi_1 \vee \psi_2) \wedge (\psi_2 \vee \psi_3) \wedge (\psi_3 \vee \psi_1) \\ \alpha_1 &:= (\chi_2 \vee \chi_3) \wedge (\chi_3 \vee \chi_1 \vee \psi_3 \vee \psi_1) \wedge (\chi_1 \vee \chi_2 \vee \psi_1 \vee \psi_2) \\ \alpha_2 &:= (\chi_3 \vee \chi_1) \wedge (\chi_1 \vee \chi_2 \vee \psi_1 \vee \psi_2) \wedge (\chi_2 \vee \chi_3 \vee \psi_2 \vee \psi_3) \\ \alpha_3 &:= (\chi_1 \vee \chi_2) \wedge (\chi_2 \vee \chi_3 \vee \psi_2 \vee \psi_3) \wedge (\chi_3 \vee \chi_1 \vee \psi_3 \vee \psi_1) \\ \chi &:= \alpha_1 \vee \alpha_2 \vee \alpha_3 \end{aligned}$$

and observe also that

$$(-\psi_1 \vee -\psi_2) \wedge (-\psi_2 \vee -\psi_3) \wedge (-\psi_3 \vee -\psi_1) \vdash_{\mathcal{BD}} -\psi.$$

It follows that

$$\gamma_1 \vee \gamma_2 \vee \gamma_3 \vdash_{\mathcal{BD}} \chi \vee \psi \qquad \text{and} \qquad \gamma_1 \vee \gamma_2 \vee \gamma_3 \vdash_{\mathcal{BD}} -\psi \vee \varphi,$$

therefore the implication holds for $\gamma := \gamma_1 \vee \gamma_2 \vee \gamma_3$.

Secondly, we show that if the implication (i) $\implies$ (ii) holds for each $\varphi \in \{\varphi_1, \varphi_2\}$, then it holds for $\varphi := \varphi_1 \wedge \varphi_2$. The assumption that the implication holds for $\varphi \in \{\varphi_1, \varphi_2\}$ yields formulas ψ_i and classical contradictions χ_i for $1 \leq i \leq 2$ such that

$$\gamma \vdash_{\mathcal{BD}} \chi_i \vee \psi_i \qquad \text{and} \qquad \gamma \vdash_{\mathcal{BD}} -\psi_i \vee \varphi_i.$$

But then taking

$$\chi := \chi_1 \vee \chi_2$$
$$\psi := \psi_1 \wedge \psi_2$$

yields that for $\varphi := \varphi_1 \wedge \varphi_2$ we have

$$\gamma \vdash_{\mathcal{BD}} \chi \vee \psi \qquad \text{and} \qquad \gamma \vdash_{\mathcal{BD}} -\psi \vee \varphi.$$

We now prove the implication (i) $\implies$ (ii) for arbitrary γ. By the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) the formula γ is equivalent in $\mathcal{BD}$ to a formula in disjunctive normal form $\bigvee_{1 \leq i \leq n} \gamma_i$, and the formula φ is equivalent in $\mathcal{BD}$ to a formula in conjunctive normal form $\bigwedge_{1 \leq j \leq m} \varphi_j$. By the second auxiliary claim it suffices to prove the implication for $m = 1$, i.e. under the assumption that φ is a disjunctive clause. (The case of $m = 0$, i.e. $\varphi = \mathsf{t}$, is trivial.) By the first auxiliary claim it suffices to prove the implication for $n \leq 2$. Without loss of generality we take $n = 2$.

If γ is a classical contradiction, the implication holds trivially for $\chi := \gamma$ and $\psi := -\gamma$. Otherwise, we may suppose that without loss of generality the conjunctive clause γ_2 is not a classical contradiction.

Suppose now that the right-hand side of the equivalence fails and γ_1 is not a classical contradiction. Taking $\psi := \gamma$, either $\gamma_1 \nvdash_{\mathcal{BD}} -\gamma \vee \varphi$ or $\gamma_2 \nvdash_{\mathcal{BD}} -\gamma \vee \varphi$. In particular, either $\gamma_1 \nvdash_{\mathcal{BD}} \varphi$ or $\gamma_2 \nvdash_{\mathcal{BD}} \varphi$. Suppose without loss of generality that $\gamma_2 \nvdash_{\mathcal{BD}} \varphi$. Then γ_2 has no literal in common with φ, therefore there is a valuation $v : \mathbf{Fm} \to \mathbf{DM_8}$ such that $v(l) = \mathsf{t}$ for each literal l of γ_2 while $v(l) \in \{\mathsf{f}, \mathsf{b}, \mathsf{c}\}$ for each literal l of φ: take $v(l) = \mathsf{t}$ for each literal l of γ_2 and take $v(l) \in \{\mathsf{b}, \mathsf{c}\}$ for each literal l of φ such that $-l$ is not (equivalent to) a literal of γ_2.

Finally, suppose that the right-hand side of the equivalence fails and γ_1 is a classical contradiction. Taking $\chi := \gamma_1$ and $\psi := \gamma_2$, either $\gamma_1 \nvdash_{\mathcal{BD}} -\gamma_2 \vee \varphi$ or $\gamma_2 \nvdash_{\mathcal{BD}} -\gamma_2 \vee \varphi$. The latter case, where $\gamma_2 \nvdash_{\mathcal{BD}} \varphi$, has already been dealt with. Suppose therefore that $\gamma_1 \nvdash_{\mathcal{BD}} -\gamma_2 \vee \varphi$.

Now consider the following valuation $v : \mathbf{Fm} \to \mathbf{DM_8}$. If p and $-p$ are both literals of γ_1, take $v(p) := \mathsf{a}$. If p but not $-p$ is a literal of γ_1, take $v(p) := \mathsf{t}$, while if $-p$ but not p is a literal of γ_1, take $v(p) := \mathsf{f}$. For atoms such that neither p nor $-p$ is a literal of γ_1, take $v(p) := \mathsf{b}$ if p is a literal of γ_2 and $v(p) := \mathsf{c}$ if $-p$ is a literal of γ_2. (These two subcases are mutually exclusive, since γ_2 is not a classical contradiction.) For other atoms p take arbitrary $v(p) \in \{\mathsf{b}, \mathsf{c}\}$.

We have $v(\gamma_1) = \mathsf{a}$, since γ_1 contains for some atom p both p and $-p$. Moreover, $v(\gamma_2) \in \{\mathsf{t}, \mathsf{b}\}$, since γ_2 is a conjunction of literals l with $v(l) \in \{\mathsf{t}, \mathsf{b}\}$: if l is a literal of both γ_1 and γ_2, then $-l$ is not a literal of γ_1, since $\gamma_1 \nvdash_{\mathcal{BD}} -\gamma_2$. Thus $v(\gamma) = v(\gamma_1 \vee \gamma_2) = \mathsf{t}$. Finally, $v(\varphi) \in \{\mathsf{f}, \mathsf{b}, \mathsf{c}\}$. To see this, observe that all literals take values in $\{\mathsf{f}, \mathsf{b}, \mathsf{c}, \mathsf{t}\}$ and no literal of φ takes the value t because $\gamma_1 \nvdash_{\mathcal{BD}} \varphi$. Thus $\gamma \nvdash_{\mathcal{K}_-} \varphi$. $\square$

Using the above proposition, it is now easy to identify the logic $\mathcal{K}_-$ as the limit of an infinite chain of logics. Let $\mathcal{SDS}_n$ for $n \geq 1$ (for "strong disjunctive syllogism") be the extension of $\mathcal{BD}$ by the rule

$$\chi_n \vee q, -q \vee r \vdash r,$$

where again $\chi_n := (p_1 \wedge -p_1) \vee \dots (p_n \wedge -p_n)$. In particular, $\mathcal{SDS}_1$ is axiomatized by the rule $(p \wedge -p) \vee q, -q \vee r \vdash r$. Clearly $\mathcal{SDS}_n \leq \mathcal{SDS}_{n+1}$.

The join of this chain of logics will be denoted $\mathcal{SDS}_\omega$. As with $\mathcal{ECQ}_\omega$ and $\mathcal{ETL}_\omega$, observe that

$$\Gamma \vdash_{\mathcal{SDS}_\omega} \varphi \iff \Gamma \vdash_{\mathcal{SDS}_n} \varphi \text{ for some } n.$$

It is easy to see that $\mathcal{ETL}_n \leq \mathcal{SDS}_n$. In fact, even $\mathcal{ETL}_{n+1} \leq \mathcal{SDS}_n$.

Fact 5.16. $\mathcal{ETL}_{n+1} \leq \mathcal{SDS}_n$.

Proof. We have $\chi_{n+1} \vdash_{\mathcal{BD}} \chi_n \vee p_{n+1}$ and $\chi_{n+1} \vdash_{\mathcal{BD}} -p_{n+1} \vee \chi_n$, hence $\chi_{n+1} \vdash_{\mathcal{SDS}_n} \chi_n$ and $\chi_n \vdash_{\mathcal{SDS}_n} \emptyset$. □

The inequalities $\mathcal{SDS}_n \leq \mathcal{SDS}_{n+1}$ and $\mathcal{ETL}_{n+1} \leq \mathcal{SDS}_n$ are in fact strict. However, we postpone the proofs of these facts until the appropriate tools to separate these logics are introduced in Chapter 7 (Super-Belnap logics and finite graphs).

Proposition 5.17 (Completeness for $\mathcal{K}_-$).
$\mathcal{K}_- = \mathcal{SDS}_\omega$.

Proof. This follows immediately from Proposition 5.15 and the description of classical contradictions (Corollary 5.2). □

In Chapter 7 (Super-Belnap logics and finite graphs) we provide an entirely different proof of the completeness theorem for $\mathcal{K}_-$ (Proposition 7.34).

We may also axiomatize the logic $\mathcal{KO}_- := \mathcal{LP} \cap \mathcal{K}_-$ by essentially the same argument that we used to axiomatize $\mathcal{LP} \cap \mathcal{ETL}$.

Proposition 5.18 (Completeness for $\mathcal{KO}_-$).
$\mathcal{KO}_-$ is axiomatized by the rules $\chi_n \vee p, -p \vee q \vee -q \vdash q \vee -q$ for $n \geq 1$.

Proof. These rules hold in $\mathcal{KO}_-$. Conversely, suppose that $\Gamma \vdash_{\mathcal{KO}_-} \varphi$ and $\Gamma \nvdash_{\mathcal{BD}} \varphi$. By the Normal Form Theorem for $\mathcal{BD}$ (Theorem 3.12) we may assume that φ is a disjunctive clause. Then $\Gamma, \{p \vee -p \mid p \in \mathrm{At}(\varphi)\} \vdash_{\mathcal{BD}} \varphi$ by Proposition 3.5, hence $p \vee -p \vdash_{\mathcal{BD}} \varphi$ for some $p \in \mathrm{At}(\varphi)$ by Proposition 3.13. It follows that $\varphi = p \vee -p \vee \alpha$ for some α. Now $\Gamma \vdash_{\mathcal{K}_-} \varphi$, i.e. $\Gamma \vdash_{\mathcal{K}_-} p \vee -p \vee \alpha$, implies $\Gamma \vdash_{\mathcal{BD}} \chi \vee p \vee -p \vee \alpha$ and $\Gamma \vdash_{\mathcal{BD}} -\psi \vee p \vee -p \vee \alpha$ for some ψ and some classical contradiction χ by Proposition 5.15. The formula φ is therefore derivable from Γ in the extension of $\mathcal{BD}$ by the rules $\chi_n \vee p, -p \vee q \vee -q \vee r \vdash q \vee -q \vee r$ for $n \in \omega$. But these rules are derivable from the simpler rules $\chi_n \vee p, -p \vee q \vee -q \vdash q \vee -q$ as in the proof of Proposition 5.14. □

Chapter 6

The lattice of super-Belnap logics

The current chapter studies the landscape of super-Belnap logics from a purely lattice-theoretic perspective, abstracting away from the individual properties of the logics. We investigate the structure of the lattice $\operatorname{Ext}\mathcal{BD}$ by means of so-called splitting pairs of logics, which allow us to decompose $\operatorname{Ext}\mathcal{BD}$ into simpler parts. In particular, we show that the lattice of non-trivial super-Belnap logics has a certain vertical and horizontal structure. The horizontal structure consists in splitting into the disjoint intervals

$$[\mathcal{BD}, \mathcal{LP}], \qquad [\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}], \qquad [\mathcal{ETL}, \mathcal{CL}],$$

while the vertical structure consists in splitting into the disjoint intervals

$$[\mathcal{BD}, \mathcal{ETL}], \qquad [\mathcal{LP} \cap \mathcal{ECQ}_2, \mathcal{K}_{-}], \qquad [\mathcal{KO}, \mathcal{K}], \qquad [\mathcal{LP}, \mathcal{CL}].$$

The basic structure of the lattice of super-Belnap logics $\operatorname{Ext}\mathcal{BD}$ is depicted in Figure 6.1. The main goal of this chapter will be to show that this figure faithfully reflects the structure of $\operatorname{Ext}\mathcal{BD}$. Moreover, we shall see that the lattice $\operatorname{Ext}_\omega \mathcal{BD}$ is non-modular.

We also show that each of the intervals $[\mathcal{BD}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $[\mathcal{ETL}, \mathcal{CL}]$ contains an isomorphic copy of the lattice $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$. In particular, the lattices $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$ and $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$ are isomorphic, as are $\operatorname{Exp}\operatorname{Ext}\mathcal{ECQ}$ and $\mathcal{LP} \cap \operatorname{Exp}\operatorname{Ext}\mathcal{ECQ}$.

The study of $\operatorname{Ext}_\omega \mathcal{BD}$ will be continued in the following chapter, where we show that $\operatorname{Ext}_\omega \mathcal{BD}$ can be described in purely graph-theoretic terms, and infer that it has the cardinality of the continuum.

Our study of the splittings of $\operatorname{Ext}\mathcal{BD}$ has several important precedents. Splittings of lattices were studied already by Whitman [75], but it was McKenzie's investigation [45] of splitting pairs of equational theories (or equivalently, varieties) of lattices, along with Jankov's earlier study [35] of splittings of the lattice of super-intuitionistic logics, which established the importance of this notion in the study of lattices of equational theories and logics. A fruitful investigation of the splittings of the lattice of normal modal logics was also initiated by Blok [11].

The present investigation of the splittings of the lattice of super-Belnap logics differs from these antecedents in several respects. Firstly, we study the lattice of all extensions rather than the lattice of axiomatic extensions. Secondly, although we point out several important splittings of $\operatorname{Ext} \mathcal{BD}$, we do not aim here to fully characterize the splittings of this lattice. Finally, and perhaps most importantly, unlike super-intuitonistic logics and normal modal logics, super-Belnap logics are not directly amenable to a purely algebraic treatment.

The main challenge in the study of the lattice of super-Belnap logics lies in the fact that $\mathcal{BD}$ is not even protoalgebraic, much less algebraizable. The link between logic and algebra is therefore too weak in the super-Belnap realm to allow us to directly apply the algebraic methods used to study the lattices of super-intuitionistic logics and normal modal logics.

For example, the lattice of super-intuitionistic logics may be identified with the lattice of varieties of Heyting algebras and studied directly with the tools of universal algebra. A similar remark can be made regarding the lattice of normal modal logics and the lattice of varieties of Boolean algebras with operators. Apart from its intrinsic interest, the present investigation therefore also has some value as a contribution to the study of lattices of non-protoalgebraic logics.

6.1 Splitting the super-Belnap lattice

Throughout the present section $\mathcal{L}$ will always denote a super-Belnap logic.

A pair of super-Belnap logics $\mathcal{L}_1$ and $\mathcal{L}_2$ will be called a *splitting pair* if $\mathcal{L}_1 \nleq \mathcal{L}_2$ and each super-Belnap logic lies either above $\mathcal{L}_1$ or below $\mathcal{L}_2$, i.e. either $\mathcal{L}_1 \leq \mathcal{L}$ or $\mathcal{L} \leq \mathcal{L}_2$. In other words, $\operatorname{Ext} \mathcal{BD}$ splits into two disjoint intervals $\operatorname{Ext} \mathcal{L}_1$ and $[\mathcal{BD}, \mathcal{L}_2]$. In that case $\mathcal{L}_1$ is completely join prime in the sense that

$$\mathcal{L}_1 \leq \bigvee_{i \in I} \mathcal{L}_i \implies \mathcal{L}_1 \leq \mathcal{L}_i \text{ for some } i \in I,$$

and $\mathcal{L}_2$ is completely meet prime in the dual sense. It follows that $\mathcal{L}_1$ is the extension of $\mathcal{BD}$ by a single rule, say $\Gamma \vdash \varphi$, and $\mathcal{L}_2$ is determined by a single reduced model of $\mathcal{BD}$, say $\mathbb{A}$.

How does one prove that $\mathcal{L}_1$ and $\mathcal{L}_2$ indeed form a splitting pair? We always follow the same template. Suppose that $\mathcal{L}_1 \nleq \mathcal{L}$. Then $\mathcal{L}$ has a reduced model $\mathbb{M}$ where the rule $\Gamma \vdash \varphi$ fails, as witnessed by a valuation $v : \mathbf{Fm} \to \mathbb{M}$. The failure of $\Gamma \vdash \varphi$ in $\mathbb{M}$ tells us that $\mathbb{M}$ contains a submatrix $\mathbb{N}$ whose structure we can partly infer from the failure of the rule in the valuation v. The computational heart of the proof now consists in showing that if $\mathbb{N}$ is a model of $\mathcal{L}$, then so is the matrix $\mathbb{A}$ with respect to which $\mathcal{L}_2$ is complete, and therefore $\mathcal{L} \leq \mathcal{L}_2$. In the simplest cases, the main trick is to identify $\mathbb{A}$ as the Leibniz reduct of a submatrix of $\mathbb{N}$.

Our goal in the present section will be to demonstrate that Figure 6.1 faithfully represents the structure of Ext $\mathcal{BD}$. Since we have already proved completeness theorems for many super-Belnap logics, it is easy to verify the following strict inequalities.

Fact 6.1.

(i) $\mathcal{BD} < \mathcal{LP} \cap \mathcal{ECQ} < \mathcal{ECQ} < \mathcal{ETL}$.

(ii) $\mathcal{KO}_- < \mathcal{KO}_- \vee \mathcal{ECQ} < \mathcal{K}_-$.

(iii) $\mathcal{KO} < \mathcal{KO} \vee \mathcal{ECQ} < \mathcal{K}$.

(iv) $\mathcal{LP} < \mathcal{LP} \vee \mathcal{ECQ} < \mathcal{CL}$.

Fact 6.2. $\mathcal{LP} \cap \mathcal{ECQ}_{n+1} < \mathcal{ECQ}_{n+1} < \mathcal{ETL}_{n+1} < \mathcal{SDS}_n$ *for* $n \geq 1$.

Proof. Clearly $\mathcal{LP} \cap \mathcal{ECQ}_{n+1} \leq \mathcal{ECQ}_{n+1} \leq \mathcal{ETL}_{n+1}$. We have already seen (Fact 5.16) that $\mathcal{ETL}_{n+1} \leq \mathcal{SDS}_n$. Moreover, $\mathcal{ECQ}_{n+1} \geq \mathcal{ECQ} \nleq \mathcal{LP} \geq \mathcal{LP} \cap \mathcal{ECQ}_{n+1}$, hence $\mathcal{LP} \cap \mathcal{ECQ}_{n+1} < \mathcal{ECQ}_{n+1}$. Likewise, $\mathcal{ETL}_{n+1} \geq \mathcal{ETL} \nleq \mathcal{LP} \vee \mathcal{ECQ} \geq \mathcal{ECQ}_{n+1}$, hence $\mathcal{ECQ}_{n+1} < \mathcal{ETL}_{n+1}$. Finally, $\mathcal{SDS}_n \geq \mathcal{SDS}_1 \nleq \mathcal{ETL}_\omega \geq \mathcal{ETL}_{n+1}$, hence $\mathcal{ETL}_{n+1} < \mathcal{SDS}_n$. □

We shall see later (Fact 7.35) that $\mathcal{LP} \cap \mathcal{ECQ}_{n+2} \nleq \mathcal{SDS}_n$ for each $n \geq 1$, therefore also

$$\begin{aligned} \mathcal{LP} \cap \mathcal{ECQ}_n &< \mathcal{LP} \cap \mathcal{ECQ}_{n+1}, \\ \mathcal{ECQ}_n &< \mathcal{ECQ}_{n+1}, \\ \mathcal{ETL}_n &< \mathcal{ETL}_{n+1}, \\ \mathcal{SDS}_n &< \mathcal{SDS}_{n+1}. \end{aligned}$$

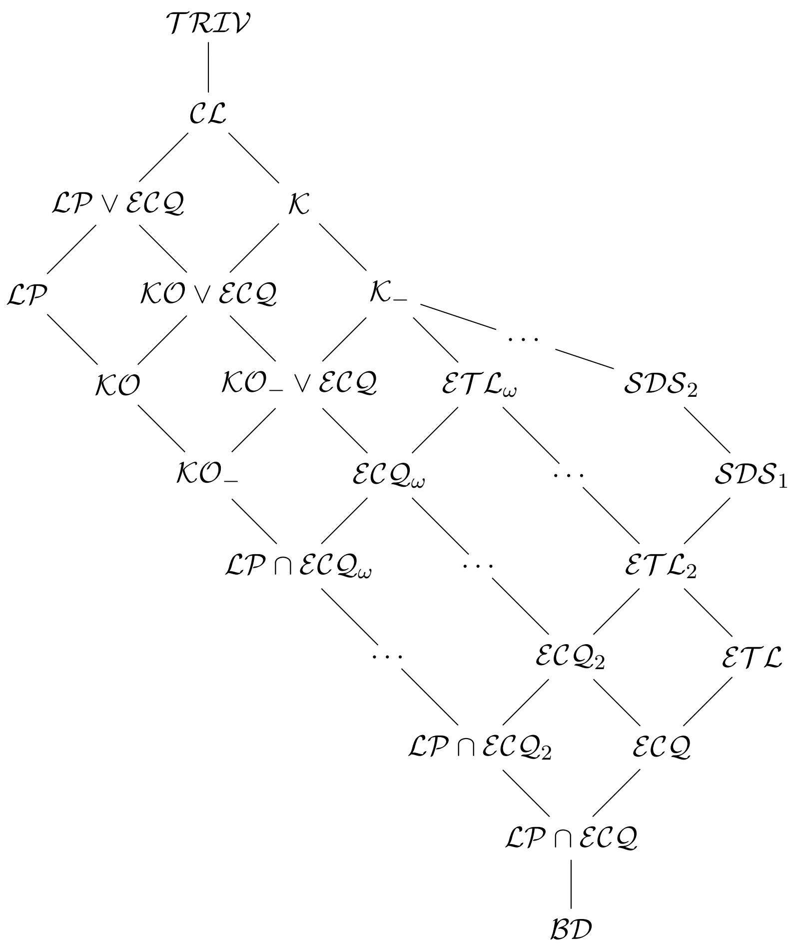
In particular, it follows that the logics $\mathcal{LP} \cap \mathcal{ECQ}_\omega$, $\mathcal{ECQ}_\omega$, $\mathcal{ETL}_\omega$, and $\mathcal{SDS}_\omega$ are not finitely axiomatizable.

As a first step in our study of Ext $\mathcal{BD}$, let us determine the largest non-trivial extension and the smallest proper extension of $\mathcal{BD}$.

Proposition 6.3 (Largest non-trivial extension of $\mathcal{BD}$).
$\mathcal{CL}$ is the largest non-trivial extension of $\mathcal{BD}$.

Proof. If $\mathcal{L}$ is a non-trivial extension of $\mathcal{BD}$, then it has a non-trivial reduced model $\langle \mathbf{A}, F \rangle$. We know that $\mathsf{t} \in F$ and moreover $\mathsf{f} \notin F$, since the matrix $\langle \mathbf{A}, F \rangle$ is non-trivial. The submatrix of $\langle \mathbf{A}, F \rangle$ with the universe $\{\mathsf{f}, \mathsf{t}\}$ is thus isomorphic to $\mathbb{B}_\mathbf{2}$. Therefore $\mathbb{B}_\mathbf{2}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{CL}$. □

Figure 6.1: The lattice of super-Belnap logics



Recall that by Proposition 4.20 the logic $\mathcal{LP} \cap \mathcal{ECQ}$ is axiomatized by the rule $p, -p \vdash q \vee -q$.

Proposition 6.4 (Smallest proper extension of $\mathcal{BD}$).
$\mathcal{LP} \cap \mathcal{ECQ}$ is the smallest proper extension of $\mathcal{BD}$.

Proof. Suppose that $\mathcal{LP} \cap \mathcal{ECQ} \nleq \mathcal{L}$. Then $p \wedge -p \nvdash_{\mathcal{L}} q \vee -q$, therefore $\mathcal{L}$ has a reduced model $\langle \mathbf{A}, F \rangle$ with elements $a \in F$ and $b \notin F$ such that $a \leq -a$ and $-b \leq b$. The principal congruence $\theta := \mathrm{Cg}^{\mathbf{A}} \langle a, -a \rangle$ is compatible with F: if $x \in F$ and $\langle x, y \rangle \in \theta$, then $x \wedge a = y \wedge a$ by the equational description of principal congruence of De Morgan lattices (Theorem 2.10), therefore $x \wedge a \in F$ and $y \in F$. Since the matrix $\langle \mathbf{A}, F \rangle$ is reduced, θ is the identity congruence and $a = -a$.

Consider now the submatrix $\langle \mathbf{B}, G \rangle$ of $\langle \mathbf{A}, F \rangle$ generated by the elements a and $c = (a \wedge b) \vee -b$. Note that $-c = (a \vee -b) \wedge b = c$. We have $c \notin F$ because $c \leq b \notin F$. The elements a and c are distinct because $a \in F$ but $c \notin F$. Since $a = -a$ and $c = -c$, the universe of the algebra $\mathbf{B}$ is the set $\{\mathsf{f}, a \wedge c, a, c, a \vee c, \mathsf{t}\}$ and $G = \{a, a \vee c, \mathsf{t}\}$. The congruence $\phi := \mathrm{Cg}^{\mathbf{B}} \langle a \vee c, \mathsf{t} \rangle$ is compatible with G and the matrix $\langle \mathbf{B}, G \rangle / \phi$ is isomorphic to $\mathbb{BD}_{\mathbf{4}}$. Therefore $\mathbb{BD}_{\mathbf{4}}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{BD}$. □

The lattice $\mathrm{Ext}\,\mathcal{BD}$ can also be split into those logics which share the theorems of $\mathcal{BD}$ and those which do not. Recall that by Proposition 3.4 the logic $\mathcal{L}$ has the same theorems as $\mathcal{CL}$ in case $\mathcal{LP} \leq \mathcal{L}$, and the same theorems as $\mathcal{BD}$ in case $\mathcal{L} \leq \mathcal{K}$. Moreover, $\mathcal{LP} \nleq \mathcal{K}$.

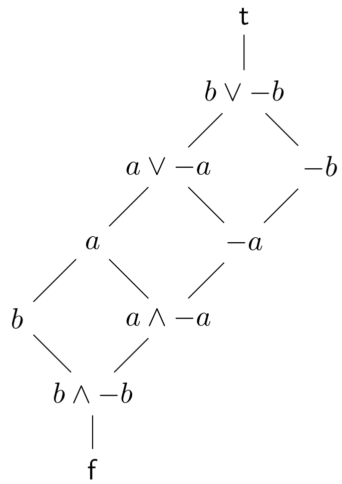
Proposition 6.5 (Splitting by theorems).
Either $\mathcal{LP} \leq \mathcal{L}$ or $\mathcal{L} \leq \mathcal{K}$.

Proof. Suppose that $\mathcal{LP} \nleq \mathcal{L}$. Then $\emptyset \nvdash_{\mathcal{L}} p \vee -p$ and $\mathcal{L}$ has a reduced model $\langle \mathbf{A}, F \rangle$ such that $a \notin F$ for some $a \in \mathbb{A}$ such that $-a \leq a$. Consider the submatrix $\langle \mathbf{B}, G \rangle$ of $\langle \mathbf{A}, F \rangle$ generated by a. The universe of $\mathbf{B}$ is the set $\{\mathsf{f}, -a, a, \mathsf{t}\}$ and $G = \{\mathsf{t}\}$. The congruence $\theta := \mathrm{Cg}^{\mathbf{B}} \langle -a, a \rangle$ is compatible with G and the matrix $\langle \mathbf{B}, G \rangle / \theta$ is isomorphic to $\mathbb{K}_{\mathbf{3}}$. Therefore $\mathbb{K}_{\mathbf{3}}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{K}$. □

Similarly, we can split $\mathrm{Ext}\,\mathcal{BD}$ into those logics which share the antitheorems of $\mathcal{BD}$ and those whic do not. Recall that $\mathrm{Exp}_{\mathcal{BD}}\,\mathcal{LP} = \mathcal{BD}$ (Proposition 5.8), therefore the logic $\mathcal{L}$ has the same antitheorems as $\mathcal{BD}$ if $\mathcal{L} \leq \mathcal{LP}$. On the other hand, if $\mathcal{ECQ} \leq \mathcal{L}$, then $\mathcal{L}$ has strictly more antitheorems than $\mathcal{BD}$, in particular $p \wedge -p$ is an antitheorem of $\mathcal{L}$ but not of $\mathcal{BD}$. Moreover, $\mathcal{ECQ} \nleq \mathcal{LP}$.

Proposition 6.6 (Splitting by antitheorems).
Either $\mathcal{ECQ} \leq \mathcal{L}$ or $\mathcal{L} \leq \mathcal{LP}$.

Figure 6.2: The free DMA generated by a and b modulo $b \leq a$ and $a \leq -a \vee b$



Proof. Suppose that $\mathcal{ECQ} \not\leq \mathcal{L}$. Then $p \wedge -p \nvdash_{\mathcal{L}} q$ and $\mathcal{L}$ has a non-trivial reduced model $\langle \mathbf{A}, F \rangle$ such that $a \in F$ for some $a \in \mathbf{A}$ with $a \leq -a$. Consider the submatrix $\langle \mathbf{B}, G \rangle$ of $\langle \mathbf{A}, F \rangle$ generated by a. The universe of $\mathbf{B}$ is the set $\{\mathsf{f}, a, -a, \mathsf{t}\}$ and $G = \{a, -a, \mathsf{t}\}$. The congruence $\theta := \mathrm{Cg}^{\mathbf{B}} \langle a, -a \rangle$ is compatible with G and the matrix $\langle \mathbf{B}, G \rangle / \theta$ is isomorphic to $\mathbb{P}_{\mathbf{3}}$. Therefore $\mathbb{P}_{\mathbf{3}}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{LP}$. $\square$

It remains to establish one more splitting in order to show that $\operatorname{Ext} \mathcal{BD}$ splits into three disjoint intervals which form its "horizontal" structure. We shall see in Chapter 7 (Super-Belnap logics and finite graphs) that all of these intervals have the cardinality of the continuum when restricted to finitary logics, and moreover that the horizontal structure of $\operatorname{Ext} \mathcal{BD}$ is important when it comes to describing the finite reduced models of super-Belnap logics.

Lemma 6.7.
The algebra shown in Figure 6.2 is the free De Morgan algebra generated by a and b modulo the inequalities $b \leq a$ and $a \leq -a \vee b$.

Proof. The algebra is clearly a distributive lattice and the (unique) order-inverting involution on this lattice yields a De Morgan algebra. It now suffices to show that all equalities which hold in the algebra in Figure 6.2 are consequences of the inequalities $b \leq a$ and $a \leq -a \vee b$. In particular, it suffices to check that these inequalities imply that $b \wedge -a = b \wedge -b$ and $a \wedge -a = a \wedge -b$, therefore dually $-b \vee a = -b \vee b$ and $-a \vee a = -a \vee b$. Moreover, $a = b \vee (a \wedge -a)$ and $-a = -b \wedge (a \vee -a)$. $\square$

Theorem 6.8 (Horizontal structure of $\mathrm{Ext}\,\mathcal{BD}$).
Each non-trivial proper extension of $\mathcal{BD}$ lies in exactly one of the intervals $[\mathcal{LP} \cap \mathcal{ECQ}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, $[\mathcal{ETL}, \mathcal{CL}]$.

Proof. The disjointness of these intervals is easy to check since we have completeness theorems for all six logics. We have already shown that each non-trivial proper extension of $\mathcal{BD}$ lies in the interval $[\mathcal{LP} \cap \mathcal{ECQ}, \mathcal{CL}]$, and if it does not lie below $\mathcal{LP}$, then it lies above $\mathcal{ECQ}$. It remains to split the interval $[\mathcal{ECQ}, \mathcal{CL}]$ into $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$ and $[\mathcal{ETL}, \mathcal{CL}]$.

Suppose therefore that $\mathcal{ETL} \nleq \mathcal{L}$. Then $p, -p \vee q \nvdash_{\mathcal{L}} q$ and $\mathcal{L}$ has a reduced model $\langle \mathbf{A}, F \rangle$ such that $a \in F$ and $-a \vee b \in F$ but $b \notin F$ for some $a, b \in \mathbf{A}$. Without loss of generality we may take $b := a \wedge b$ and $a := a \wedge (-a \vee b)$, i.e. we may assume that $b \leq a$ and $a \leq -a \vee b$.

Consider the submatrix $\langle \mathbf{B}, G \rangle$ of $\langle \mathbf{A}, F \rangle$ generated by a and b. Let $\mathbf{C}$ be the algebra shown in Figure 6.2. By the previous lemma there is a homomorphism $h : \mathbf{C} \to \mathbf{B}$. Expand $\mathbf{C}$ to a matrix $\langle \mathbf{C}, H \rangle$ such that $H = h^{-1}[G]$. In particular, $a \in H$ and $b \notin H$. The matrix $\langle \mathbf{C}, H \rangle$ is a model of $\mathcal{L}$ by virtue of being a strict homomorphic pre-image of $\langle \mathbf{B}, G \rangle$.

Now H is the principal lattice filter generated either by a or by $a \wedge -a$. In the former case the congruence $\phi := \mathrm{Cg}^{\mathbf{C}} \langle a, a \vee -a \rangle$ is compatible with H and the matrix $\langle \mathbf{C}, H \rangle / \phi$ is isomorphic to $\mathbb{P}_{\mathbf{3}} \times \mathbb{B}_{\mathbf{2}}$. In the latter case the congruence $\psi := \mathrm{Cg}^{\mathbf{C}} \langle a \wedge -a, \mathsf{t} \rangle$ is compatible with H and the matrix $\langle \mathbf{C}, H \rangle / \psi$ is isomorphic to $\mathbb{P}_{\mathbf{3}}$. Thus either $\mathbb{P}_{\mathbf{3}} \times \mathbb{B}_{\mathbf{2}}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{LP} \vee \mathcal{ECQ}$ by the completeness theorem for $\mathcal{LP} \vee \mathcal{ECQ}$ (Proposition 5.9), or $\mathbb{P}_{\mathbf{3}}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{LP} \leq \mathcal{LP} \vee \mathcal{ECQ}$. $\square$

It immediately follows from the above theorem that lattice of non-trivial extensions of $\mathcal{LP}$ consists of the logics $\mathcal{LP}$, $\mathcal{LP} \vee \mathcal{ECQ}$, and $\mathcal{LP} \vee \mathcal{ETL} = \mathcal{CL}$. This was in fact already proved by Pynko [64].

Proposition 6.9 [64, Thm 4.13] (Extensions of $\mathcal{LP}$).
$\mathcal{LP}$ has exactly two non-trivial proper extensions: $\mathcal{LP} \vee \mathcal{ECQ}$ and $\mathcal{CL}$.

We can also infer that classical logic can in a certain sense be canonically decomposed into a join of $\mathcal{LP}$ and $\mathcal{ETL}$. To be more precise, let us say that $c = \bigvee_{i \in I} a_i$ is a *canonical decomposition* of an element c of a lattice if $c = \bigvee_{j \in J} b_j$ implies that for each $i \in I$ there is some $j \in J$ such that $a_i \leq b_j$. In other words, if c is obtained as a join of elements, then in a sense the elements a_i cannot be avoided among the joinands. (This definition is essentially taken from [37].)

Proposition 6.10 (Canonical decomposition of $\mathcal{CL}$).
$\mathcal{CL} = \mathcal{LP} \vee \mathcal{ETL}$ is a canonical decomposition of $\mathcal{CL}$ in $\mathrm{Ext}\,\mathcal{BD}$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be super-Belnap logics. If $\mathcal{L}_1 \ngeq \mathcal{LP}$ and $\mathcal{L}_2 \ngeq \mathcal{LP}$, then $\mathcal{L}_1 \leq \mathcal{K}$ and $\mathcal{L}_2 \leq \mathcal{K}$, therefore $\mathcal{L}_1 \vee \mathcal{L}_2 \leq \mathcal{K}$. If $\mathcal{L}_2 \ngeq \mathcal{ETL}$ and

$\mathcal{L}_2 \not\geq \mathcal{ETL}$, then $\mathcal{L}_1 \leq \mathcal{LP} \vee \mathcal{ECQ}$ and $\mathcal{L}_2 \leq \mathcal{LP} \vee \mathcal{ECQ}$, therefore $\mathcal{L}_1 \vee \mathcal{L}_2 \leq \mathcal{LP} \vee \mathcal{ECQ}$. Therefore, if $\mathcal{L}_1 \vee \mathcal{L}_2 = \mathcal{CL}$, then at least one of the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ is an extension of $\mathcal{LP}$ and at least one is an extension of $\mathcal{ETL}$. $\square$

A further splitting theorem will enable us to provide a similar canonical decomposition for $\mathcal{K}$. Recall that the logic $\mathcal{K}_-$ was introduced in Chapter 5 (Completeness theorems) by means of the matrix $\mathbb{M}_\mathbf{8}$ (see Figure 5.1).

Lemma 6.11.
The algebra shown in Figure 6.3 is the free De Morgan algebra generated by a and b modulo the inequalities $a \leq -a$ and $b \leq -b$.

Proof. We omit the tedious but straightforward proof of this claim. $\square$

Proposition 6.12 (Splitting by $\mathcal{K}_-$).
Either $\mathcal{KO} \leq \mathcal{L}$ or $\mathcal{L} \leq \mathcal{K}_-$.

Proof. Suppose that $\mathcal{KO} \nleq \mathcal{L}$. Then $(p \wedge -p) \vee r \nvdash_\mathcal{L} (q \vee -q) \vee r$ and $\mathcal{L}$ has a reduced model $\langle \mathbf{A}, F \rangle$ such that $a \vee d \in F$ and $c \vee d \notin F$ for some $a, c, d \in \mathbf{A}$ with $a \leq -a$ and $-c \leq c$. Without loss of generality we may take $d := c \vee d$ and $b := -d$. It follows that $b \leq -b$ and $-b \notin F$.

Consider the submatrix $\langle \mathbf{B}, G \rangle$ of $\langle \mathbf{A}, F \rangle$ generated by the elements a and b. Let $\mathbf{C}$ be the algebra shown in Figure 6.3. By the previous lemma there is a homomorphism $h : \mathbf{C} \to \mathbf{B}$. Let us expand $\mathbf{C}$ to a matrix $\langle \mathbf{C}, H \rangle$ such that $H = h^{-1}[G]$. In particular, $a \vee -b \in H$ and $-b \notin H$. The matrix $\langle \mathbf{C}, H \rangle$ is a model of $\mathcal{L}$ by virtue of being a strict homomorphic pre-image of $\langle \mathbf{B}, G \rangle$. The congruence $\theta := \mathrm{Cg}^\mathbf{C} \langle -a \vee -b, \mathsf{t} \rangle$ is compatible with H, hence the matrix $\langle \mathbf{D}, I \rangle := \langle \mathbf{C}, H \rangle / \theta$ is a model of $\mathcal{L}$.

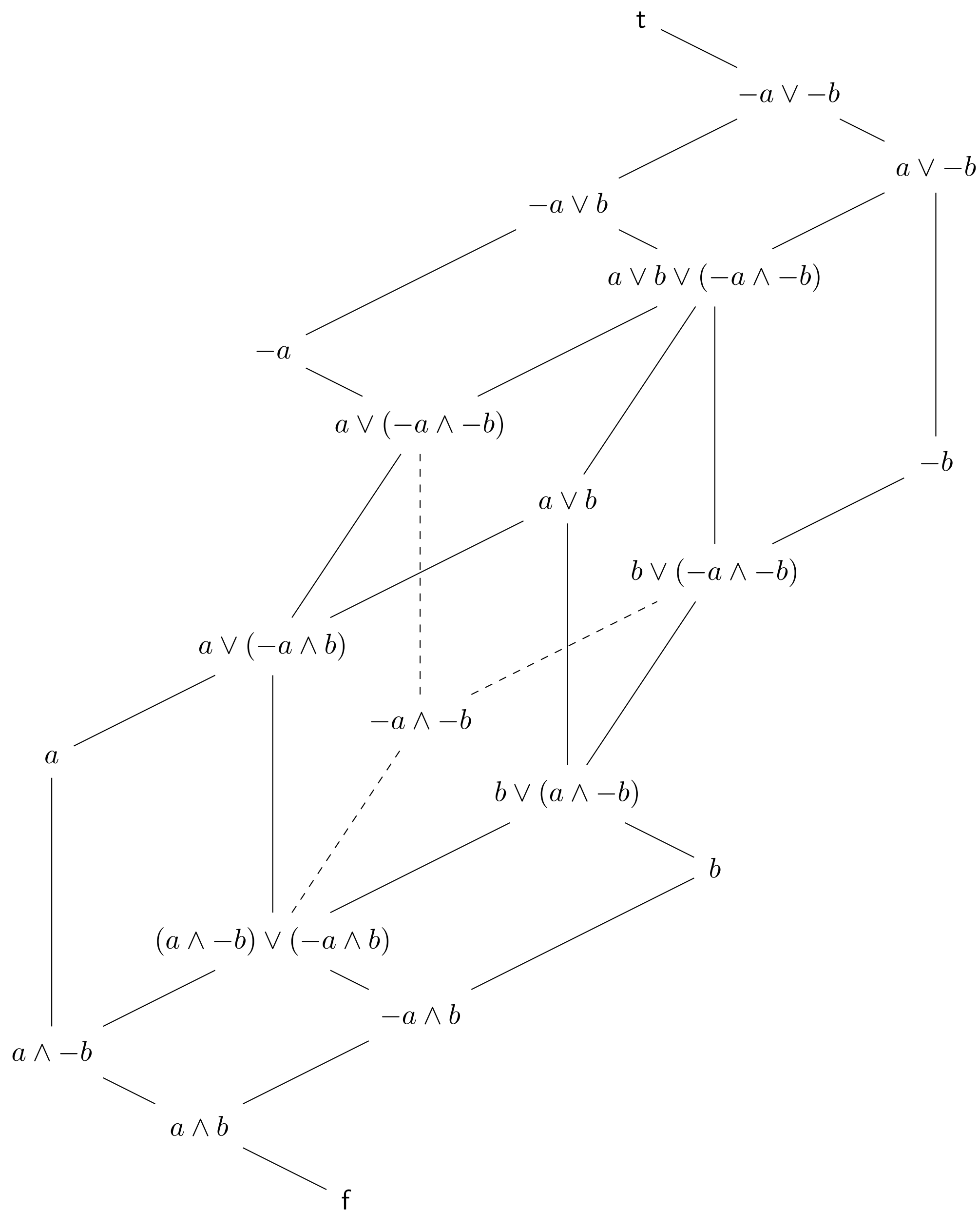
There are now several cases to consider. If $-a \vee b \notin I$, then the congruence $\phi := \mathrm{Cg}^\mathbf{D} \langle a, -a \rangle$ is compatible with I and $\langle \mathbf{D}, I \rangle / \phi$ is isomorphic to the matrix $\mathbb{M}_\mathbf{8}$ (see Figure 5.1). Then $\mathbb{M}_\mathbf{8}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{K}_-$. On the other hand, if $a \vee (-a \wedge -b) \in I$, then the rule $p \wedge -p \vdash q \vee -q$ fails in $\langle \mathbf{D}, I \rangle$ under the valuation $v(p) := a \vee (-a \wedge -b)$ and $v(q) := -b$. Therefore $\mathcal{LP} \cap \mathcal{ECQ} \nleq \mathrm{Log} \langle \mathbf{D}, I \rangle$ and $\mathcal{L} \leq \mathrm{Log} \langle \mathbf{D}, I \rangle \leq \mathcal{BD}$ by Proposition 6.4.

Finally, suppose that $-a \vee b \in I$ and $a \vee (-a \wedge b) \notin I$. Then the congruence $\psi := \mathrm{Cg}^\mathbf{D} \langle a \vee b \vee (-a \wedge -b), \mathsf{t} \rangle$ is compatible with I and the matrix $\langle \mathbf{D}, I \rangle / \psi$ is isomorphic either to $\mathbb{BD}_\mathbf{4} \times \mathbb{B}_\mathbf{2}$ or to $\mathbb{ETL}_\mathbf{4} \times \mathbb{B}_\mathbf{2}$. Thus either $\mathcal{L} \leq \mathrm{Log}\, \mathbb{BD}_\mathbf{4} \times \mathbb{B}_\mathbf{2} = \mathcal{ECQ}_\omega \leq \mathcal{K}_-$ by Proposition 5.11 or $\mathcal{L} \leq \mathrm{Log}\, \mathbb{ETL}_\mathbf{4} \times \mathbb{B}_\mathbf{2} = \mathcal{ETL}_\omega \leq \mathcal{K}_-$ by Proposition 5.12. $\square$

Proposition 6.13 (Canonical decomposition of $\mathcal{K}$).
$\mathcal{K} = \mathcal{KO} \vee \mathcal{ETL}$ is a canonical decomposition of $\mathcal{K}$ in $\mathrm{Ext}\, \mathcal{BD}$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be super-Belnap logics. If $\mathcal{L}_1 \not\geq \mathcal{KO}$ and $\mathcal{L}_2 \not\geq \mathcal{KO}$, then $\mathcal{L}_1 \leq \mathcal{K}_-$ and $\mathcal{L}_2 \leq \mathcal{K}_-$, thus $\mathcal{L}_1 \vee \mathcal{L}_2 \leq \mathcal{K}_-$. If $\mathcal{L}_2 \not\geq \mathcal{ETL}$ and $\mathcal{L}_2 \not\geq \mathcal{ETL}$, then $\mathcal{L}_1 \leq \mathcal{LP} \vee \mathcal{ECQ}$ and $\mathcal{L}_2 \leq \mathcal{LP} \vee \mathcal{ECQ}$, therefore $\mathcal{L}_1 \vee \mathcal{L}_2 \leq$

Figure 6.3: The free DMA generated by a and b modulo $a \leq -a$ and $b \leq -b$



$\mathcal{LP} \vee \mathcal{ECQ}$. Thus if $\mathcal{L}_1 \vee \mathcal{L}_2 = \mathcal{CL}$, then at least one of the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ is an extension of $\mathcal{KO}$ and at least one is an extension of $\mathcal{ETL}$. $\square$

The upper part of Ext $\mathcal{BD}$ can now be described explicitly.

Fact 6.14. $\mathcal{KO}_- \vee \mathcal{ETL} = \mathcal{K}_-$.

Proof. Let $\chi_n := (p_1 \wedge -p_1) \vee \cdots \vee (p_n \wedge -p_n)$ and let $\Gamma := \{\chi_n \vee p, -p \vee q\}$. To derive the rule $\Gamma \vdash q$ from the rules valid in $\mathcal{KO}_-$ and $\mathcal{ETL}$, it suffices to derive in $\mathcal{KO}_-$ the rule $\Gamma \vdash -(\chi_n \vee p) \vee q$. This is equivalent to showing that $\Gamma \vdash_{\mathcal{KO}_-} -p \vee q$ and $\Gamma \vdash_{\mathcal{KO}_-} -\chi_n \vee q$. The first rule is trivially valid and the second rule is equivalent to $\Gamma \vdash_{\mathcal{KO}_-} p_i \vee -p_i \vee q$ for each p_i. It therefore suffices to show that $\Gamma \vdash_{\mathcal{KO}_-} q \vee r \vee -r$. But $-p \vee q \vdash_{\mathcal{BD}} -p \vee \alpha \vee -\alpha$ for $\alpha := q \vee (r \wedge -r)$, therefore $\Gamma \vdash_{\mathcal{KO}_-} \alpha \vee -\alpha$, and $\alpha \vee -\alpha \vdash_{\mathcal{BD}} q \vee r \vee -r$. $\square$

Theorem 6.15 (Extensions of $\mathcal{KO}_-$).
$\mathcal{KO}_-$ has exactly nine non-trivial extensions: $\mathcal{KO}_-$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{KO}_- \vee \mathcal{ECQ}$, $\mathcal{KO} \vee \mathcal{ECQ}$, $\mathcal{LP} \vee \mathcal{ECQ}$, $\mathcal{K}_-$, $\mathcal{K}$, $\mathcal{CL}$.

Proof. We have $\mathcal{KO} \vee \mathcal{ECQ} = (\mathcal{LP} \cap \mathcal{K}) \vee \mathcal{ECQ} = (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{K}$ and $\mathcal{KO}_- \vee \mathcal{ECQ} = (\mathcal{LP} \cap \mathcal{K}_-) \vee \mathcal{ECQ} = (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{K}_-$ by Proposition 5.6.

We have already described the extensions of $\mathcal{LP}$ (Proposition 6.9). If $\mathcal{L}$ is not an extension of $\mathcal{LP}$, then $\mathcal{L} \leq \mathcal{K}$. If $\mathcal{KO} < \mathcal{L} \leq \mathcal{K}$, then $\mathcal{L} \nleq \mathcal{LP}$ (otherwise $\mathcal{KO} < \mathcal{L} \leq \mathcal{LP} \cap \mathcal{K}$), hence $\mathcal{KO} \vee \mathcal{ECQ} \leq \mathcal{L}$. If $\mathcal{KO} \vee \mathcal{ECQ} < \mathcal{L} \leq \mathcal{K}$, then $\mathcal{L} \nleq \mathcal{LP} \vee \mathcal{ECQ}$ (otherwise $\mathcal{KO} \vee \mathcal{ECQ} < \mathcal{L} \leq (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{K}$), hence $\mathcal{K} = \mathcal{KO} \vee \mathcal{ETL} \leq \mathcal{L}$.

Now suppose that $\mathcal{KO} \nleq \mathcal{L}$. Then $\mathcal{L} \leq \mathcal{K}_-$. If $\mathcal{KO}_- < \mathcal{L}$, then $\mathcal{L} \nleq \mathcal{LP}$ (otherwise $\mathcal{KO}_- < \mathcal{L} \leq \mathcal{LP} \cap \mathcal{K}_-$), hence $\mathcal{KO}_- \vee \mathcal{ECQ} \leq \mathcal{L}$. If $\mathcal{KO}_- \vee \mathcal{ECQ} < \mathcal{L}$, then $\mathcal{L} \nleq \mathcal{LP} \vee \mathcal{ECQ}$ (otherwise $\mathcal{KO}_- \vee \mathcal{ECQ} < \mathcal{L} \leq (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{K}_-$), hence $\mathcal{KO}_- \vee \mathcal{ETL} \leq \mathcal{L}$. But $\mathcal{KO}_- \vee \mathcal{ETL} = \mathcal{K}_-$. $\square$

It remains to establish one more splitting in order to show that Ext $\mathcal{BD}$ splits into four disjoint intervals which form its "vertical structure". We shall see in Chapter 8 (Metalogical properties of super-Belnap logics) that the algebraic counterpart Alg $\mathcal{L}$ of a super-Belnap logic $\mathcal{L}$ depends on its position in this vertical structure.

Proposition 6.16 (Splitting by $\mathcal{ETL}$).
Either $\mathcal{LP} \cap \mathcal{ECQ}_2 \leq \mathcal{L}$ or $\mathcal{L} \leq \mathcal{ETL}$.

Proof. Suppose that $\mathcal{LP} \cap \mathcal{ECQ}_2 \nleq \mathcal{L}$. Then $(p \wedge -p) \vee (q \wedge -q) \nvdash_{\mathcal{L}} r \vee -r$ and $\mathcal{L}$ has a reduced model $\langle \mathbf{A}, F \rangle$ such that $a \vee b \in F$ for some $a, b \in \mathbf{A}$ with $a \leq -a$ and $b \leq -b$, and $c \notin F$ for some $c \in \mathbf{A}$ with $-c \leq c$.

Consider the submatrix $\langle \mathbf{B}, G \rangle$ of $\langle \mathbf{A}, F \rangle$ generated by the elements a and b. Let $\mathbf{C}$ be the algebra shown in Figure 6.3. As in the proof of the splitting $(\mathcal{KO}, \mathcal{K}_-)$, there is a homomorphism $h : \mathbf{C} \to \mathbf{B}$ and $\mathbf{C}$ may be expanded to a model $\langle \mathbf{C}, H \rangle$ of $\mathcal{L}$ such that $H = h^{-1}[G]$ and $a \vee b \in H$.

If $a \vee (-a \wedge -b) \in H$ or $b \vee (a \wedge -b) \in H$, then there is some $d \in F$ such that $d \leq -d$: either $d = h(a \vee (-a \wedge -b))$ or $d = h(b \vee (a \wedge -b))$. Since $c \notin F$ for some $c \in \mathbf{A}$ such that $-c \leq c$, it follows that $p \wedge -p \vdash q \vee -q$ fails in $\langle \mathbf{A}, F \rangle$, hence $\mathcal{L} \leq \mathrm{Log}\langle \mathbf{A}, F \rangle \leq \mathcal{BD}$ by Proposition 6.4.

Suppose therefore that $a \vee (-a \wedge -b) \notin H$ or $b \vee (a \wedge -b) \notin H$. In that case H is the principal filter generated by $a \vee b$ because $a \vee b \in H$. The congruence $\theta := \mathrm{Cg}^{\mathbf{C}}\langle a \vee b, \mathsf{t} \rangle$ is then compatible with H and the matrix $\langle \mathbf{C}, H \rangle / \theta$ is isomorphic to $\mathbb{ETL}_{\mathbf{4}}$. Therefore $\mathbb{ETL}_{\mathbf{4}}$ is a model of $\mathcal{L}$ and $\mathcal{L} \leq \mathcal{ETL}$. □

The following corollary was already proved by Rivieccio [67]. Indeed it was Rivieccio's result which suggested that the above splitting might exist.

Corollary 6.17 [67]**.** *$\mathcal{ETL}_2$ is the smallest proper extension of $\mathcal{ETL}$.*

Theorem 6.18 (Vertical structure of $\mathrm{Ext}\,\mathcal{BD}$**).**
Each non-trivial proper extension of $\mathcal{BD}$ lies in exactly one of the intervals $[\mathcal{LP} \cap \mathcal{ECQ}, \mathcal{ETL}]$, $[\mathcal{LP} \cap \mathcal{ECQ}_2, \mathcal{K}_{-}]$, $[\mathcal{KO}, \mathcal{K}]$, $[\mathcal{LP}, \mathcal{CL}]$.

Proof. The disjointness claim is again easy to establish because we have completeness theorems for all the logic involved. The rest now follows from the splittings established above. □

We establish one more splitting of $\mathrm{Ext}\,\mathcal{BD}$, although we shall not make use of this splitting in what follows.

Proposition 6.19 (Splitting by $\mathcal{ETL}_\omega$).
Either $(p \wedge -p) \vee q \vee -q, (q \wedge -q) \vee p \vee -p \vdash_{\mathcal{L}} p \vee -p$ or $\mathcal{L} \leq \mathcal{ETL}_\omega$.

Proof. Suppose that $(p \wedge -p) \vee q \vee -q, (q \wedge -q) \vee p \vee -p \nvdash_{\mathcal{L}} p \vee -p$. Then $\mathcal{L}$ has a reduced model $\langle \mathbf{A}, F \rangle$ such that $a \vee -b \in F$ and $b \vee -a \in F$ but $-b \notin F$ for some $a, b \in \mathbf{A}$ with $a \leq -a$ and $b \leq -b$.

We now proceed as in the proofs of the previous two splittings. Again, if $a \vee (-a \wedge b) \in H$, then the rule $p \wedge -p \vdash q \vee -q$ fails in $\langle \mathbf{C}, H \rangle$, hence $\mathcal{L} \leq \mathrm{Log}\langle \mathbf{C}, H \rangle = \mathcal{BD}$. Suppose instead that $a \vee (-a \wedge b) \notin H$. Then H is a principal filter generated either by $a \vee (-a \wedge -b)$ or by $a \vee b$ or by $a \vee b \vee (-a \wedge -b)$. In the first two cases, the Leibniz reduct of $\langle \mathbf{C}, H \rangle$ is isomorphic to the matrix $\mathbb{BD}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$, while in the third case it is isomorphic to them matrix $\mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$. But we know that $\mathrm{Log}\,\mathbb{BD}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}} = \mathcal{ECQ}_\omega \leq \mathcal{ETL}_\omega$ and $\mathrm{Log}\,\mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}} = \mathcal{ETL}_\omega$ by Propositions 5.11 and 5.12. Therefore $\mathcal{L} \leq \mathrm{Log}\langle \mathbf{C}, H \rangle \leq \mathcal{ETL}_\omega$. □

We do not have a semantic description of the logic axiomatized by the rule $(p \wedge -p) \vee q \vee -q, (q \wedge -q) \vee p \vee -p \vdash_{\mathcal{L}} p \vee -p$. In particular, we do not know whether it is the logic of a finite set of finite matrices. However, one can check that this rule fails in $\mathcal{ETL}_\omega = \mathrm{Log}\,\mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$, therefore the above proposition really does establish a splitting of $\mathrm{Ext}\,\mathcal{BD}$.

We end our study of the lattice of super-Belnap logics by showing that it is non-modular, or more precisely that $\mathrm{Ext}_\omega\,\mathcal{BD}$ is non-modular.

Proposition 6.20.
$(\mathcal{LP} \cap \mathcal{ETL}) \vee \mathcal{ECQ} < (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{ETL}$.

Proof. We have

$$\begin{aligned}(\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{ETL} &= (\mathcal{LP} \cup \mathcal{ECQ}_\omega) \cap \mathcal{ETL} \\ &= (\mathcal{LP} \cap \mathcal{ETL}) \cup (\mathcal{ECQ}_\omega \cap \mathcal{ETL})\end{aligned}$$

and by Proposition 4.28

$$\begin{aligned}(\mathcal{LP} \cap \mathcal{ETL}) \vee \mathcal{ECQ} &= (\mathcal{LP} \cap \mathcal{ETL}) \vee \operatorname{Exp}_{\mathcal{BD}} \mathcal{ETL} \\ &= (\mathcal{LP} \cap \mathcal{ETL}) \cup \operatorname{Exp}_{\mathcal{BD}} \mathcal{ETL} = (\mathcal{LP} \cap \mathcal{ETL}) \cup \mathcal{ECQ}.\end{aligned}$$

It therefore suffices to find Γ and φ such that $\Gamma \vdash_{\mathcal{ECQ}_\omega \cap \mathcal{ETL}} \varphi$ but $\Gamma \nvdash_{\mathcal{ECQ}} \varphi$ and $\Gamma \nvdash_{\mathcal{LP}} \varphi$. Take $\Gamma := (p_1 \wedge -p_1) \vee (p_2 \wedge -p_2), q, -q \vee r$ and $\varphi := r$. Then clearly $\Gamma \vdash_{\mathcal{ECQ}_\omega \cap \mathcal{ETL}} \varphi$. Moreover, $\Gamma \nvdash_{\mathcal{LP}} \varphi$, as witnessed by the valuation $v : \mathbf{Fm} \to \mathbf{P_3}$ such that $v(p_1) = v(p_2) = q = \mathsf{b}$ and $v(r) = \mathsf{f}$. To see that, $\Gamma \nvdash_{\mathcal{ECQ}} \varphi$, recall that $\mathcal{ECQ} = \operatorname{Log} \mathbb{BD}_\mathbf{4} \times \mathbb{ETL}_\mathbf{4}$ by Proposition 5.13. It is easy to check that there is a valuation on $\mathbb{BD}_\mathbf{4} \times \mathbb{ETL}_\mathbf{4}$ which designates $(p_1 \wedge -p_1) \vee (p_2 \wedge -p_2)$, therefore $\Gamma \vdash_{\mathcal{ECQ}} \varphi$ only if $q, -q \vee r \vdash_{\mathcal{ECQ}} r$. But $\mathcal{ECQ} < \mathcal{ETL}$, i.e. $q, -q \vee r \nvdash_{\mathcal{ECQ}} r$. □

Corollary 6.21. *The lattice $\operatorname{Ext}_\omega \mathcal{BD}$ is not modular.*

6.2 Lattices of explosive extensions

In this brief section we show that each of the three intervals $[\mathcal{BD}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $[\mathcal{ETL}, \mathcal{CL}]$ of $\operatorname{Ext}_\omega \mathcal{BD}$ contains an isomorphic copy of $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ETL}$. In combination with the result proved in the following chapter (Corollary 7.47) that $\mathcal{ETL}$ in fact has continuum many finitary explosive extensions, this shows that each of the three intervals contains a continuum of finitary logics.

Let us first observe that the (finitary) explosive extensions of $\mathcal{BD}$ consist precisely of $\mathcal{BD}$ and the (finitary) explosive extensions of $\mathcal{ECQ}$. The lattice $\operatorname{Exp} \operatorname{Ext}_{(\omega)} \mathcal{ECQ}$ is therefore precisely the lattice of proper (finitary) explosive extensions of $\mathcal{BD}$.

Fact 6.22. $\operatorname{Exp} \operatorname{Ext}_{(\omega)} \mathcal{BD} = \{\mathcal{BD}\} \cup \operatorname{Exp} \operatorname{Ext}_{(\omega)} \mathcal{ECQ}$.

Proof. We know that $\operatorname{Exp}_{\mathcal{BD}} \mathcal{LP} = \mathcal{BD}$ (Proposition 5.8), thus $\mathcal{L}_{\mathrm{exp}} \nleq \mathcal{LP}$ for each proper explosive extension $\mathcal{L}_{\mathrm{exp}}$ of $\mathcal{BD}$. But then the splitting pair $\langle \mathcal{ECQ}, \mathcal{LP} \rangle$ (Proposition 6.6) yields that $\mathcal{ECQ} \leq \mathcal{L}_{\mathrm{exp}}$. □

Theorem 6.23.
The lattices $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ETL}$ and $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ECQ}$ are isomorphic via the maps $\mathcal{L} \mapsto \operatorname{Exp}_{\mathcal{BD}} \mathcal{L}$ and $\mathcal{L} \mapsto \mathcal{ETL} \vee \mathcal{L}$.

Proof. We have $\mathcal{L}_{\text{exp}} = \operatorname{Exp}_{\mathcal{ETL}} \mathcal{L}_{\text{exp}} = \mathcal{ETL} \vee \operatorname{Exp}_{\mathcal{BD}} \mathcal{L}_{\text{exp}}$ for each $\mathcal{L}_{\text{exp}} \in \operatorname{Exp}\operatorname{Ext} \mathcal{ETL}$. Moreover, $\mathcal{L}_{\text{exp}} = \operatorname{Exp}_{\mathcal{BD}} \mathcal{L}_{\text{exp}} \leq \operatorname{Exp}_{\mathcal{BD}}(\mathcal{ETL} \vee \mathcal{L}_{\text{exp}})$ for each $\mathcal{L}_{\text{exp}} \in \operatorname{Exp}\operatorname{Ext} \mathcal{ECQ}$. It remains to prove that $\operatorname{Exp}_{\mathcal{BD}}(\mathcal{ETL} \vee \mathcal{L}_{\text{exp}}) \leq \mathcal{L}_{\text{exp}}$ for each $\mathcal{L}_{\text{exp}} \in \operatorname{Exp}\operatorname{Ext}_{\omega} \mathcal{ECQ}$. The logic $\operatorname{Exp}_{\mathcal{BD}}(\mathcal{ETL} \vee \mathcal{L}_{\text{exp}})$ is finitary, therefore it suffices to show that each finitary explosive rule which fails in $\mathcal{L}_{\text{exp}}$ also fails in $\mathcal{ETL} \vee \mathcal{L}_{\text{exp}}$.

Suppose therefore that $\Gamma \nvdash_{\mathcal{L}_{\text{exp}}} \emptyset$ for some finite Γ. Then $\Gamma \vdash \emptyset$ fails in some finite reduced model $\langle \mathbf{A}, F \rangle$ of $\mathcal{L}_{\text{exp}}$. Our goal will be to produce a model of $\mathcal{ETL} \vee \mathcal{L}_{\text{exp}}$ where $\Gamma \vdash \emptyset$ fails.

The matrix $\langle \mathbf{A}, F \rangle$ is a witnessed finite subdirect product of the matrices $\mathbb{B}_{\mathbf{2}}$, $\mathbb{K}_{\mathbf{3}}$, $\mathbb{P}_{\mathbf{3}}$, $\mathbb{BD}_{\mathbf{4}}$, and $\mathbb{ETL}_{\mathbf{4}}$ by Proposition 3.22. (See Section 3.4 for the definition of a witnessed subdirect product.) We now show that there is some $b \in \mathbf{A}$ such that

$$\pi_i(b) = \mathsf{b} \text{ if } \langle \mathbf{B}_i, G_i \rangle \in \{\mathbb{P}_{\mathbf{3}}, \mathbb{BD}_{\mathbf{4}}\},$$
$$\pi_i(b) = \mathsf{f} \text{ if } \langle \mathbf{B}_i, G_i \rangle \in \{\mathbb{B}_{\mathbf{2}}, \mathbb{K}_{\mathbf{3}}, \mathbb{ETL}_{\mathbf{4}}\}.$$

Because the subdirect product is witnessed, for each subdirect factor of the form $\mathbb{P}_{\mathbf{3}}$ or $\mathbb{BD}_{\mathbf{4}}$ with index $i \in I$ there is some $a_i \in F$ such that $\pi_i(a_i) = \mathsf{b}$, where $\pi_i : \Pi_{i \in I} \mathbf{B}_i \to \mathbf{B}_i$ is the projection map. But then $\pi_j(a_i) = \mathsf{t}$ if $\langle \mathbf{B}_j, G_j \rangle \in \{\mathbb{B}_{\mathbf{2}}, \mathbb{K}_{\mathbf{3}}, \mathbb{ETL}_{\mathbf{4}}\}$ and $\pi_j(a_i) \geq \mathsf{b}$ if $\langle \mathbf{B}_j, G_j \rangle \in \{\mathbb{P}_{\mathbf{3}}, \mathbb{BD}_{\mathbf{4}}\}$. It follows that for $b_i := a_i \wedge -a_i$ we have

$$\pi_i(b_i) = \mathsf{b},$$
$$\pi_j(b_i) = \mathsf{f} \text{ if } \langle \mathbf{B}_j, G_j \rangle \in \{\mathbb{B}_{\mathbf{2}}, \mathbb{K}_{\mathbf{3}}, \mathbb{ETL}_{\mathbf{4}}\},$$
$$\pi_j(b_i) \leq \mathsf{b} \text{ if } \langle \mathbf{B}_j, G_j \rangle \in \{\mathbb{P}_{\mathbf{3}}, \mathbb{BD}_{\mathbf{4}}\}.$$

Finally, let $b := \bigvee_{i \in I} b_i$ (recall that I is finite). Then b satisfies the condition that $\pi_i(b) = \mathsf{b}$ if $\mathbb{M}_i \in \{\mathbb{P}_{\mathbf{3}}, \mathbb{BD}_{\mathbf{4}}\}$ and $\pi_i(b) = \mathsf{f}$ if $\mathbb{M}_i \in \{\mathbb{B}_{\mathbf{2}}, \mathbb{K}_{\mathbf{3}}, \mathbb{ETL}_{\mathbf{4}}\}$.

Grouping together the factors $\mathbb{B}_{\mathbf{2}}$, $\mathbb{K}_{\mathbf{3}}$, $\mathbb{ETL}_{\mathbf{4}}$ and the factors $\mathbb{P}_{\mathbf{3}}$, $\mathbb{BD}_{\mathbf{4}}$ yields a subdirect decomposition of $\langle \mathbf{A}, F \rangle$ into a De Morgan matrix $\langle \mathbf{B}, \{\mathsf{t}\} \rangle$ and a De Morgan matrix $\langle \mathbf{C}, G \rangle$. We have moreover established in the previous paragraph the existence of a pair $z = \langle x, y \rangle \in \mathbf{A}$ such that $x = \mathsf{f}$ and $y = -y \in G$. Because the rule $\Gamma \vdash \emptyset$ fails in $\langle \mathbf{A}, F \rangle$, it also fails in $\langle \mathbf{B}, \{\mathsf{t}\} \rangle$. It remains to show that the matrix $\langle \mathbf{B}, \{\mathsf{t}\} \rangle$, which is a model of $\mathcal{ETL}$, is also a model of $\mathcal{L}_{\text{exp}}$ given that $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{L}_{\text{exp}}$.

Suppose therefore that an explosive rule $\Delta \vdash \emptyset$ fails in $\langle \mathbf{B}, \{\mathsf{t}\} \rangle$ under some valuation $v_{\mathbf{B}} : \mathbf{Fm} \to \mathbf{B}$, where without loss of generality Δ does not contain the constants t and f. Then by subdirectness there is some valuation $v_{\mathbf{A}} : \mathbf{Fm} \to \mathbf{A}$ such that $v_{\mathbf{A}}(p) = \langle v_{\mathbf{B}}(p), v_{\mathbf{C}}(p) \rangle$ for some valuation $v_{\mathbf{C}} : \mathbf{Fm} \to \mathbf{C}$. Consider the valuation $w_{\mathbf{A}} : \mathbf{Fm} \to \mathbf{A}$ with $w_{\mathbf{A}}(p) := (v_{\mathbf{A}}(p) \vee z) \wedge -z$. Then $w_{\mathbf{A}}(p) = \langle v_{\mathbf{A}}(p), y \rangle$ for each atom p, hence $w_{\mathbf{A}}(\varphi) = \langle v_{\mathbf{A}}(\varphi), y \rangle$ for each formula φ. But then the valuation $w_{\mathbf{A}}$ witnesses the failure of the rule $\Delta \vdash \emptyset$ in $\langle \mathbf{A}, F \rangle$.

Note that the proof of this theorem could be simplified by appealing to the description of finite reduced models of $\mathcal{BD}$ provided in Section 7.3 (Proposition 7.17) instead of the witnessed subdirect decomposition. $\square$

Corollary 6.24. *Each finitary explosive extension of $\mathcal{ETL}$ has the form $\mathcal{ETL} \cup \mathcal{L}_{exp}$ for some finitary explosive extension $\mathcal{L}_{exp}$ of $\mathcal{BD}$.*

Proof. For $\mathcal{L}_{\mathrm{exp}} \in \mathrm{Exp\,Ext}_{\omega}\, \mathcal{ETL}$ we have $\mathcal{L}_{\mathrm{exp}} = \mathcal{ETL} \vee \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{L}_{\mathrm{exp}} = \mathcal{ETL} \cup \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{L}_{\mathrm{exp}}$ by Proposition 4.28. $\square$

In the following theorem $\mathcal{LP} \cap \mathrm{Exp\,Ext}_{\omega}\, \mathcal{ECQ}$ denotes the poset of all logics which are intersections of $\mathcal{LP}$ with an explosive extension of $\mathcal{ECQ}$.

Theorem 6.25.
The lattices $\mathrm{Exp\,Ext}_{(\omega)}\, \mathcal{ECQ}$ and $\mathcal{LP} \cap \mathrm{Exp\,Ext}_{(\omega)}\, \mathcal{ECQ}$ are isomorphic via the maps $\mathcal{L}_{exp} \mapsto \mathcal{LP} \cap \mathcal{L}_{exp}$ and $\mathcal{L} \mapsto \mathcal{ECQ} \vee \mathcal{L}$ (with $\mathcal{LP} \mapsto \mathcal{TRIV}$).

Proof. It suffices to prove that $\mathcal{ECQ} \vee (\mathcal{LP} \cap \mathcal{L}) = \mathcal{L}$ for each non-trivial $\mathcal{L} \in \mathrm{Exp\,Ext}_{\omega}\, \mathcal{ECQ}$. The inclusion $\mathcal{ECQ} \vee (\mathcal{LP} \cap \mathcal{L}) \leq \mathcal{L}$ is trivial. Conversely, by Proposition 4.22 we have

$$\begin{aligned}
\mathrm{Mod}(\mathcal{ECQ} \vee (\mathcal{LP} \cap \mathcal{L}_{\mathrm{exp}})) &= \mathrm{Mod}\, \mathcal{ECQ} \cap \mathrm{Mod}\, \mathcal{LP} \vee \mathcal{L}_{\mathrm{exp}} \\
&= \mathrm{Mod}\, \mathcal{ECQ} \cap (\mathrm{Mod}\, \mathcal{LP} \cup \mathrm{Mod}\, \mathcal{L}_{\mathrm{exp}}) \\
&\subseteq (\mathrm{Mod}\, \mathcal{ECQ} \cap \mathrm{Mod}\, \mathcal{LP}) \cup \mathrm{Mod}\, \mathcal{L}_{\mathrm{exp}} \\
&= \mathrm{Mod}(\mathcal{LP} \vee \mathcal{ECQ}) \cup \mathrm{Mod}\, \mathcal{L}_{\mathrm{exp}} \\
&\subseteq \mathrm{Mod}\, \mathcal{L}_{\mathrm{exp}}
\end{aligned}$$

because $\mathcal{L}_{\mathrm{exp}} = \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{L}_{\mathrm{exp}} \leq \mathrm{Exp}_{\mathcal{BD}}\, \mathcal{CL} = \mathcal{ECQ}_{\omega} \leq \mathcal{LP} \vee \mathcal{ECQ}$ for each non-trivial $\mathcal{L}_{\mathrm{exp}} \in \mathrm{Exp\,Ext}_{\omega}$. $\square$

Chapter 7

Super-Belnap logics and finite graphs

The current chapter continues the study of the lattice of finitary super-Belnap logics started in the previous chapter. Whereas our goal in the previous chapter was to decompose this lattice into smaller chunks, here our goal will be to describe this lattice fully in terms of finite graphs. We then apply well-known theorems of graph theory in order to prove non-trivial results about super-Belnap logics.

The link between super-Belnap logics and finite graphs rests on the observation that finite reduced models of $\mathcal{BD}$ correspond precisely to triples $\langle G, H, k \rangle$, where G and H are finite undirected graphs and $k \in \omega$ (we admit loops). But each finitary super-Belnap logic is complete as a finitary logic with respect to its finite models. By describing in intrinsic terms which classes of triples correspond to classes of finite reduced models of a finitary super-Belnap logic, we obtain a purely graph-theoretic description of the lattice $\operatorname{Ext}_\omega \mathcal{BD}$.

Although this description turns out to be somewhat unwieldy when it comes to arbitrary finitary extensions of $\mathcal{BD}$, it simplifies substantially if we restrict our attention to $\operatorname{Ext}_\omega \mathcal{ETL}$, and even more so if we focus on the interval $[\mathcal{ETL}, \mathcal{ETL}_\omega]$. In particular, this interval can be described as the lattice of all classes of finite graphs without loops closed under surjective homomorphisms, finite disjoint unions, and replacing isolated edges by isolated vertices. The lattice of non-trivial proper finitary explosive extensions of $\mathcal{BD}$ is then nothing but the order dual of the lattice of all classes of non-empty finite graphs closed under homomorphisms. From this we can infer that there is a continuum of finitary explosive extensions of $\mathcal{BD}$ and of $\mathcal{ETL}$ and a continuum of antivarieties of De Morgan algebras.

The above bridge between the realm of super-Belnap logics and the realm of graphs will also enable us to exploit graph-theoretic results in order to prove results about super-Belnap logics. We show two examples of such applications of graph-theoretic results to super-Belnap logics. Firstly, we use the countable universality of the homomorphism order on finite graphs to show the existence of a non-finitary super-Belnap logic, as well as the existence of continuum many finitary super-Belnap logics. Secondly, we use the Girth–Chromatic Number Theorem of Erdős to prove that, unlike $\mathcal{ECQ}$ and $\mathcal{ETL}$, the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ are not complete with respect to any finite set of finite matrices for $n \geq 2$.

The chapter begins by introducing the necessary graph-theoretic notions and results. The duality theory for De Morgan algebras due to Cornish and Fowler is then recalled and extended to a duality theory for De Morgan matrices. (We in fact restrict to finite algebras and matrices.) As stated above, the duals of finite reduced De Morgan matrices admit a simple description in terms of a pair of graphs G and H and a parameter $k \in \omega$. We then determine which of these triples correspond to models of $\mathcal{ECQ}_n$, $\mathcal{ETL}_n$, and $\mathcal{SDS}_n$, obtaining completeness theorems for these logics. The final section then provides a graph-theoretic description of $\mathrm{Ext}_\omega \mathcal{BD}$ and further exploits the link between graphs and super-Belnap logics.

7.1 Graph-theoretic preliminaries

We first introduce some basic graph-theoretic notions which will be used in the current chapter. Note that some of the terminology is non-standard.

By a *graph* $G = \langle X, R \rangle$ we shall mean a finite set of vertices X equipped with a symmetric binary relation $R \subseteq X^2$. We write uRv for $\langle u, v \rangle \in R$, in which case we say that u and v are neighbours and that $\langle u, v \rangle$ is an *edge* of the graph. The set of all neighbours of a vertex u will be denoted $R[u]$, i.e.

$$R[u] := \{v \in X \mid uRv\}.$$

The set of all neighbours of a set of vertices U will be denoted $R[U]$, i.e.

$$R[U] := \bigcup_{u \in U} R[u].$$

A *graph homomorphism* is an edge-preserving map between (vertices of) graphs. The notation $G \to H$ ($G \twoheadrightarrow H$) abbreviates the claim that there is a (surjective) graph homomorphism from G to H.

In addition to ordinary homomorphisms, it will be convenient to introduce relational homomorphisms. Consider a pair of graphs $G = \langle X, R \rangle$ and $H = \langle Y, S \rangle$. An *edge-preserving relation* from G to H is a relation $P \subseteq X \times Y$ such that uPu' and vPv' and uRv imply $u'Sv'$. A *relational homomorphism* is then an edge-preserving relation from G to H such that for each $u \in X$ there is some $v \in Y$ with uPv. Clearly the graph of an ordinary graph homomorphism is a relational homomorphism. Conversely, if in a relational homomorphism we pick for each $u \in X$ some $h(u) \in Y$ such that $uPh(u)$, then $h : G \to H$ is an ordinary graph homomorphism.

If uRu, we say that the vertex u has a *loop*. Otherwise, u is called *loopless*. Note that each vertex with a loop is its own neighbour. Vertices with (without) loops will also be called *reflexive* (*irreflexive*).

A vertex is called *isolated* if it has no neighbours. In particular, such a vertex is loopless. An *isolated loop* is a vertex u such that $R[u] = \{u\}$. An *isolated edge* is a pair of distinct vertices u and v such that $R[u] = \{v\}$ and $R[v] = \{u\}$. The loopless singleton graph will be denoted $\bullet$.

We now define several graph-theoretic constructions which will be used in this chapter. The *subgraph* of $G = \langle X, R\rangle$ *induced by* $Y \subseteq X$ is the graph $\langle Y, R \cap Y^2\rangle$. The subgraph induced by the non-isolated vertices of G will be denoted $\overline{G}$. The *disjoint union* $G \sqcup H$ of graphs $G = \langle X, R\rangle$ and $H = \langle Y, S\rangle$ is the graph $\langle X \sqcup Y, R \sqcup S\rangle$.

The *complete graph* of n vertices, i.e. the graph $\langle \{1, \dots, n\}, \neq\rangle$, will be denoted K_n. A graph G is called *n-colourable* if there is a homomorphism $G \to K_n$. A well-known result of Erdős states that there are graphs of arbitrarily high girth among graphs which are not n-colourable, where the *girth* of a graph is the length of its shortest cycle. We will use this theorem to show that the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ are not complete with respect to a finite set of finite matrices.

Theorem 7.1 [23] **(Girth–Chromatic Number Theorem).**
For each n and k there is a graph of girth at least k which is not n-colourable.

A graph G will be called *weakly n-colourable* if there is a homomorphism $H \to K_n$ from an induced subgraph H of G such that not every vertex of G is a neighbour of H (i.e. a neighbour of some vertex in H).

The *homomorphism pre-order* on graphs is defined as follows: $G \leq H$ if and only if there is a graph homomorphism $G \to H$. The *homomorphism order* on graphs is obtained by collapsing the homomorphism pre-order down to a partial order, i.e. by identified equivalent graphs in the homomorphism pre-order.[1] We shall say that a class of graphs is *closed under homomorphisms* if it is upward closed in the homomorphism pre-order. Note that the homomorphisms in question need not be surjective.

A remarkable property of this order is its countable universality, i.e. the fact that it contains every countable poset. We will use this property to construct a non-finitary super-Belnap logic.

Theorem 7.2 [33] **(Countable universality of the hom-order).**
Every countable poset embeds into the homomorphism order on graphs.

[1] The homomorphism order is usually studied on loopless graphs. It is, however, easy to see that the homomorphism order on graphs which may include loops is just the homomorphism order on loopless graphs extended by a new top element consisting of the equivalence class of all graphs with a loop.

7.2 Duality for De Morgan matrices

Our description of the finite reduced models of $\mathcal{BD}$ will rely on the duality for De Morgan algebras due to Cornish and Fowler [13] which we now recall.

The set prime filters of a De Morgan algebra $\mathbf{A}$ will be denoted Prime $\mathbf{A}$, and the set of upsets of a partial order $(W, \leq)$ will be denoted $\mathrm{Up}\,(W, \leq)$.

Definition 7.3 (De Morgan frames).
A *De Morgan frame* $\mathcal{F}$ is a poset $(W, \leq)$ equipped with an order-inverting involution ∂, i.e. with a map $\partial : W \to W$ such that

$$u \leq v \implies \partial v \leq \partial u \qquad \text{and} \qquad \partial\partial u = 0.$$

A *morphism of De Morgan frames* is a monotone map commuting with ∂.

Definition 7.4 (Complex algebras).
The *complex algebra* of the De Morgan frame $\mathcal{F} = (W, \leq, \partial)$ is the algebra

$$\mathcal{F}^{+} := (\mathrm{Up}\,(W, \leq), \cap, \cup, W, \emptyset, -) \text{ where } -U = W \setminus \partial[U],$$

i.e. the bounded distributive lattice of upsets of $(W, \leq)$ equipped with the operation $- : U \mapsto W \setminus \partial[U]$.

Definition 7.5 (Dual frames).
The *dual frame* of a De Morgan algebra $\mathbf{A}$ is the De Morgan frame

$$\mathbf{A}_{+} := (\mathrm{Prime}\,\mathbf{A}, \subseteq, \partial) \text{ where } \partial F = \mathbf{A} \setminus -[F].$$

The above constructions extend to functors. Each morphism $f : \mathcal{F} \to \mathcal{G}$ of De Morgan frames yields a homomorphism $f^{+} := f^{-1} : \mathcal{G}^{+} \to \mathcal{F}^{+}$ of their complex algebras. Conversely, each homomorphism of De Morgan algebras $h : \mathbf{A} \to \mathbf{B}$ yields a morphism $h_{+} := h^{-1} : B_{+} \to A_{+}$ of their dual frames.

Moreover, these functors in fact yield an adjunction between De Morgan frames and De Morgan algebras whose unit and counit are the maps

$$\eta_{\mathcal{F}} : \mathcal{F} \to (\mathcal{F}^{+})_{+} \text{ such that } u \mapsto \{U \in \mathrm{Up}\,\mathcal{F} \mid u \in U\},$$
$$\iota_{\mathbf{A}} : \mathbf{A} \to (\mathbf{A}_{+})^{+} \text{ such that } a \mapsto \{U \in \mathrm{Prime}\,\mathbf{A} \mid a \in U\}.$$

Theorem 7.6 (Duality for De Morgan algebras).
The complex algebra and dual frame functors yield a dual equivalence between the categories of finite De Morgan algebras and homomorphisms and finite De Morgan frames and their morphisms with unit $\eta_{\mathcal{F}}$ and counit $\iota_{\mathbf{A}}$.

To extend this duality to finite De Morgan matrices, we only need to take care of the filters of designated values. These filters are principal in finite De Morgan matrices, therefore it suffices to extend De Morgan frames by an upset (upward closed set) representing this principal filter.

Definition 7.7 ($\mathcal{BD}$-frames).
A *$\mathcal{BD}$-frame* $\langle \mathcal{F}, D \rangle$ is a De Morgan frame $\mathcal{F}$ equpped with an upset of designated points D. A *morphism of $\mathcal{BD}$-frames* $f : \langle \mathcal{F}, D \rangle \to \langle \mathcal{G}, E \rangle$ is a morphism of De Morgan frames which preserves designation, i.e. $u \in D \implies f(u) \in E$. The morphism f is *strict* if for each designated v in $\mathcal{G}$ there is some designated u in $\mathcal{G}$ such that $f(u) \leq v$.

Definition 7.8 (Complex matrices).
The *complex matrix* $\langle \mathcal{F}, D \rangle^+$ of $\langle \mathcal{F}, D \rangle$ is the complex algebra $\mathcal{F}^+$ equipped with the principal filter generated by D.

Definition 7.9 (Dual $\mathcal{BD}$-frames).
The *dual $\mathcal{BD}$-frame* of a De Morgan matrix $\langle \mathbf{A}, F \rangle$ is the dual frame of $\mathbf{A}$ such that $U \in \operatorname{Prime} \mathbf{A}$ is designated if and only if $F \subseteq U$.

These constructions extend to functors in the same way as before, namely by taking the inverse images of (homo)morphisms.

The complex matrix of a $\mathcal{BD}$-frame is clearly a De Morgan matrix, and conversely the dual $\mathcal{BD}$-frame of a De Morgan matrix is indeed a $\mathcal{BD}$-frame. To obtain a duality for De Morgan matrices, it therefore suffices to show that morphisms of $\mathcal{BD}$-frames correspond precisely to homomorphisms of De Morgan algebras. We also show that strict morphisms of $\mathcal{BD}$-frames correspond precisely to strict homomorphisms of De Morgan matrices.

(Recall that a homorphism $h : \langle \mathbf{A}, F \rangle \to \langle \mathbf{B}, G \rangle$ is merely required to preserve designation, while for strict homomorphisms we have $h^{-1}[G] = F$.)

Theorem 7.10 (Duality for De Morgan matrices).
The complex matrix and dual $\mathcal{BD}$-frame functors yield a dual equivalence between the categories of finite De Morgan matrices and (strict) homomorphisms and finite $\mathcal{BD}$-frames and their (strict) morphisms with unit $\eta_{\mathcal{F}}$ and counit $\iota_{\mathbf{A}}$.

Proof. It suffices to check that (strict) morphisms of $\mathcal{BD}$-frames are sent to (strict) homomorphisms of De Morgan matrices and vice versa.

Let $f : \langle \mathcal{F}, D \rangle \to \langle \mathcal{G}, E \rangle$ be a (strict) morphism of $\mathcal{BD}$ frames. We first show that $f^+ : \langle \mathbf{B}, G \rangle \to \langle \mathbf{A}, F \rangle$ is a (strict) homomorphism of De Morgan matrices, where $\langle \mathbf{A}, F \rangle := \langle \mathcal{F}, D \rangle^+$ and $\langle \mathbf{B}, G \rangle := \langle \mathcal{G}, E \rangle^+$.

Given an upset U of a $\mathcal{BD}$-frame, let U^+ be the corresponding element of the complex matrix. For each upset U of $\mathcal{G}$ we have

$$\begin{aligned} U^+ \in G \iff E^+ \leq U^+ \iff E \subseteq U \implies f^{-1}[E] \subseteq f^{-1}[U] \\ \implies D \subseteq f^{-1}[U] \iff D^+ \leq f^+(U^+) \iff f^+(U^+) \in F \end{aligned}$$

because $D \subseteq f^{-1}[E]$ by the definition of a morphism of $\mathcal{BD}$-frames, theefore f^+ is a homomorphism of De Morgan matrices. If f is strict, it suffices to show that moreover $D \subseteq f^{-1}[U] \implies E \subseteq U$, i.e. that $f[D] \subseteq U \implies E \subseteq$

U. But this holds if E is contained in the upward closure of $f[D]$, which is precisely what the definition of a strict morphism of $\mathcal{BD}$-frames states.

Conversely, let $h : \langle \mathbf{A}, F \rangle \to \langle \mathbf{B}, G \rangle$ be a (strict) homomorphism of De Morgan matrices. We show that $h_+ : \langle \mathcal{G}, E \rangle \to \langle \mathcal{F}, D \rangle$ is a (strict) morphism of $\mathcal{BD}$-frames.

Given a prime filter U on a De Morgan matrix, let U_+ be the corresponding element of the dual $\mathcal{BD}$-frame. For each prime filter U of $\mathbf{B}$

$$\begin{aligned} U_+ \in E \iff G \subseteq U &\implies h^{-1}[G] \subseteq h^{-1}[U] \\ &\implies F \subseteq h^{-1}[U] \iff h_+(U_+) \in D \end{aligned}$$

because $F \subseteq h^{-1}[G]$ by the definition of a homomorphism of De Morgan matrices, therefore f_+ is a morphism of $\mathcal{BD}$-frames.

Now suppose that h is strict and consider $V_+ \in D$. Let $I := \mathbf{A} \setminus V$. Then $F \subseteq V$, hence $I \cap F = \emptyset$. The strictness of h implies that $h[I] \cap G = \emptyset$. Now let J be the ideal generated by $h[I]$. Then $J \cap G = \emptyset$. By the Filter–Ideal Separation Lemma (Lemma 1.2) G extends to a prime filter U on $\mathbf{B}$ disjoint from J. But then $h^{-1}[V] \cap I = \emptyset$, hence $h^{-1}[U] \subseteq V$, i.e. $h_+(U_+) \leq V_+$ for some $U_+ \in E$. □

Submatrices, products of matrices, and strict homomorphic images of matrices will be crucial constructions in the following. We therefore wish to describe these constructions in dual terms. In the rest of the section, $\mathbb{A}$ and $\mathbb{B}$ will denote finite De Morgan matrices.

Products of De Morgan matrices are easily seen to correspond to disjoint unions of $\mathcal{BD}$-frames, denoted $\langle \mathcal{F}, D \rangle \sqcup \langle \mathcal{G}, E \rangle$ and defined in the expected way. We omit the easy proof of the following observation.

Fact 7.11. $(\mathbb{M} \times \mathbb{N})_+ = \mathbb{M}_+ \sqcup \mathbb{N}_+$.

Strict homomorphic images correspond to subframes. Recall that $\mathbb{M}^*$ denotes the Leibniz reduct of the matrix $\mathbb{M}$.

Definition 7.12 (Subframes of $\mathcal{BD}$-frames).
A *subframe* of a finite $\mathcal{BD}$-frame $\langle \mathcal{F}, D \rangle$ is the restriction of $\langle \mathcal{F}, D \rangle$ to a subset closed under ∂ which contains the minimal points of D. Each finite $\mathcal{BD}$-frame has a smallest subframe $\langle \mathcal{F}, D \rangle^*$, called the *Leibniz subframe*, which consists precisely of those points u such that either u or ∂u is a minimal point of D.

Proposition 7.13 (Strict homomorphic images and subframes).
$\mathbb{N}$ is a strict homomorphic image of a $\mathbb{M}$ if and only if $\mathbb{N}_+$ is isomorphic to a subframe of $\mathbb{M}_+$. In particular, $(\mathbb{M}^*)_+ = (\mathbb{M}_+)^*$.

Proof. It is known that $h : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism of De Morgan algebras if and only if h_+ is an embedding of De Morgan frames, i.e.

an order-reflecting morphism of De Morgan frames. Up to isomorphism, De Morgan frames which embed into a given De Morgan frame may be identified with its subsets closed under ∂. A surjective homomorphism $h : \langle \mathbf{A}, F \rangle \to \langle \mathbf{B}, G \rangle$ is strict if and only if $a \in F \iff h(a) \in G$ for all $a \in \mathbf{A}$. But $a \in F$ if and only if a as an upset of the dual frame contains each minimal designated element. Thus $a \in F \iff h(a) \in G$ holds for all $a \in \mathbf{A}$ if and only if the subframe contains all minimal elements of D. $\square$

Corollary 7.14 (Reduced $\mathcal{BD}$-frames).
The complex matrix of a $\mathcal{BD}$-frame $\langle \mathcal{F}, D \rangle$ is reduced if and only if for each u in $\mathcal{F}$ either u or ∂u is minimal in D.

Finally, submatrices correspond to quotient frames.

Definition 7.15 (Quotients of $\mathcal{BD}$-frames).
A *quotient* of a $\mathcal{BD}$-frame is its strict surjective image. That is, $\langle \mathcal{G}, E \rangle$ is a quotient of $\langle \mathcal{F}, D \rangle$ if there is a surjective morphism of De Morgan frames $f : \mathcal{F} \to \mathcal{G}$ and E is the upset generated by $f[D]$.

Proposition 7.16 (Submatrix and quotient frames).
$\mathbb{B}$ embeds into $\mathbb{A}$ if and only if $\mathbb{B}_+$ is a quotient of $\mathbb{A}_+$.

Proof. It is known that $h : \mathbf{A} \to \mathbf{B}$ is an embedding of De Morgan algebras if and only if h_+ is surjective. The claim now follows by virtue of the fact that an embedding of matrices is an embedding of algebras which is a strict homomorphism of matrices. $\square$

Quotients may be identified with pre-orders $\leqq$ extending $\leq$ such that $u \leqq v \implies \partial v \leqq \partial u$. A *principal quotient* is then a smallest quotient in this sense, i.e. a quotient obtained by postulating that $u \leqq v$ for some $u \nleq v$. It will be useful to observe that a $\mathcal{BD}$-frame is a quotient of a finite $\mathcal{BD}$-frame $\langle \mathcal{F}, D \rangle$ if and only if it lies in the closure of $\langle \mathcal{F}, D \rangle$ under principal quotients.

7.3 Reduced models of $\mathcal{BD}$

This section is devoted to describing the finite reduced models of $\mathcal{BD}$ in graph-theoretic terms and using this description to provide completeness theorems for the logics $\mathcal{ECQ}_n$, $\mathcal{ETL}_n$, and $\mathcal{SDS}_n$. We shall also provide an alternative proof of the completeness theorem for $\mathcal{K}_-$ which uses the dual description of the finite reduced models of $\mathcal{BD}$.

We first show how to assign a De Morgan frame to a given finite graph. Let $G = \langle X, R \rangle$ be a finite graph and let $X \sqcup \partial X$ denote the disjoint union of two copies of X, denoted X and ∂X, with ∂ being an involution on $X \sqcup \partial X$ switching between the two copies. A partial order $u \leq_G v$ may be defined on this set such that

$$u \leq_G v \iff \text{either } v = u \text{ or } v = \partial w \text{ for some } w \in X \text{ and } uRw.$$

This yields the De Morgan frame $\mathcal{F}(G)$.

This frame can be extended by one of the two designated sets

$$D_+(G) := X \sqcup \partial X \qquad \text{and} \qquad D_-(G) := X.$$

to yield the $\mathcal{BD}$-frames $\mathcal{F}_+G$ and $\mathcal{F}_-G$. The complex matrices of these two $\mathcal{BD}$-frames will be denoted

$$\boldsymbol{\mu}_+(G) := \langle \mathcal{F}(G), D_+(G) \rangle^+ \qquad \text{and} \qquad \boldsymbol{\mu}_- := \langle \mathcal{F}(G), D_-(G) \rangle^+.$$

In particular, $\boldsymbol{\mu}_\pm(\emptyset)$ is the trivial reduced matrix and

$$\boldsymbol{\mu}_+(\bullet) = \mathbb{ETL}_\mathbf{4} \qquad \text{and} \qquad \boldsymbol{\mu}_-(\bullet) = \mathbb{BD}_\mathbf{4}.$$

We now combine the two maps $\boldsymbol{\mu}_+$ and $\boldsymbol{\mu}_-$ into a single map $\boldsymbol{\mu}$. Given a pair of graphs G and H and some $k \in \omega$, let

$$\boldsymbol{\mu}(G, H, k) := \boldsymbol{\mu}_+(G) \times \boldsymbol{\mu}_-(H) \times \mathbf{B}_\mathbf{2}^k.$$

The dual frame of this matrix will be denoted $\mathcal{F}(G, H, k)$, i.e. $\mathcal{F}(G, H, k)$ is the disjoint union of $\mathcal{F}_+(G)$ and $\mathcal{F}_-(H)$ and k isolated loops.

Theorem 7.17 (Finite reduced models of $\mathcal{BD}$).
The finite reduced models of $\mathcal{BD}$ are precisely the matrices of the form $\boldsymbol{\mu}(G, H, k)$ for some graphs G and H and some $k \in \omega$.

Proof. Each finite reduced model of $\mathcal{BD}$ is the complex matrix of a reduced $\mathcal{BD}$-frame $\langle \mathcal{F}, D \rangle$, i.e. for each u in $\mathcal{F}$ either u or ∂u is minimal in D. Since a disjoint union of two $\mathcal{BD}$-frames is reduced if and only if they are both reduced, each reduced $\mathcal{BD}$-frame is a disjoint union of connected reduced $\mathcal{BD}$-frames, where a $\mathcal{BD}$-frame is called *connected* if for each pair of elements u and v there is a sequence $u \leq w_1 \geq \cdots \leq w_n \geq w_{n+1}$ such that $w_{n+1} = v$ or $w_{n+1} = \partial v$. Let us therefore consider a connected reduced $\mathcal{BD}$-frame $\langle \mathcal{F}, D \rangle$ with $\mathcal{F} = (W, \leq, \partial)$. It suffices to prove that it is either a designated isolated loop or it has one of the forms $\mathcal{F}_+(G)$ or $\mathcal{F}_-(G)$.

Firstly, $u < v$ implies $v \in D$. If $v \notin D$, then $\partial v \in D$. But $\partial v < \partial u$, hence ∂u is not a minimal element of D. Thus $u \in D$ and also $v \in D$.

Secondly, $\mathcal{F}$ contains no chain of length three. If $u < v < w$, then w is not a minimal element of D because $v \in D$, thus ∂w is a minimal element of D. But then $\partial w < \partial v < \partial u$, therefore ∂u is not a minimal element of D and $u \in D$. But then neither v nor ∂v is a minimal element of D.

Finally, we show that if $u < v > w$ and $u \in D$, then $w \in D$. If $u \in D$, then v is not a minimal element of D, hence $\partial v \in D$. But $\partial v < \partial w$, hence ∂w is not a minimal element of D and $w \in D$.

The frame $\mathcal{F}$ only consists of elements of heights 0 and 1 by the second observation. If there are no elements of height 1, then by connectedness $\mathcal{F}$

consists either of a designated isolated loop or of two incomparable elements u and ∂u, in which case $\mathcal{F} = \mathcal{F}(\bullet)$, where $\bullet$ is the loopless singleton.

If there are some elements of height 1, then ∂ is an bijection between elements of height 0 and height 1, therefore the order relation on $\mathcal{F}$ in fact has the form $\leq_G$ for some graph G. All elements of height 1 are designated by the first observation. By connectedness and the third observation either all elements are designated or all elements of height 1 are designated, i.e. the $\mathcal{BD}$-frame has either the form $\mathcal{F}_+(G)$ or $\mathcal{F}_-(G)$. $\square$

Proposition 7.18 (Finite reduced models of $\mathcal{ETL}$).
The finite reduced models of $\mathcal{ETL}$ are precisely the matrices of the form $\boldsymbol{\mu}(G, \emptyset, k)$ for some graph G and some $k \in \omega$.

Proof. By the description of reduced models of $\mathcal{ETL}$ (Proposition 3.20) the matrix $\boldsymbol{\mu}(G, H, k)$ is a reduced model of $\mathcal{ETL}$ if and only if its designated filter is a singleton. $\square$

Proposition 7.19 (Finite reduced models of $\mathcal{ECQ}$).
The finite reduced models of $\mathcal{ECQ}$ are precisely the matrices of the form $\boldsymbol{\mu}(G, H, k)$ such that either G is non-empty or $H = \emptyset$ or $k \geq 1$.

Proof. A product of non-trivial matrices is a model of $\mathcal{ECQ}$ if and only if one of the factors is a model of $\mathcal{ECQ}$ by Corollary 4.26. $\square$

Let us now make some easy observations about the matrices $\boldsymbol{\mu}(G, H, k)$.

Fact 7.20. $\operatorname{Log} \boldsymbol{\mu}(G, H, k) = \operatorname{Log} \boldsymbol{\mu}(G, H, 1)$ *for* $k \geq 1$.

Fact 7.21. $\boldsymbol{\mu}(G_1 \sqcup G_2, H_1 \sqcup H_2, k_1 + k_2) = \boldsymbol{\mu}(G_1, H_1, k_1) \times \boldsymbol{\mu}(G_2, H_2, k_2)$.

To understand which rules are valid in the matrices $\boldsymbol{\mu}_+(G)$, let us introduce another matrix in the signature of $\mathcal{BD}$ called the graph matrix $\boldsymbol{\gamma}(G)$. Although these matrices will typically not be a model of $\mathcal{BD}$, it will be very useful for understanding which rules hold in $\boldsymbol{\mu}_+(G)$.

Definition 7.22 (Graph matrices).
The *graph matrix* of a graph $G = \langle X, R \rangle$ is the matrix

$$\boldsymbol{\gamma}(G) := \langle (2^X, \cap, \cup, X, \emptyset, -), X \rangle \text{ where } -U = X \setminus R[U],$$

i.e. the bounded distributive lattice of all subsets of X equipped with the operation $- : U \mapsto X \setminus R[U]$ and the set of designated values $\{X\}$.

Lemma 7.23 (Graph Matrix Lemma).
Let $\Gamma \cup \{\varphi\}$ be a set formulas of $\mathcal{BD}$ where negation is only applied to atoms, and let $G = \langle X, R \rangle$ be a graph without isolated vertices. If the rule $\Gamma \vdash \varphi$ is valid in $\boldsymbol{\mu}_+(G)$, then it is valid in $\boldsymbol{\gamma}(G)$. If negation does not occur in φ, then the opposite implication also holds.

Proof. For each formula φ where negation is only applied to atoms, there is a set of disjunctive clauses (disjunctions of atoms and negated atoms) Φ such that φ is equivalent to Φ in both $\mathcal{BD}$ and in the logic determined by $\boldsymbol{\gamma}(G)$. We can therefore assume without loss of generality that $\Gamma \cup \{\varphi\}$ is a set of disjunctive clauses.

Now recall that X is identified with the elements of $\mathcal{F}_+(G)$ of height 0. We define the auxiliary maps

$$\uparrow_G : \boldsymbol{\gamma}(G) \to \boldsymbol{\mu}_+(G) \qquad \text{and} \qquad \downarrow_G : \boldsymbol{\mu}_+(G) \to \boldsymbol{\gamma}(G)$$

as follows: $\uparrow_G U$ for $U \subseteq X$ is the upward closure of U in $\mathcal{F}_+(G)$, while $\downarrow_G U$ for $U \subseteq \mathcal{F}_+(G)$ is the restriction of U to X, i.e. $U \cap X$. Then clearly

$$\downarrow_G \uparrow_G U = U \text{ for each } U \subseteq \boldsymbol{\gamma}(G).$$

These maps are not homomorphisms, but they retain enough of the properties of homomorphisms to be useful. In particular, observe that

$$\downarrow_G - \uparrow_G U = -U \qquad \text{and} \qquad \downarrow_G -U \subseteq - \downarrow_G U.$$

Both maps also preserve joins. Moreover,

$$\begin{aligned} U = \mathsf{t} \text{ in } \boldsymbol{\gamma}(G) &\iff \uparrow_G U = \mathsf{t} \text{ in } \boldsymbol{\mu}_+(G), \\ U = \mathsf{t} \text{ in } \boldsymbol{\mu}_+(G) &\iff \downarrow_G U = \mathsf{t} \text{ in } \boldsymbol{\gamma}(G). \end{aligned}$$

Here we use the assumption that G does not contain isolated vertices.

To prove that $\Gamma \vdash \varphi$ fails in $\boldsymbol{\mu}_+(G)$ whenever it fails in $\boldsymbol{\gamma}(G)$, it suffices to transform each valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(G)$ into a valuation $\overline{v} : \mathbf{Fm} \to \boldsymbol{\mu}_+(G)$ such that $\overline{v}(\varphi) = \mathsf{t}$ in $\boldsymbol{\mu}_+(G)$ if and only if $v(\varphi) = \mathsf{t}$ in $\boldsymbol{\gamma}(G)$ for each disjunctive clause $\varphi := \bigvee_{i \in I} l_i$. By the observations above $\overline{v}(\varphi) = \mathsf{t}$ in $\boldsymbol{\mu}_+(G)$ if and only if $\downarrow_G \overline{v}(\varphi) = \mathsf{t}$ in $\boldsymbol{\gamma}(G)$, but $\downarrow_G \overline{v}(\varphi) = \downarrow_G \bigvee_{i \in I} \overline{v}(l_i) = \bigvee_{i \in I} \downarrow_G \overline{v}(l_i) = \bigvee_{i \in I} v(l_i) = v(\varphi)$ because $\downarrow_G \overline{v}(p) = \downarrow_G \uparrow_G v(p) = v(p)$ and $\downarrow_G \overline{v}(-p) = \downarrow_G - \uparrow_G v(p) = -v(p) = v(-p)$.

Conversely, to prove that $\Gamma \vdash \varphi$ fails in $\boldsymbol{\gamma}(G)$ whenever it fails in $\boldsymbol{\mu}_+(G)$, it suffices to transform each valuation $v : \mathbf{Fm} \to \boldsymbol{\mu}_+(G)$ into a valuation $\overline{v} : \mathbf{Fm} \to \boldsymbol{\gamma}(G)$ such that $\overline{v}(\varphi) = \mathsf{t}$ in $\boldsymbol{\gamma}(G)$ if and only if $v(\varphi) = \mathsf{t}$ in $\boldsymbol{\mu}_+(G)$ for each disjunctive clause $\varphi := \bigvee_{i \in I} l_i$. Suppose that $v(\varphi) = \mathsf{t}$ in $\boldsymbol{\mu}_+(G)$. Then by the observations above in $\boldsymbol{\gamma}(G)$ we have $\mathsf{t} = \downarrow_G v(\varphi) = \downarrow_G (\bigvee_{i \in I} v(l_i)) = \bigvee \downarrow_G v(l_i) \subseteq \bigvee_{i \in I} w(l_i) = w(\bigvee_{i \in I} l_i) = w(\varphi)$, where the inclusion in the middle uses the observation that $\downarrow_G -U \subseteq - \downarrow_G U$. In other words, each disjunctive clause designated by v is also designated by w. Moreover, if φ is a positive clause, then the inclusion in the middle is an equality, therefore we may reverse this reasoning: if $\mathsf{t} = w(\varphi)$, then $\mathsf{t} = \downarrow_G v(\varphi)$ in $\boldsymbol{\gamma}(G)$, hence $\mathsf{t} = v(\varphi)$ in $\boldsymbol{\mu}_+(G)$. In other words, a positive disjunctive clause is designated by v if and only if it is designated by w. $\square$

To obtain a completeness theorem for a finitary super-Belnap logic, it now suffices to determine, using the previous lemma, which of the matrices of the form $\boldsymbol{\mu}(G, H, k)$ are its models. We use this method to obtain completeness theorems for the logics $\mathcal{ECQ}_n$, $\mathcal{ETL}_n$, and $\mathcal{SDS}_n$. We also show an alternative proof of the completeness theorem for $\mathcal{K}_- = \mathcal{SDS}_\omega$ originally proved in Section 5.2 (Proposition 5.17).

Proposition 7.24 (Finite reduced models of $\mathcal{ECQ}_n$).
$\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{ECQ}_n$ for $n \geq 2$ if and only if either $k \geq 1$ or G is not n-colourable.

Proof. By Corollary 4.26 a non-trivial matrix $\boldsymbol{\mu}(G, H, k) = \boldsymbol{\mu}_+(G) \times \boldsymbol{\mu}_-(H) \times \mathbb{B}_{\mathbf{2}}^k$ is a model of $\mathcal{ECQ}_n$ if and only if one of the matrices $\boldsymbol{\mu}_+(G)$, $\boldsymbol{\mu}_-(H)$, $\mathbb{B}_{\mathbf{2}}^k$ is a non-trivial model of $\mathcal{ECQ}_n$. But $\mathbb{B}_{\mathbf{2}}^k$ is a non-trivial model of $\mathcal{ECQ}_n$ if and only if $k \geq 1$, and $\boldsymbol{\mu}_-(H)$ is never a non-trivial model of $\mathcal{ECQ}$, as witnessed by the valuation $v : \mathbf{Fm} \to \boldsymbol{\mu}_-(H)$ such that $v(p)$ is interpreted as the set of all designated points of $\mathcal{F}_-(H)$. Since a point u is designated in $\mathcal{F}_-(H)$ if and only if ∂u is not, $v(-p) = v(p)$, hence $v(p \wedge -p)$ is designated in $\boldsymbol{\mu}_-(H)$.

Finally, we show that $\boldsymbol{\mu}_+(G)$ is a non-trivial model of $\mathcal{ECQ}_n$ if and only if either $n = 1$ or G is not n-colourable. If $n = 1$, then $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{ECQ}_n$ by virtue of being a model of $\mathcal{ETL}$. If G consists entirely of isolated vertices, then G is n-colourable and $\boldsymbol{\mu}_+(G) = \mathbb{BD}_{\mathbf{4}}^k$ for some k, hence $\boldsymbol{\mu}_+(G)$ is not a model of $\mathcal{ECQ}_n$ for $n \geq 2$. It thus suffices to prove the equivalence for $n \geq 2$ and for G not consisting entirely of isolated vertices. In that case G is the disjoint union of a set of isolated vertices and a non-empty graph $\overline{G}$ with no isolated vertices. Then $\boldsymbol{\mu}_+(G) = \boldsymbol{\mu}_+(\overline{G}) \times \mathbb{BD}_{\mathbf{4}}^k$ for some k, hence $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{ECQ}_n$ if and only if $\boldsymbol{\mu}_+(\overline{G})$ is a model of $\mathcal{ECQ}_n$ by Corollary 4.26. But the Graph Matrix Lemma (Lemma 7.23) applies to $\boldsymbol{\mu}_+(\overline{G})$, thus $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{ECQ}_n$ if and only if $\boldsymbol{\gamma}(\overline{G})$ is a model of $\mathcal{ECQ}_n$.

It remains to show that $\boldsymbol{\gamma}(\overline{G})$ is a model of $\mathcal{ECQ}_n$ for $n \geq 2$ and $\overline{G}$ non-empty if and only if $\overline{G}$ is not n-colourable. (Observe that $\overline{G}$ is n-colourable if and only if G is n-colourable.) But if $h : \overline{G} \to K_n$ is a colouring of $\overline{G}$, then the valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(\overline{G})$ with $v(p_i) := h^{-1}\{i\}$ witnesses the failure of $\mathcal{ECQ}_n$ in $\boldsymbol{\gamma}(\overline{G})$. Conversely, suppose that the valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(\overline{G})$ witnesses the failure of the rule $(p_1 \wedge -p_1) \vee \dots \vee (p_n \wedge -p_n) \vdash \emptyset$ in $\boldsymbol{\gamma}(\overline{G})$. Then the sets $v(p_i \wedge -p_i)$ form an n-colouring of $\overline{G}$. □

Proposition 7.25 (Finite reduced models of $\mathcal{ETL}_n$).
$\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{ECQ}_n$ for $n \geq 2$ if and only if $H = \emptyset$ and either $k \geq 1$ or G is not n-colourable.

Proof. This follows from Propositions 7.18 and 7.24. □

Proposition 7.26 (Finite reduced models of $\mathcal{ECQ}_\omega$).
$\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{ECQ}_\omega$ if and only if either $k \geq 1$ or G contains a loop.

Proof. By Proposition 7.24 the matrix $\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{ECQ}_\omega$ if and only if either $k \geq 1$ or G is not n-colourable for any n. But a graph is not n-colourable for any n if and only if it contains a loop. □

Proposition 7.27 (Finite reduced models of $\mathcal{ETL}_\omega$).
$\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{ECQ}_\omega$ if and only if $H = \emptyset$ and either $k \geq 1$ or G contains a loop.

Proof. This follows from Propositions 7.18 and 7.26. □

Proposition 7.28 (Finite reduced models of $\mathcal{SDS}_n$).
$\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{SDS}_n$ if and only if $H = \emptyset$ and G is not weakly n-colourable and contains no isolated vertex.

Proof. If $\boldsymbol{\mu}(G, H, k)$ is a model of $\mathcal{SDS}_n$, then it is a model of $\mathcal{ETL}$, hence $H = \emptyset$. Moreover, $\boldsymbol{\mu}(G, \emptyset, k) = \boldsymbol{\mu}_+(G) \times \mathbb{B}_{\mathbf{2}}^k$ is a model of $\mathcal{SDS}_n$ if and only if $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{SDS}_n$ by the description of logics of products (Theorem 4.25). It thus suffices to prove that $\boldsymbol{\mu}_+(G)$ is a non-trivial model of $\mathcal{SDS}_n$ if and only if G is not weakly n-colourable. If G contains an isolated vertex, then either $\boldsymbol{\mu}_+(G) = \mathbb{BD}_\mathbf{4}$, in which case $\boldsymbol{\mu}_+(G)$ is not a model of $\mathcal{SDS}_n$, or $\boldsymbol{\mu}_+(G) = \mathbb{M} \times \mathbb{BD}_\mathbf{4}$ for some non-trivial model $\mathbb{M}$ of $\mathcal{BD}$, hence $\operatorname{Log} \boldsymbol{\mu}_+(G) = \operatorname{Exp}_{\mathcal{BD}} \operatorname{Log} \mathbb{M} \leq \operatorname{Exp}_{\mathcal{BD}} \mathcal{CL} = \mathcal{ECQ}_\omega$ by Theorem 4.25. But then $\boldsymbol{\mu}_+(G)$ is not a model of $\mathcal{SDS}_n$.

Suppose therefore that $G := \langle X, R \rangle$ contains no isolated vertices. Then the Graph Matrix Lemma (Lemma 7.23) applies and $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{SDS}_n$ if and only if $\boldsymbol{\gamma}(G)$ is a model of $\mathcal{SDS}_n$. We now show that $\boldsymbol{\gamma}(G)$ is a model of $\mathcal{SDS}_n$ if and only if G is not weakly n-colourable. If $h : H \to K_n$ is a weak n-colouring of G for some subgraph H of G induced by the set of vertices $Y \subseteq X$, then the valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(G)$ with $v(p_i) := h^{-1}\{i\}$ and $v(q) := X \setminus Y$ and $v(r) := X \setminus -[X \setminus Y]$ witnesses the failure of the rule $\chi_n \vee p, -p \vee q \vdash q$ in $\boldsymbol{\gamma}(G)$. Conversely, suppose that the valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(G)$ witnesses the failure of this rule in $\boldsymbol{\gamma}(G)$. Then the sets $v(p_i \wedge -p_i)$ form a weak n-colouring of G. □

Proposition 7.29 (Finite reduced models of $\mathcal{SDS}_\omega$).
$\boldsymbol{\mu}(G, H, k)$ is a non-trivial model of $\mathcal{SDS}_\omega$ if and only if $H = \emptyset$ and each vertex of G has a reflexive neighbour.

Proof. By Proposition 7.28 it suffices to show that G is not weakly n-colourable if and only if each irreflexive vertex of G has a reflexive neighbour. Right-to-left, a partial n-colouring is always undefined on loops, hence if each element of G has a reflexive neighbour, then each vertex of G is a neighbour of a vertex on which each partial n-colouring is undefined. Conversely, let u be a vertex of G with no reflexive neighbours. Then for some n there is a partial n-colouring of G which is defined on all neighbours of u, therefore it is a weak n-colouring of G. $\square$

Recall that a finitary logic $\mathcal{L}$ is said to be ω-complete with respect to a class of matrices K if it is complete with respect to K as a finitary logic, i.e. if each finitary rule is valid in $\mathcal{L}$ if and only if it is valid in each matrix in K. The reduced models of super-Belnap logics are locally finite, therefore each finitary super-Belnap logic is ω-complete with respect to its finite reduced models. We now use the above descriptions of finite reduced models of selected super-Belnap logics to prove completeness theorems for them.

Proposition 7.30 (Completeness for $\mathcal{ECQ}_n$).
The logic $\mathcal{ECQ}_n$ for $n \geq 2$ is ω-complete with respect to the class of all matrices $\boldsymbol{\mu}_+(G) \times \mathbb{BD}_\mathbf{4}$ where G is a graph which is not n-colourable.

Proof. Recall the description of the finite reduced models of $\mathcal{ECQ}_n$ above (Proposition 7.24). Clearly all matrices $\boldsymbol{\mu}_+(G) \times \mathbb{BD}_\mathbf{4}$, where G is a graph which is not n-colourable, satisfy this description. Conversely, suppose that a finitary rule fails in $\mathcal{ECQ}_n$. Then it fails in some finite reduced model of $\mathcal{ECQ}_n$, i.e. in some matrix $\boldsymbol{\mu}(G, H, k)$ such that either $k \geq 1$ or G is not n-colourable. We have $\operatorname{Log} \boldsymbol{\mu}(G, H, 0) \geq \operatorname{Log} \boldsymbol{\mu}(G, \bullet, 0)$, even if $H = \emptyset$, therefore each rule which fails in $\boldsymbol{\mu}(G, H, 0)$ also fails in $\boldsymbol{\mu}(G, \bullet, 0) = \boldsymbol{\mu}_+(G) \times \mathbb{BD}_\mathbf{4}$. On the other hand, for $k \geq 1$ we have $\operatorname{Log} \boldsymbol{\mu}(G, H, k) = \operatorname{Log} \boldsymbol{\mu}(G, H, 1) \geq \mathcal{ECQ}_\omega$ by the description of the finite reduced models of $\mathcal{ECQ}_\omega$ (Proposition 7.26) and $\mathcal{ECQ}_\omega \geq \operatorname{Log} \boldsymbol{\mu}_+(G) \times \mathbb{BD}_\mathbf{4}$ since $\operatorname{Log} \boldsymbol{\mu}_+(G) \times \mathbb{BD}_\mathbf{4} = \operatorname{Exp}_{\mathcal{BD}} \operatorname{Log} \boldsymbol{\mu}_+(G) \leq \operatorname{Exp}_{\mathcal{BD}} \mathcal{CL} = \mathcal{ECQ}_\omega$ by Corollary 4.26. Thus each rule which fails in some matrix of the form $\boldsymbol{\mu}(G, H, k)$ for $k \geq 1$ also fails in each matrix of the form $\boldsymbol{\mu}_+(G) \times \mathbb{BD}_\mathbf{4}$. $\square$

Proposition 7.31 (Completeness for $\mathcal{ETL}_n$).
The logic $\mathcal{ETL}_n$ for $n \geq 2$ is ω-complete with respect to the class of all matrices $\boldsymbol{\mu}_+(G)$ where G is a graph which is not n-colourable.

Proof. The proof is entirely analogous to the proof of Proposition 7.30 if we take into account that $\boldsymbol{\mu}(G, H, k)$ is a model of $\mathcal{ETL}$ if and only if $H = \emptyset$ and replace $\mathcal{ECQ}_n$ and $\mathbb{BD}_\mathbf{4}$ by $\mathcal{ETL}_n$ and $\mathbb{ETL}_\mathbf{4}$. $\square$

Proposition 7.32 (Completeness for $\mathcal{SDS}_n$).
The logic $\mathcal{SDS}_n$ for $n \geq 1$ is ω-complete with respect to the class of all $\boldsymbol{\mu}_+(G)$ where G is a graph without isolated vertices which is not weakly n-colourable.

Proof. This follows from Proposition 7.28 in view of the fact that $\operatorname{Log} \boldsymbol{\mu}_+(G) \leq \operatorname{Log} \boldsymbol{\mu}_+(G) \times \mathbb{B}_\mathbf{2} = \operatorname{Log} \boldsymbol{\mu}(G, \emptyset, k)$. $\square$

Proposition 7.33 (Completeness for $\mathcal{SDS}_\omega$).
The logic $\mathcal{SDS}_\omega$ is ω-complete with respect to the class of all matrices $\boldsymbol{\mu}_+(G) \times \mathbb{B}_\mathbf{2}$ such that each vertex of G has a reflexive neighbour.

Proof. By Proposition 7.29 we have ω-completeness with respect to the class of all matrices $\boldsymbol{\mu}_+(G) \times \mathbb{B}_\mathbf{2}^k$ such that each vertex of G has a reflexive neighbour. But these matrices yield the same logics for all $k \geq 1$. Moreover, $\operatorname{Log} \boldsymbol{\mu}_+(G) \leq \operatorname{Log} \boldsymbol{\mu}_+(G) \times \mathbb{B}_\mathbf{2}$ holds by the description of logics of products (Theorem 4.25). □

We now have a completeness theorem for each logic introduced so far except for the logic mentioned in Proposition 6.19. Of course, this last batch of completeness theorems is somewhat less satisfying than the previous ones, but we shall see in the following section that for the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ this is unavoidable: apart from $\mathcal{ECQ}$ and $\mathcal{ETL}$, none of them are complete with respect to a finite set of finite matrices. We do not know whether this holds for $\mathcal{SDS}_n$.

Using the tools developed in this chapter, we can also provide an alternative proof of the completeness theorem for $\mathcal{K}_-$ (Proposition 5.17), which relies on the dual description of the finite reduced models of $\mathcal{K}_-$. The previous proof relied on an explicit description of the consequence relation of $\mathcal{K}_-$, while the current proof is semantic and show that each finite reduced model of $\mathcal{K}_-$ may be obtained as a submatrix of a finite product of $\mathbb{M}_\mathbf{8}$.

Proposition 7.34 (Completeness for $\mathcal{K}_-$ — alternative proof).
$\mathcal{SDS}_\omega = \operatorname{Log} \mathbb{M}_\mathbf{8}$.

Proof. By Proposition 7.33 it suffices to show that if $\mathbb{M}_\mathbf{8} \in \operatorname{Mod} \mathcal{L}$ for some super-Belnap logic $\mathcal{L}$, then $\boldsymbol{\mu}_+(G) \in \operatorname{Mod} \mathcal{L}$ whenever G is a graph in which each vertex has a reflexive neighbour. Observe that $\mathbb{M}_\mathbf{8} = \boldsymbol{\mu}_+(G_2)$, where G_2 is the graph obtained by adding one loop to K_2, i.e. it consists of a reflexive and an irreflexive vertex which are neighbours. But each graph in which each vertex has a reflexive neighbour can be obtained by taking a suitable quotient of a disjoint union of finitely many copies of G_2 and adding some edges to it. Since $\operatorname{Mod} \mathcal{L}$ is closed under finite products and submatrices, it now suffices to observe that taking a disjoint union of finitely many copies of G_2 corresponds to taking a finite power of $\mathbb{M}_\mathbf{8}$, and taking a quotient and adding some edges to it corresponds to taking a submatrix. □

An example of a matrix separating the logics $\mathcal{LP} \cap \mathcal{ECQ}_{n+2}$ and $\mathcal{SDS}_n$, whose existence was asserted in Section 6.1, may now also be supplied.

Fact 7.35. *$\mathcal{LP} \cap \mathcal{ECQ}_{n+2} \nleq \mathcal{SDS}_n$.*

Proof. The graph K_{n+2} is $(n+2)$-colourable but not weakly n-colourable, therefore $\boldsymbol{\mu}_+(K_{n+2})$ is a model of $\mathcal{SDS}_n$ but not a model of $\mathcal{ECQ}_{n+2}$ by

Propositions 7.24) and 7.28). Moreover, $\boldsymbol{\mu}_+(K_{n+2})$ is not a model of $\mathcal{LP}$, hence not a model of $\mathcal{LP} \cap \mathcal{ECQ}_2$ by Proposition 4.22. $\square$

Recall that $\mathcal{LP} \cap \mathcal{ECQ}_{n+1} < \mathcal{ECQ}_{n+1} < \mathcal{ETL}_{n+1} < \mathcal{SDS}_n$.

Corollary 7.36. *For each $n \geq 1$ we have:*

(i) $\mathcal{LP} \cap \mathcal{ECQ}_n < \mathcal{LP} \cap \mathcal{ECQ}_{n+1}$,

(ii) $\mathcal{ECQ}_n < \mathcal{ECQ}_{n+1}$,

(iii) $\mathcal{ETL}_n < \mathcal{ETL}_{n+1}$,

(iv) $\mathcal{SDS}_n < \mathcal{SDS}_{n+1}$.

7.4 Describing $\operatorname{Ext} \mathcal{BD}$ graph-theoretically

Having described the finite reduced models of $\mathcal{BD}$ in terms of finite graphs, we may now describe the lattice of finitary super-Belnap logics in terms of classes of finite graphs. Although the description of the whole lattice $\operatorname{Ext}_\omega \mathcal{BD}$ turns out to be somewhat complicated, it yields a reasonably simple description of the lattices $\operatorname{Ext}_\omega \mathcal{ETL}$, $[\mathcal{ETL}, \mathcal{ETL}_\omega]$, and $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ETL}$ and $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{BD}$. In particular, the lattice of finitary explosive extensions of $\mathcal{BD}$ turns out to be nothing but the order dual of the lattice of classes of finite graphs closed under homomorphisms, give or take an element at the top and bottom. These results will then enable us to exploit theorems of graph theory to prove facts about super-Belnap logics and De Morgan algebras.

To start with, we show that in order to describe the lattice of finitary extensions of a finitary logic satisfying a suitable local finiteness condition, it suffices to describe the $\mathbb{RS}$-*order* on its finite reduced models, i.e. the order $\mathbb{M} \leq_{\mathbb{RS}} \mathbb{N}$ such that

$$\mathbb{M} \leq_{\mathbb{RS}} \mathbb{N} \iff \mathbb{M} \in \mathbb{RS}(\mathbb{N}).$$

The following theorem is essentially a reformulation of a theorem of Grätzer and Quackenbush [32, Thm 2.3] for matrices.

Theorem 7.37 (Extensions of finitary locally finite logics). *Let $\mathcal{B}$ be a finitary logic such that $\mathbb{SPP}_U(\operatorname{Alg} \mathcal{B})$ is locally finite. Then $\operatorname{Ext}_\omega \mathcal{B}$ is dually isomorphic to the lattice of classes K of finite reduced models of $\mathcal{B}$ closed under finite products such that $\mathbb{A} \in \mathbb{RS}(\mathbb{B})$ & $\mathbb{B} \in \mathsf{K} \implies \mathbb{A} \in \mathsf{K}$.*

Proof. If $\operatorname{Alg} \mathcal{B}$ is locally finite, then each finitary extension $\mathcal{L}$ of $\mathcal{B}$ is complete as a finitary logic with respect to the class $\operatorname{Mod}^*_\omega \mathcal{L}$ of its finite reduced models. $\operatorname{Ext}_\omega \mathcal{B}$ is therefore dually isomorphic to the lattice of classes K of finite reduced models of $\mathcal{B}$ such that $\mathsf{K} = \operatorname{Mod}^*_\omega \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Ext}_\omega \mathcal{B}$. It now suffices to show that $\mathsf{K} = \mathrm{Mod}^*_\omega \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Ext}_\omega \mathcal{B}$ if and only if K satisfies the conditions of the theorem.

A theorem due to Czelakowski [15], formulated in greater generality by Dellunde and Jansana [16, Thm 6], implies that $\mathrm{Mod}^* \mathrm{Log}_\omega \mathsf{K} = \mathbb{R}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ for each class of matrices K. It follows that $\mathrm{Mod}^* \mathcal{L}$ satisfies the conditions of the theorem for each finitary extension $\mathcal{L}$ of $\mathcal{B}$.

Conversely, let K be a class of finite reduced models of $\mathcal{B}$ satisfying the conditions of the theorem. It now suffices to show that $\mathbb{R}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathsf{K}) \subseteq \mathsf{K}$. Suppose therefore that $\mathbb{M} \in \mathbb{R}(\mathbb{N})$ and $\mathbb{N} \in \mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. Since $\mathbb{M}$ is finitely generated, we may assume without loss of generality that $\mathbb{N}$ is also finitely generated, hence by the local finiteness of $\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathrm{Alg}\,\mathcal{B})$ it is finite. But a finite submatrix of a product of matrices in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ is in fact a finite submatrix of a product of finitely many finite submatrices of matrices in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. But again by the local finiteness assumption a finite submatrix of an ultraproduct of K (which are finite) is in fact a finite submatrix of some matrix in K. It follows that $\mathbb{N} \in \mathbb{S}(\mathsf{K})$ and $\mathbb{M} \in \mathbb{R}\mathbb{S}(\mathsf{K})$. But then $\mathbb{M} \in \mathsf{K}$ by the closure of K under $\mathbb{R}\mathbb{S}$. □

The $\mathbb{R}\mathbb{S}$-order extends to triples $\langle G, H, k\rangle$ in the obvious way, namely

$$\langle G_1, H_1, k_1\rangle \leq_{\mathbb{R}\mathbb{S}} \langle G_2, H_2, k_2\rangle \iff \boldsymbol{\mu}(G_1, H_1, k_1) \in \mathbb{R}\mathbb{S}(\boldsymbol{\mu}(G_2, H_2, k_2)).$$

Theorem 7.38 (The lattice $\mathrm{Ext}_\omega \mathcal{BD}$).
The lattice $\mathrm{Ext}_\omega \mathcal{BD}$ is dually isomorphic via the map $\mathcal{L} \mapsto \boldsymbol{\mu}^{-1}[\mathrm{Mod}^* \mathcal{L}]$ to the lattice of classes K of triples $\langle G, H, k\rangle$ such that $\langle \emptyset, \emptyset, 0\rangle \in \mathsf{K}$ and

(i) $\langle G_1, H_1, k_1\rangle \leq_{\mathbb{R}\mathbb{S}} \langle G_2, H_2, k_2\rangle \in \mathsf{K}$ *implies* $\langle G_1, H_1, k_1\rangle \in \mathsf{K}$,

(ii) $\langle G_i, H_i, k_i\rangle \in \mathsf{K}$ *for* $i \in \{1, 2\}$ *implies* $\langle G_1 \sqcup G_2, H_1 \sqcup H_2, k_1 + k_2\rangle \in \mathsf{K}$.

Proof. Let $\mathsf{M} := \boldsymbol{\mu}[\mathsf{K}]$. The conditions on K state precisely that M closed under finite products (including the nullary product, which yields the trivial singleton matrix), and $\mathbb{A} \in \mathsf{M}$ whenever $\mathbb{A} \in \mathbb{R}\mathbb{S}(\mathbb{B})$ and $\mathbb{B} \in \mathsf{M}$. The claim now holds by the previous theorem, since each finite reduced model of $\mathcal{BD}$ has the form $\boldsymbol{\mu}(G, H, k)$. □

In order for the above theorem to be informative, it remains to provide a concrete description of the $\mathbb{R}\mathbb{S}$-order. Recall that $\bullet$ denotes the loopless singleton graph and K_2 denotes the isolated edge. The isolated loop will be denoted $\circlearrowright$ and the graph obtained by adding a loop to one of the edges of K_2 will be denoted L_2.

Proposition 7.39 (The $\mathbb{R}\mathbb{S}$-order on finite reduced models of $\mathcal{BD}$).
The downward closure of a triples in the $\mathbb{R}\mathbb{S}$-order is the smallest class K containing this triple such that:

(i) if $\langle G, H, k\rangle \in \mathsf{K}$ *and there is a pair of surjective graph homomorphisms* $G \twoheadrightarrow G'$ *and* $H \twoheadrightarrow H'$, *then* $\langle G', H', k\rangle \in \mathsf{K}$,

(ii) *if* $\langle G, H, k\rangle \in \mathsf{K}$ *and* $1 \leq k' \leq k$*, then* $\langle G, H, k'\rangle \in \mathsf{K}$*,*

(iii) *if* $\langle G, H, k\rangle \in \mathsf{K}$ *and* G *contains a loop, then* $\langle G, H, 0\rangle \in \mathsf{K}$*,*

(iv) *if* $\langle G, H, k\rangle \in \mathsf{K}$ *and* H' *is an induced subgraph of* H *such that* $\langle G, H', k\rangle$ *is distinct from* $\langle \emptyset, \emptyset, 0\rangle$*, then* $\langle G, H', k\rangle \in \mathsf{K}$*,*

(v) *if* $\langle G \sqcup K_2, H, k\rangle \in \mathsf{K}$*, then* $\langle G \sqcup \bullet, H, k\rangle \in \mathsf{K}$*,*

(vi) *if* $\langle G \sqcup \circlearrowright, H, k\rangle \in \mathsf{K}$*, then* $\langle G, H, k+1\rangle \in \mathsf{K}$*,*

(vii) *if* $\langle G, H \sqcup \circlearrowright, k\rangle \in \mathsf{K}$*, then* $\langle G, H, k+1\rangle \in \mathsf{K}$*,*

(viii) *if* $\langle G, H \sqcup \bullet, k\rangle \in \mathsf{K}$ *and* G' *is obtained by adding a non-isolated vertex to* G*, then* $\langle G', H, k\rangle \in \mathsf{K}$*,*

(ix) *if* $\langle G, H \sqcup L_2, k\rangle \in \mathsf{K}$ *or* $\langle G, H \sqcup \circlearrowright \sqcup \bullet, k\rangle \in \mathsf{K}$*, then* $\langle G \sqcup \circlearrowright, H, k\rangle \in \mathsf{K}$*,*

(x) *if* $\langle G, H \sqcup K_2, k\rangle \in \mathsf{K}$*, then* $\langle G \sqcup \bullet, H, k\rangle \in \mathsf{K}$*,*

(xi) *if* $\langle G, H \sqcup \bullet \sqcup \bullet, k\rangle \in \mathsf{K}$*, then* $\langle G \sqcup K_2, H, k\rangle \in \mathsf{K}$.

Proof. We need to verify that each of these operations yields a triple which lies below the original triple in the $\mathbb{RS}$-order, and moreover that each triple which lies below $\langle G, H, k\rangle$ in the $\mathbb{RS}$-order may be obtained from $\langle G, H, k\rangle$ by a sequence of these operations. The proof of the first claim is easier and essentially contained in the proof of the second claim, therefore we omit it.

In dual terms the relation $\langle G_1, H_1, k_1\rangle \leq_{\mathbb{RS}} \langle G_2, H_2, k_2\rangle$ states that the frame $\mathcal{F}(G_1, H_1, k_1)$ is the Leibniz subframe of a quotient frame of $\mathcal{F}(G_2, H_2, k_2)$. But it is known that $\mathbb{RSRS}(\mathbf{A}) = \mathbb{RS}(\mathbf{A})$. Downward closure in the $\mathbb{RS}$-order therefore amounts to closure under Leibniz subframes of *principal* quotient frames. It now suffices to show that the Leibniz subframe of each principal quotient of a frame of the form $\mathcal{F}(G, H, k)$ is obtained by using closure under the operations (i)–(xi). The rest of the proof is a tedious but fairly straightforward case analysis.

Consider the Leibniz subframe of the principal quotient of $\mathcal{F}(G, H, k)$ obtained by adding some pair $\langle x, y\rangle$ with $x \nleq y$ to the order $\leq$. We show that in each case this frame may be obtained from $\mathcal{F}(G, H, k)$ using the operations (i)–(xi). We have the following cases:

$\langle u, \partial v\rangle$ for $\langle u, v\rangle \in G \times G$: add an edge between $u \in G$ and $v \in G$.
$\langle u, \partial v\rangle$ for $\langle u, v\rangle \in H \times H$: add an edge between $u \in H$ and $v \in H$.
$\langle u, v\rangle \in G \times G$: take a quotient identifying u and v.
$\langle \partial u, \partial v\rangle$ for $\langle u, v\rangle \in G \times G$: same quotient frame as $\langle v, u\rangle$.
$\langle u, v\rangle \in H \times H$: delete $u \in H$.
$\langle \partial u, \partial v\rangle \in H \times H$ for $\langle u, v\rangle \in H \times H$: same quotient frame as $\langle v, u\rangle$.
$\langle u, v\rangle \in H \times G$: delete $u \in H$.
$\langle \partial u, \partial v\rangle$ for $\langle u, v\rangle \in G \times H$: same quotient frame as $\langle v, u\rangle$.

$\langle u, \partial v\rangle$ for $\langle u, v\rangle \in G \times H$: delete $v \in H$.

$\langle u, \partial v\rangle$ for $\langle u, v\rangle \in H \times G$: same quotient frame as $\langle v, \partial u\rangle$.

$\langle u, v\rangle \in G \times H$: delete v and all neighbours of v. If v is loopless, add a loopless neighbour to u, i.e. use (viii).

$\langle \partial u, \partial v\rangle$ for $\langle u, v\rangle \in H \times G$: same quotient frame as $\langle v, u\rangle$.

$\langle \partial u, v\rangle$ for $\langle u, v\rangle \in G \times G$: if u and v both have some neighbour outside the set $\{u, v\}$, then the Leibniz subframe is obtained by (i) deleting u and v, (ii) adding an edge between x and y whenever xRu and vRy, (iv) adding a loop to each neighbour of u in case v has a loop, and (v) adding a loop to each neighbour of v in case u has a loop. This effect can be achieved by adding edges between each neighbour of u and each neighbour of v, adding loops to each neighbour of u or v in the appropriate cases, and then collapsing u with some neighbour of v and v with some neighbour of u.

Now suppose that one of the vertices u, v has no neighbour outside the set $\{u, v\}$. Withous loss of generality we may assume that this vertex is u, taking $\langle \partial v, u\rangle$ instead of $\langle \partial u, v\rangle$ otherwise.

If $u = v$, then the Leibniz subframe is obtained either by adding a loop to an isolated vertex of G or by removing a loop from G and increasing k by one, i.e. using (i) or (vi). Let us therefore assume that $u \neq v$.

If u is an isolated vertex, instead of $\langle \partial u, v\rangle$ we may take $\langle u, v\rangle$. If $R[u] = \{v\}$, then the Leibniz subframe is obtained by removing u. This effect can be achieved by collapsing u with a neighbour of v other than u (if there are some) or replacing u and v by an isolated vertex (if u and v form an isolated edge), i.e. using (i) or (v). If $R[u] = \{u\}$ or $R[u] = \{u, v\}$ and v has a neighbour outside of $\{u, v\}$, then the Leibniz subframe is obtained by adding an edge between each pair of neighbours of v other than u and v and collapsing u and v with some such neighbour of v, i.e. using (i).

Finally, suppose that u has a loop and v has no neighbour outside of $\{u, v\}$. If $R[v] = \emptyset$, then instead of $\langle \partial u, v\rangle$ we may take $\langle v, u\rangle$. If v has a loop, then the Leibniz subframe is obtained by collapsing u and v into a single loop and removing this loop while increasing k, i.e. using (i) and (vi). If $R[v] = \{u\}$ and $R[u] = \{u, v\}$, then the Leibniz subframe is obtained by collapsing u and v into a single loop, i.e. using (i).

$\langle \partial u, v\rangle$ for $\langle u, v\rangle \in H \times H$: in every case the Leibniz subframe removes all neighbours of u and v outside the set $\{u, v\}$. We may therefore assume without loss of generality that u and v have no neighbours outside of $\{u, v\}$.

If $u = v$, then the Leibniz subframe moreover either removes an isolated vertex from H and adds a loop to G or it removes an isolated loop from H while increasing k, i.e. it may be obtained using (vii) or (viii). Suppose therefore that $u \neq v$.

If both u and v have a loop, then the Leibniz subframe is obtained by collapsing u and v into a single loop and removing this loop while increasing k, i.e. using (i) and (vii). If u has a loop and v does not, then the Leibniz subframe is obtained by removing u and v and adding an isolated loop to

G, i.e. using (ix). If v has a loop and u does not, taking $\langle \partial v, u\rangle$ instead of $\langle \partial u, v\rangle$ reduces the situation to the previous case.

If neither u nor v has a loop but u and v are neighbours, the Leibniz subframe is obtained using (x). If u and v are both isolated, then the Leibniz subframe is obtained using (xi).

$\langle \partial u, v\rangle$ for $\langle u, v\rangle \in H \times G$: in every case the Leibniz subframe removes all neighbours of u other than u. We may therefore assume without loss of generality that u is either an isolated vertex or an isolated loop. If u is an isolated vertex, then the Leibniz subframe is obtained by removing u from H. If u is an isolated loop, then the Leibniz subframe is obtained by removing u and taking the principal quotient by $\langle \partial v, v\rangle$.

$\langle \partial u, v\rangle$ for $\langle u, v\rangle \in G \times H$: same quotient as $\langle \partial v, u\rangle$.

Now suppose that $x = \partial x$ is a point of $\mathcal{F}(G, H, k)$.

$\langle y, x\rangle$ for $y = \partial y$ distinct from x: decreasing k by one.

$\langle u, x\rangle$ for $u \in G$: add a loop to u and decrease k by one.

$\langle u, x\rangle$ for $u \in H$: remove u from H.

$\langle \partial u, x\rangle$ for $u \in G$: if u has a neighbour other than u, add an edge between each pair of neighbours of u and decrease k by one. If u is an isolated vertex, add a loop to u and decrease k by one. If u is an isolated loop, then remove u from G. This effect may be achieved by using (vi) and (ii).

$\langle x, \partial u\rangle$ for $u \in H$: remove all neighbours of u other than u. If u has a loop, remove u from H. If u is loopless, decrease k by one, remove u from H and add a loop to G. This may be achieved using (viii) and (ii) or (iii).

The cases $\langle u, x\rangle$ and $\langle \partial u, x\rangle$ reduce to the previous four cases. □

The above proposition give us a description, however unwieldy, of the lattice of finitary super-Belnap logics in purely graph-theoretic terms.

Fortunately, the picture simplifies if we restrict to extensions of $\mathcal{ETL}$. Recall that $\boldsymbol{\mu}(G, H, k)$ is a model of $\mathcal{ETL}$ if and only if $H = \emptyset$. We can therefore reduce the representation of finite reduced models of $\mathcal{ETL}$ in terms of triples $\langle G, H, k\rangle$ to a representation in terms of pairs $\langle G, k\rangle$ consisting of a graph G and some $k \in \omega$.

Abusing our notation slightly, we shall identify pairs $\langle G, k\rangle$ with triples $\langle G, \emptyset, k\rangle$. That is, $\boldsymbol{\mu}(G, k) := \boldsymbol{\mu}(G, \emptyset, k)$ and

$$\langle G_1, k_1\rangle \leq_{\mathbb{RS}} \langle G_2, k_2\rangle \iff \boldsymbol{\mu}(G_1, \emptyset, k_1) \in \mathbb{RS}(\boldsymbol{\mu}(G_2, \emptyset, k_2)).$$

Proposition 7.40 (The $\mathbb{RS}$-order on finite reduced models of $\mathcal{ETL}$). *The downward closure of a pair in the order $\leq_{\mathbb{RS}}$ is the smallest class* K *containing this pair such that:*

(i) if $\langle G, k\rangle \in \mathsf{K}$ and $G \twoheadrightarrow H$, then $\langle H, k\rangle \in \mathsf{K}$,

(ii) if $\langle G, k\rangle \in \mathsf{K}$ and $1 \leq l \leq k$, then $\langle G, l\rangle \in \mathsf{K}$,

(iii) if $\langle G, k\rangle \in \mathsf{K}$ and G contains a loop, then $\langle G, 0\rangle \in \mathsf{K}$,

(iii) *if* $\langle G \sqcup K_2, k\rangle \in \mathsf{K}$, *then* $\langle G \sqcup \bullet, k\rangle \in \mathsf{K}$,

(iv) *if* $\langle G \sqcup \circlearrowright, k\rangle \in \mathsf{K}$, *then* $\langle G, k+1\rangle \in \mathsf{K}$.

Proof. This is an immediate corollary to Proposition 7.39. □

In fact, we can restrict to triples and pairs with $k \in \{0, 1\}$, since for each $k \geq 1$ we have $\boldsymbol{\mu}(G, H, 1) \in \operatorname{Mod} \mathcal{L} \iff \boldsymbol{\mu}(G, H, k) \in \operatorname{Mod} \mathcal{L}$. A class of such restricted pairs may be identified with a pair of classes of graphs K_0 and K_1 such that $G \in \mathsf{K}_i \iff \langle G, i\rangle \in \mathsf{K}$ for $i \in \{0, 1\}$. Let $\boldsymbol{\mu}_{0,1}$ be the restriction of the map $\boldsymbol{\mu}$ to such pairs.

Theorem 7.41 (The lattice $\operatorname{Ext}_\omega \mathcal{ETL}$).
The lattice $\operatorname{Ext}_\omega \mathcal{ETL}$ *is dually isomorphic via the map* $\mathcal{L} \mapsto \boldsymbol{\mu}_{0,1}^{-1}[\operatorname{Mod}^* \mathcal{L}]$ *to the lattice of pairs of classes of graphs* $\langle \mathsf{K}_0, \mathsf{K}_1\rangle$ *such that* $\emptyset \in \mathsf{K}_0$ *and*

(i) *if* $G \in \mathsf{K}_i$ *and* $G \twoheadrightarrow H$, *then* $H \in \mathsf{K}_i$ *for* $i \in \{0, 1\}$,

(ii) *if* $G, H \in \mathsf{K}_i$, *then* $G \sqcup H \in \mathsf{K}_i$ *for* $i \in \{0, 1\}$,

(iii) *if* $G \in \mathsf{K}_0$ *and* $\emptyset \in \mathsf{K}_1$, *then* $G \in \mathsf{K}_1$,

(iv) *if* $G \in \mathsf{K}_1$ *and* G *contains a loop, then* $G \in \mathsf{K}_0$,

(v) *if* $G \sqcup K_2 \in \mathsf{K}_i$, *then* $G \sqcup \bullet \in \mathsf{K}_i$ *for* $i \in \{0, 1\}$,

(vi) *if* $G \sqcup \circlearrowright \in \mathsf{K}_i$, *then* $G \in \mathsf{K}_1$ *for* $i \in \{0, 1\}$.

Proof. This follows from Theorem 7.38 and Proposition 7.40 using the fact that each finite reduced model of $\mathcal{ETL}$ has the form $\boldsymbol{\mu}(G, k)$. □

Restricting to the interval $[\mathcal{ETL}, \mathcal{ETL}_\omega]$ in the lattice $\operatorname{Ext}_\omega \mathcal{ETL}$ yields a graph-theoretic description which is very simple to grasp.

Theorem 7.42 (The interval $[\mathcal{ETL}, \mathcal{ETL}_\omega]$).
The interval $[\mathcal{ETL}, \mathcal{ETL}_\omega]$ *of the lattice* $\operatorname{Ext}_\omega \mathcal{ETL}$ *is dually isomorphic via the map* $\mathcal{L} \mapsto \boldsymbol{\mu}_+^{-1}[\operatorname{Mod} \mathcal{L}]$ *to the lattice of classes of graphs* K *such that*

(i) *if* $G \in \mathsf{K}$ *and* $G \twoheadrightarrow H$, *then* $H \in \mathsf{K}$,

(ii) *if* $G, H \in \mathsf{K}$, *then* $G \sqcup H \in \mathsf{K}$,

(iii) *if* G *contains a loop, then* $G \in \mathsf{K}$,

(iv) *if* $G \sqcup K_2 \in \mathsf{K}$, *then* $G \sqcup \bullet \in \mathsf{K}$.

Proof. This follows from Theorem 7.41: each matrix of the form $\boldsymbol{\mu}(G, 1)$ is a model of $\mathcal{ETL}_\omega$, therefore $\mathcal{L}_1 \leq \mathcal{L}_2$ obtains in the interval $[\mathcal{ETL}, \mathcal{ETL}_\omega]$ if and only if each model of $\mathcal{L}_2$ of the form $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{L}_1$. □

Instead of assuming that all graphs with loops belong to K, we could alternatively restrict to loopless graphs and obtain the same lattice.

Finally, we describe the lattice $\operatorname{Exp}\operatorname{Ext}_{\omega}\mathcal{BD}$. In particular, we prove that $\operatorname{Exp}\operatorname{Ext}_{\omega}\mathcal{BD}$ is, give or take an element at the top and bottom, dually isomorphic to the lattice of finite graphs closed under homomorphisms, i.e. to the lattice of upsets of the homomorphism pre-order on graphs.

We could provide a semantic proof of this fact which would rely on the characterization of explosive extensions of $\mathcal{BD}$ as precisely those which are closed under homomorphic preimages (recall Theorem 4.9). However, we shall instead provide a proof which yields, given a class K of graphs closed under homomorphisms (satisfying a minor technical assumption), an axiomatization of the finitary explosive extension of $\mathcal{BD}$ whose finite reduced models are precisely the matrices $\boldsymbol{\mu}_{+}(G)$ for $G \in \mathsf{K}$.

Given a graph $G = \langle X, R \rangle$ which does not consist entirely of isolated vertices, we show that the class of all matrices $\boldsymbol{\mu}_{+}(H)$ such that $H \nleq G$ in the homomorphism pre-order is axiomatized by the rule $\alpha_G \vdash \emptyset$ for a suitable formula α. To construct this formula, let us label a set of propositional atoms by the vertices of a graph G and define for each $u \in X$ the formula

$$\varphi_u := \bigwedge p_u \wedge \bigwedge_{\substack{v \in X \\ \neg uRv}} -p_v.$$

Now consider the formula

$$\alpha_G := \bigvee_{u \in X} \varphi_u \vdash \emptyset.$$

Recall that $\overline{G}$ denotes the graph obtained by removing all isolated vertices from G.

Lemma 7.43 (Graph Homomorphism Lemma).
Let G and H be graphs with $\overline{G}$ non-empty. Then $\alpha_G \vdash \emptyset$ fails in $\boldsymbol{\mu}_{+}(H)$ if and only if there is a graph homomorphism $H \to G$.

Proof. If there is a graph homomorphism $H \to G$, then there is a graph homomorphism $H \to \overline{G}$, provided that $\overline{G}$ is non-empty. But then there is a morphism of $\mathcal{BD}$-frames $\mathcal{F}_{+}(H) \to \mathcal{F}_{+}(\overline{G})$, hence by the duality for de morgan matrices (Theorem 7.10) there is a homomorphism of De Morgan matrices $\boldsymbol{\mu}_{+}(\overline{G}) \to \boldsymbol{\mu}_{+}(H)$. If $\boldsymbol{\mu}_{+}(\overline{G})$ is not a model of $\alpha_G \vdash \emptyset$, then neither is $\boldsymbol{\mu}_{+}(H)$ by Fact 4.7. It therefore suffices to show that $\boldsymbol{\mu}_{+}(\overline{G})$ is not a model of $\alpha_G \vdash \emptyset$, i.e. by the Graph Matrix Lemma (Lemma 7.23) that $\boldsymbol{\gamma}(\overline{G})$ is not a model of $\alpha_G \vdash \emptyset$. But this is witnessed by the valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(\overline{G})$ with $v(p_u) := \{u\}$ for $u \in \overline{G}$ and $v(p_u) := \mathsf{f}$ for $u \in G \setminus \overline{G}$.

Conversely, suppose that the rule $\alpha_G \vdash \emptyset$ fails in $\boldsymbol{\mu}_{+}(H)$. If $\overline{H} = \emptyset$, then there is trivially a homomorphism $H \to G$, since G is non-empty.

Otherwise $\boldsymbol{\mu}_+(H) = \boldsymbol{\mu}_+(\overline{H}) \times \mathbb{BD}_{\mathbf{4}}^k$ for some k, hence the rule $\alpha_G \vdash \emptyset$ fails in $\boldsymbol{\mu}_+(\overline{H})$. Then by the Graph Matrix Lemma it fails in $\boldsymbol{\gamma}(\overline{H})$, as witnessed by a valuation $v : \mathbf{Fm} \to \boldsymbol{\gamma}(\overline{H})$. Since G is non-empty, to prove the existence of a homomorphism $H \to G$ it suffices to provide a homomorphism $\overline{H} \to G$.

Let $G = \langle X, R \rangle$ and $H = \langle Y, S \rangle$ and consider the relation $P \subseteq Y \times X$ such that $u'Pu$ if and only if $u' \in v(\varphi_u)$. Since $v(\bigvee_{u \in X} \varphi_u) = Y$, there is for each $u' \in Y$ some $u \in X$ such that $u'Pu$. Moreover, the relation P is edge-preserving: if $u'Pu$ and $v'Pv$ and $\neg uRv$, then $u' \in v(p_u) \cap v(-p_v)$ and $v' \in v(p_v) \cap v(-p_u)$, therefore $\neg u'Sv'$. □

Theorem 7.44 (The lattice $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$).
The lattice $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL} \setminus \{\mathcal{TRIV}\}$ is dually isomorphic via the mapping $\mathcal{L}_{exp} \mapsto \boldsymbol{\mu}_+^{-1}[\operatorname{Mod} \mathcal{L}_{exp}]$ to the lattice of classes K of non-empty graphs closed under homomorphisms such that $K_2 \in \mathsf{K} \iff \bullet \in \mathsf{K}$. The corresponding to K is axiomatized by the rules $\alpha_G \vdash \emptyset$ for $G \notin \mathsf{K}$ with non-empty $\overline{G}$.

Proof. Firstly, observe that each non-trivial explosive extension of $\mathcal{ETL}$ is complete with respect to a (possibly empty) class of matrices of the form $\boldsymbol{\mu}_+(G)$ for non-empty G extended by $\mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$: each explosive extension $\mathcal{L}_{\mathrm{exp}}$ of $\mathcal{ETL}$ is complete with respect to a class of matrices of the form $\boldsymbol{\mu}_+(G) \times \mathbb{B}_{\mathbf{2}}^k$ by Finite reduced models of $\mathcal{ETL}$ (Proposition 7.18). If $\mathcal{L}_{\mathrm{exp}}$ is non-trivial, then $\mathcal{L}_{\mathrm{exp}} \leq \mathcal{ETL}_\omega = \operatorname{Log} \mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$, but all matrices of the form $\boldsymbol{\mu}_+(G) \times \mathbb{B}_{\mathbf{2}}^k$ for $k \geq 1$ are models of $\mathcal{ETL}_\omega$. The trivial logic is then complete with respect to the matrix $\boldsymbol{\mu}_+(\emptyset)$.

Consider a non-trivial logic $\mathcal{L}_{\mathrm{exp}} \in \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$ and a non-empty graph G with $\boldsymbol{\mu}_+(G) \in \operatorname{Mod} \mathcal{L}_{\mathrm{exp}}$. If there is a graph homomorphism $G \to H$, then there is a morphism of $\mathcal{BD}$-frames $\mathcal{F}_+(G) \to \mathcal{F}_+(H)$, namely the unique morphism of $\mathcal{BD}$-frames extending the map $G \to H$, and therefore by Duality for De Morgan matrices (Theorem 7.10) a homomorphism of De Morgan matrices $\boldsymbol{\mu}_+(H) \to \boldsymbol{\mu}_+(G)$. Since $\operatorname{Mod} \mathcal{L}_{\mathrm{exp}}$ is closed under homomorphic pre-images by Models of explosive extensions (Fact 4.7), it follows that $\boldsymbol{\mu}_+(H) \in \operatorname{Mod} \mathcal{L}_{\mathrm{exp}}$.

Conversely, consider a class of non-empty graphs K closed under homomorphisms such that $K_2 \notin \mathsf{K}$. Let $\mathcal{L}_{\mathrm{exp}}$ be the extension of $\mathcal{ETL}$ by the explosive rules $\alpha_G \vdash \emptyset$ for $G \notin \mathsf{K}$ with non-empty $\overline{G}$. Then by the Graph Homomorphism Lemma (Lemma 7.43) $\boldsymbol{\mu}_+(H) \in \operatorname{Mod} \mathcal{L}_{\mathrm{exp}}$ if and only if there is no graph homomorphism $H \to G$ for $G \notin \mathsf{K}$ with non-empty $\overline{G}$. But there is a graph homomorphism $H \to G$ for some $G \notin \mathsf{K}$ with non-empty $\overline{G}$ if and only if there is a graph homomorphism $H \to G$ for some $G \notin \mathsf{K}$: if $\overline{G} = \emptyset$, then $\overline{H} = \emptyset$, hence there is a homomorphism $H \to K_2$ and by assumption $K_2 \notin \mathsf{K}$. Because K is closed under homomorphisms, there is a graph homomorphism $H \to G$ for some $G \notin \mathsf{K}$ if and only if $H \notin \mathsf{K}$. Thus $\boldsymbol{\mu}_+(H) \in \operatorname{Mod} \mathcal{L}_{\mathrm{exp}}$ if and only if $H \in \mathsf{K}$ for each non-empty graph H.

Finally, consider a class of non-empty graphs K closed under homomorphisms satisfying the assumption $K_2 \in \mathsf{K} \iff \bullet \in \mathsf{K}$ such that $K_2 \in \mathsf{K}$.

Then $\bullet \in \mathsf{K}$ and K contains every non-empty graph. In other words, K is precisely the class of all non-empty graphs G such that $\boldsymbol{\mu}_+(G)$ is a model of $\mathcal{ETL}$. $\square$

Corollary 7.45. *The lattice* $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{BD} \setminus \{\mathcal{TRIV}\}$ *is dually isomorphic to the lattice of all classes of non-empty graphs closed under homomorphisms.*

Proof. This holds because $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$ and $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$ are isomorphic (Theorem 6.23) and $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{BD} = \{\mathcal{BD}\} \cup \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$ (Fact 6.22). $\square$

Corollary 7.46. *There is a countable increasing chain of explosive extensions of* $\mathcal{ETL}$ *between* $\mathcal{ETL}_2$ *and* $\mathcal{ETL}_3$.

Proof. By Proposition 7.25 it suffices to find in the homomorphism order a countable decreasing chain of finite 3-colourable graphs which are not 2-colourable. The cycles of length $2n+1$ for $n \geq 1$ form such a chain. $\square$

It also follows that each of the intervals $[\mathcal{BD}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $[\mathcal{ETL}, \mathcal{CL}]$ contains continuum many finitary logics.

Corollary 7.47. *The lattice* $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$, $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{BD}$, $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$ *and* $\mathcal{LP} \cap \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$ *have the cardinality of the continuum.*

Proof. The lattice $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$ has the cardinality of the continuum because by the countable universality of the homomorphism order (Theorem 7.2) there is a countable antichain in the homomorphism pre-order on finite graphs. The other lattices have the same cardinality because the lattices $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$, $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$, and $\mathcal{LP} \cap \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}$ are isomorphic (Theorems 6.23 and 6.25). $\square$

Moreover, almost every finitary explosive rule has the form $\alpha_G \vdash \emptyset$ up to equivalence in $\mathcal{BD}$.

Corollary 7.48. *Each finitary explosive rule* $\Gamma \vdash \emptyset$ *is equivalent in* $\mathcal{BD}$ *to either* $\mathsf{f} \vdash \emptyset$ *or* $\emptyset \vdash \emptyset$ *or some rule of the form* $\alpha_G \vdash \emptyset$.

Proof. Suppose that a finitary explosive rule $\Gamma \vdash \emptyset$ axiomatizes a proper non-trivial extension of $\mathcal{BD}$, i.e. that it is equivalent to neither $p \vdash p$ nor to $\emptyset \vdash \emptyset$. To show that it is equivalent to some rule of the form $\alpha_G \vdash \emptyset$, it suffices to show that it is valid in the same finite reduced models of $\mathcal{BD}$, i.e. in the same matrices of the form $\boldsymbol{\mu}_+(G) \times \boldsymbol{\mu}_-(H) \times \mathbb{B}_\mathbf{2}^k$ by Proposition 7.17. But by Corollary 4.26 such a matrix is a model of an explosive rule if and only if $\boldsymbol{\mu}_+(G)$ and $\boldsymbol{\mu}_-(H)$ and $\mathbb{B}_\mathbf{2}$ are models of the rule. Since $\boldsymbol{\mu}_-(H)$ is not a model of any proper explosive extension of $\mathcal{BD}$ (by virtue of not being a model of $\mathcal{ECQ}$) and $\boldsymbol{\mu}_+(G)$ and $\mathbb{B}_\mathbf{2}$ are models of $\mathcal{ETL}$, it suffices to show that $\Gamma \vdash \emptyset$ is equivalent in $\mathcal{ETL}$ to some rule of the form $\alpha_G \vdash \emptyset$. But by the above description of the lattice $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$ (Theorem 7.44) the rule

$\Gamma \vdash \emptyset$ is equivalent to a set of rules of the form $\alpha_G \vdash \emptyset$. Moreover, since joins of explosive extensions are just unions (Proposition 4.16), the rule $\Gamma \vdash \emptyset$ is in fact equivalent in $\mathcal{ETL}$ to one such rule. $\square$

We can infer, using the countable universality of the homomorphism order on graphs, that there is a non-finitary super-Belnap logic.

Theorem 7.49 (Existence of non-finitary super-Belnap logics). *There is a non-finitary explosive extension of $\mathcal{ETL}$.*

Proof. Given a graph G with non-empty $\overline{G}$, let $\mathcal{L}_G$ be the extension of $\mathcal{ETL}$ by $\alpha_G \vdash \emptyset$. Given a countable set of such graphs K, let $\mathcal{L}_{\mathsf{K}} := \bigcap_{G \in \mathsf{K}} \mathcal{L}_G$. Then by Proposition 4.23 we have $\boldsymbol{\mu}_+(H) \in \operatorname{Mod} \mathcal{L}_{\mathsf{K}}$ if and only if $\boldsymbol{\mu}_+(H) \in \operatorname{Mod} \mathcal{L}_G$ for some G, i.e. if and only if H does not lie below all the graphs $G \in \mathsf{K}$ in the homomorphism order by the Graph Homomorphism Lemma (Lemma 7.43). By Proposition 4.21 the logic $\mathcal{L}_{\mathsf{K}}$ is axiomatized by the rule $\{\alpha_G \mid G \in \mathsf{K}\} \vdash \emptyset$, provided we use distinct variables in each of the formula α_G. If the logic $\mathcal{L}_{\mathsf{K}}$ is finitary, then this rule is equivalent to the rule $\{\alpha_G \mid G \in \mathsf{K}'\} \vdash \emptyset$ for some finite $\mathsf{K}' \subseteq \mathsf{K}$. But then $\mathcal{L}_{\mathsf{K}} = \mathcal{L}_{\mathsf{K}'}$, thus by the above reasoning a graph H lies below all graphs $G \in \mathsf{K}$ in the homomorphism order if and only if it lies below all graphs $G \in \mathsf{K}'$.

To find a counter-example to this condition, it now suffices to find in the homomorphism order on graphs a countable set K such that for each finite $\mathsf{K}' \subseteq \mathsf{K}$ there is some H which lies below all $G \in \mathsf{K}'$ but not below all $G \in \mathsf{K}$. This is a purely order-theoretic property, which is satisfied for example by the set of generators of a countably generated meet semilattice. But by the countable universality of the homomorphism order (Theorem 7.2) this semilattice embeds as a partial order into the homomorphism order on graphs. $\square$

We can also use Erdős's Girth–Chromatic Number Theorem to infer the following fact about the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ for $n \geq 2$.

Theorem 7.50.
The logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ are not complete with respect to any finite set of finite matrices for $n \geq 2$.

Proof. Suppose that $\mathcal{ECQ}_n$ is complete with respect to a finite set of finite reduced matrices K. We show that $\mathcal{ETL}_n$ is then complete with respect to a finite set of finite matrices as well. We know that $\mathcal{ETL}_n = \mathcal{ETL} \cup \mathcal{ECQ}_n$ by Proposition 4.28, therefore $\Gamma \nvdash_{\mathcal{ETL}_n} \varphi$ if and only if $\Gamma \nvdash_{\mathcal{ETL}} \varphi$ and $\Gamma \nvdash_{\mathcal{ECQ}_n} \varphi$, i.e. if and only if $\Gamma \nvdash_{\mathcal{ETL}} \varphi$ and $\Gamma \nvdash_{\mathcal{ECQ}_n} \emptyset$ by Proposition 4.6. But $\Gamma \nvdash_{\mathcal{ECQ}_n} \emptyset$ if and only if there is a valuation on some $\boldsymbol{\mu}(G, H, k) = \boldsymbol{\mu}_+(G) \times \boldsymbol{\mu}_-(H) \times \mathbb{B}_{\mathbf{2}}^k \in \mathsf{K}$ which designates Γ. Since $\boldsymbol{\mu}_-(H)$ is not even a model of $\mathcal{ECQ}$ if H is non-empty by Proposition 7.19, the rule $\Gamma \vdash \emptyset$ fails in $\boldsymbol{\mu}(G, H, k)$ if and only if it fails in $\boldsymbol{\mu}(G, \emptyset, k)$. It follows that $\Gamma \vdash_{\mathcal{ETL}_n} \varphi$

if and only if either $\Gamma \vdash_{\mathcal{ETL}} \varphi$ or the rule $\Gamma \vdash \emptyset$ fails in some matrix in $\mathsf{L} := \{\boldsymbol{\mu}(G, \emptyset, k) \mid \boldsymbol{\mu}(G, H, k) \in \mathsf{K}$ for some $H\}$. Thus $\mathcal{ETL}_n$ is complete with respect to the finite set of finite matrices $\{\mathbb{ETL}_{\mathbf{4}} \times \mathbb{M} \mid \mathbb{M} \in \mathsf{L}\}$.

It remains to show that $\mathcal{ETL}_n$ is not complete with respect to any finite set of finite matrices. In fact we prove a stronger claim, namely that there is no set of finite matrices K such that $\Gamma \vdash_{\mathcal{ETL}_n} \emptyset$ if and only if $\Gamma \vdash \emptyset$ in $\operatorname{Log} \mathsf{K}$ for each finite set Γ. Because $\operatorname{Log} \mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}} = \mathcal{ETL}_\omega \leq \operatorname{Log} \mathbb{M} \times \mathbb{B}_{\mathbf{2}}$ for each model $\mathbb{M}$ of $\mathcal{ETL}$, we may assume without loss of generality that K consists of $\mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$ and a finite set of matrices of the form $\boldsymbol{\mu}_+(G)$. But then by Theorem 7.44 and Proposition 7.25 it follows that for each graph G which is not n-colourable there is a surjective graph homomorphism $H \twoheadrightarrow G$ for some H with $\boldsymbol{\mu}_+(H) \in \mathsf{K}$. However, this contradicts the Girth–Chromatic Number Theorem (Theorem 7.1): a finite set of graphs has a bounded girth and the girth of a surjective homomorphic image of H is at most the girth of H. $\square$

Finally, we draw some corollaries regarding antivarieties of De Morgan algebras, where an antivariety of De Morgan algebras is a class of De Morgan algebras axiomatized relative to DMA by a set of negative universal clauses, i.e. disjunctions of negated equations.

It was proved by Adams and Dziobiak [1] that there are continuum many quasivarieties of Kleene algebras. (By contrast, the lattice of quasivarieties of De Morgan lattices was proved to be finite by Pynko [62].) We now complement the result of Adams and Dziobiak by showing that there are continuum many antivarieties of De Morgan algebras. On the other hand, there are only two non-trivial proper antivarieties of Kleene algebras.

Proposition 7.51.
The lattice of classes of De Morgan algebras axiomatized by negative clauses of the form $\mathsf{t} \not\approx t$ for some term t is dually isomorphic to $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ETL}$.

Proof. If $\mathcal{L}_{\mathrm{exp}} \in \operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ETL}$ is axiomatized by the finitary rules $\Gamma_i \vdash \emptyset$ for $i \in I$, then the class of all models of $\mathcal{L}_{\mathrm{exp}}$ of the form $\langle \mathbf{A}, \{\mathsf{t}\} \rangle$ with $\mathbf{A} \in \mathsf{DMA}$ is precisely the class of De Morgan algebras axiomatized by the negative clauses $\mathsf{t} \not\approx \bigwedge \Gamma_i$ for $i \in I$. Conversely, given a class of De Morgan algebras K axiomatized by the negative clauses $\mathsf{t} \not\approx t_i$ for $i \in I$, let $\mathcal{L}_{\mathrm{exp}}$ be the extension of $\mathcal{ETL}$ by the rule $t_i \vdash \emptyset$. Then the matrix $\langle \mathbf{A}, \{\mathsf{t}\} \rangle$ with $\mathbf{A} \in \mathsf{DMA}$ is a model of $\mathcal{L}_{\mathrm{exp}}$ if and only if $\mathbf{A} \in \mathsf{K}$.

The lattice of antivarieties of De Morgan algebras axiomatized by sets of negative clauses of the form $\mathsf{t} \not\approx t$ is thus isomorphic to the lattice of classes of De Morgan matrices of the form $\langle \mathbf{A}, \{\mathsf{t}\} \rangle$ axiomatized by sets of finitary explosive rules. But each extension of $\mathcal{ETL}$ is complete with respect to a class of such matrices, therefore this lattice is dually isomorphic to $\operatorname{Exp} \operatorname{Ext}_\omega \mathcal{ETL}$. $\square$

Corollary 7.52 (Continuum many antivarieties of DMAs).
There are continuum many antivarieties of De Morgan algebras.

Proposition 7.53 (Antivarieties of Kleene algebras).
There are exactly two non-trivial proper antivarieties of Kleene algebras, namely those axiomatized by $\mathsf{t} \not\approx \mathsf{f}$ *and by* $x \not\approx -x$*.*

Proof. Let K be an antivariety of Kleene algebras, i.e. the intersection of an antivariety and the variety KA. Recall that antivarieties are closed under homomorphic pre-images, subalgebras, and products.

If K contains the trivial Kleene algebra, then $\mathsf{K} = \mathsf{KA}$. Otherwise, the antivariety K satisfies the antiequation $\mathsf{t} \not\approx \mathsf{f}$. If K contains $\mathbf{K_3}$, then K contains all non-trivial Kleene algebras, since $\mathsf{KA} = \mathbb{SP}(\mathbf{K_3})$. Otherwise, the antiequation $x \not\approx -x$ holds in K: if there is some $a \in \mathbf{A} \in \mathsf{K}$ with $a = -a$, then $\mathbf{K_3} \leq \mathbf{A}$. But $\mathbf{B_2} \in \mathsf{K}$ whenever K is non-empty, because $\mathbf{B_2}$ is a subalgebra of each Kleene algebra. Because antivarieties are closed under homomorphic pre-images, $\mathbf{K_3} \times \mathbf{B_2} \in \mathsf{K}$. But it was proved by Pynko [62] that $\mathbf{K_3} \times \mathbf{B_2}$ generates the antivariety of Kleene algebras axiomatized by $x \not\approx x$ as a quasivariety. (More precisely, Pynko proved this for the corresponding antivariety Kleene lattices, but his proof works equally well for Kleene algebras, as observed already by Gaitán and Perea [28].) $\square$

Chapter 8

Metalogical properties of super-Belnap logics

We now use our understanding of the structure of $\operatorname{Ext}\mathcal{BD}$ to fully describe which super-Belnap logics satisfy certain metalogical properties. We consider the classification of super-Belnap logics within the Leibniz and Frege hierarchies of abstract algebraic logic and several other properties such as contraposition and the proof by cases property. The algebraic counterparts and strong versions of super-Belnap logics will also be studied. The interpolation property is not considered in the current chapter. Its discussion is postponed until Section 9.5 (Interpolation in super-Belnap logics), after we have sufficiently developed the proof theory of super-Belnap logics.

Note that in the current chapter we exclude the trivial logic from consideration in order to simplify the formulation of our results.

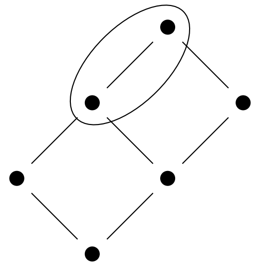
8.1 The Leibniz and Frege hierarchies

The reader unfamiliar with the Leibniz and Frege hierarchies of abstract algebraic logic should at this point consult the preliminaries (Chapter 1). In the proofs below we shall use the fact that protoalgebraicity, truth-equationality, and assertionality are preserved by extensions. We shall also make free use of the splittings of $\operatorname{Ext}\mathcal{BD}$ established in Chapter 6 (The lattice of super-Belnap logics).

The most salient point about the classification of super-Belnap logics in the Leibniz hierarchy is that apart from classical logic, none of them are protoalgebraic. The strong link which connects logic and algebra in the realm of super-intuitionistic logics or normal modal logics is therefore not available for super-Belnap logics.

Theorem 8.1 (Protoalgebraic super-Belnap logics).
The only protoalgebraic super-Belnap logic is $\mathcal{CL}$.

Figure 8.1: The matrix $\mathbb{P}_3 \times \mathbb{B}_2$



Proof. If $\mathcal{LP} \vee \mathcal{ETL} = \mathcal{CL} \nleq \mathcal{L}$, then either $\mathcal{LP} \nleq \mathcal{L}$ or $\mathcal{ETL} \nleq \mathcal{L}$, hence either $\mathcal{L} \leq \mathcal{K}$ or $\mathcal{L} \leq \mathcal{LP} \vee \mathcal{ECQ}$. It therefore suffices to show that $\mathcal{K}$ and $\mathcal{LP} \vee \mathcal{ECQ}$ are not protoalgebraic, that is, to find an algebra $\mathbf{A}$ and $\mathcal{K}$-filters $F \subseteq G \subseteq \mathbf{A}$ such that $\mathbf{\Omega^A} F \nsubseteq \mathbf{\Omega^A} F$, and likewise for $\mathcal{LP} \vee \mathcal{ECQ}$.

In the case of $\mathcal{K}$, consider the five-element De Morgan chain $\mathbf{K_5}$ with $\mathsf{f} < \mathsf{a} < \mathsf{b} < \mathsf{c} < \mathsf{t}$, and let $F := \{\mathsf{t}\}$ and $G := \{\mathsf{c}\}$. In the case of $\mathcal{LP} \vee \mathcal{ECQ}$, consider the four-element De Morgan chain $\mathbf{K_4}$ with $\mathsf{f} < \mathsf{a} < \mathsf{b} < \mathsf{t}$, and let $F := \{\mathsf{b}, \mathsf{t}\} \times \{\mathsf{t}\}$ and $G := \{\mathsf{a}, \mathsf{b}, \mathsf{t}\} \times \{\mathsf{t}\}$. $\square$

Recall that protoalgebraicity is equivalent to the existence of a so-called protoimplication set, i.e. a set of formulas in two variables $\Delta(p, q)$ which satisfies both $\emptyset \vdash \Delta(p, p)$ (Reflexivity) and $p, \Delta(p, q) \vdash q$ (Modus Ponens). Protoalgebraicity is also equivalent to a certain weak form of the deduction theorem. We have therefore shown that no non-classical super-Belnap logic enjoys the deduction theorem (for any connective) or contains a connective which satisfies both Reflexivity and Modus Ponens.

Theorem 8.2 (Truth-equational super-Belnap logics).
A super-Belnap logic $\mathcal{L}$ is truth-equational if and only if either $\mathcal{ETL} \leq \mathcal{L}$ (with respect to $\boldsymbol{\tau}(x) := x \approx \mathsf{t}$) or $\mathcal{LP} \leq \mathcal{L}$ (with respect to $x \vee -x \approx x$).

Proof. These logics were already proved to be truth-equational when we described the reduced models of $\mathcal{LP}$ and $\mathcal{ETL}$ in Proposition 3.20. Conversely, if $\mathcal{ETL} \nleq \mathcal{L}$ and $\mathcal{LP} \nleq \mathcal{L}$, then $\mathcal{L} \leq (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{K} = \mathcal{KO} \vee \mathcal{ECQ}$ by Theorem 6.15. It suffices to show that $\mathcal{KO} \vee \mathcal{ECQ}$ is not truth-equational.

To do so, consider the algebra $\mathbf{K_3} \times \mathbf{B_2}$. Then $\{\langle \mathsf{i}, \mathsf{t} \rangle, \langle \mathsf{t}, \mathsf{t} \rangle\}$ and $\{\langle \mathsf{t}, \mathsf{t} \rangle\}$ are both filters of $\mathcal{KO} \vee \mathcal{ECQ}$ on this algebra, and their Leibniz congruences are the same (namely, the identity congruence). In other words, the Leibniz operator fails to be injective on the filters of $\mathcal{KO} \vee \mathcal{ECQ}$ on $\mathbf{K_3} \times \mathbf{B_2}$. $\square$

In fact, what we have shown is that the Leibniz operator is not even injective on the filters of $\mathcal{KO} \vee \mathcal{ECQ}$, therefore a super-Belnap logic $\mathcal{L}$ is truth-equational if and only if truth is implicitly definable in $\mathrm{Mod}^* \mathcal{L}$ in the terminology of Moraschini [49].

Theorem 8.3 (Assertional super-Belnap logics).
A super-Belnap logic $\mathcal{L}$ is assertional if and only if $\mathcal{ETL} \leq \mathcal{L}$.

Proof. Each assertional logic $\mathcal{L}$ is truth-equational, thus either $\mathcal{ETL} \leq \mathcal{L}$ or $\mathcal{LP} \leq \mathcal{L}$. If $\mathcal{LP} \leq \mathcal{L}$ but $\mathcal{ETL} \nleq \mathcal{L}$, then $\mathcal{L} \leq \mathcal{LP} \vee \mathcal{ECQ}$ (Proposition 6.9). It thus suffices to show that $\mathcal{LP} \vee \mathcal{ECQ} = \operatorname{Log} \mathbb{P}_{\mathbf{3}} \times \mathbb{B}_{\mathbf{2}}$ is not assertional. But we have $p, q, -p \vee r \nvdash_{\mathcal{LP} \vee \mathcal{ECQ}} -q \vee r$, as witnessed by $v : \mathbf{Fm} \to \mathbb{P}_{\mathbf{3}} \times \mathbb{B}_{\mathbf{2}}$ such that $v(p) := \langle \mathsf{b}, \mathsf{t} \rangle$ and $v(q) := \langle \mathsf{t}, \mathsf{t} \rangle$ and $v(r) := \langle \mathsf{f}, \mathsf{t} \rangle$ (see Figure 8.1), contrary to the fact that $p, q, \varphi(p) \vdash \varphi(q)$ is valid in each assertional logic. □

Turning now to the Frege hierarchy, the general theory of selfextensional logics with conjunction due to Jansana [36] applied to super-Belnap logics implies that finitary selfextensional super-Belnap logics correspond precisely to varieties of De Morgan algebras. Because there are only three of those, we can infer that $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$ are the only three *finitary* selfextensional super-Belnap logics, as observed already by Rivieccio [67].

We now improve on this general argument and show directly that there are only three selfextensional super-Belnap logics.

Proposition 8.4 (Selfextensional super-Belnap logics).
The only selfextensional super-Belnap logics are $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$.

Proof. The logics $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$ are selfextensional by virtue of their connection to varieties of De Morgan algebras (Theorem 3.2). Conversely, let $\mathcal{L}$ be a proper selfextensional extension of $\mathcal{BD}$. Then $\mathcal{LP} \cap \mathcal{ECQ} \leq \mathcal{L}$, i.e. $p \wedge -p \vdash_{\mathcal{L}} q \vee -q$, therefore

$$p \wedge -p \dashv\vdash_{\mathcal{L}} (p \wedge -p) \wedge (q \vee -q).$$

Selfextensionality yields that

$$(p \wedge -p) \vee r \dashv\vdash_{\mathcal{L}} ((p \wedge -p) \wedge (q \vee -q)) \vee r,$$

therefore $\mathcal{L}$ is an extension of $\mathcal{KO}$.

Likewise, if $\mathcal{ECQ} \leq \mathcal{L}$, then $p \wedge -p \vdash_{\mathcal{L}} \emptyset$, hence

$$p \wedge -p \dashv\vdash_{\mathcal{L}} p \wedge -p \wedge q.$$

Selfextensionality yields that

$$(p \wedge -p) \vee q \dashv\vdash_{\mathcal{L}} (p \wedge -p \wedge q) \vee q,$$

therefore $(p \wedge -p) \vee q \vdash_{\mathcal{L}} q$ and $\mathcal{L}$ is an extension of $\mathcal{K}$. Using the list of extensions of $\mathcal{KO}$ (Theorem 6.15) it follows that $\mathcal{L} \in \{\mathcal{KO}, \mathcal{LP}, \mathcal{K}, \mathcal{CL}\}$.

It now suffices to prove that neither $\mathcal{LP}$ nor $\mathcal{K}$ is selfextensional. $\mathcal{LP}$ is not selfextensional because

$$q \dashv\vdash_{\mathcal{LP}} (p \vee -p) \wedge q \qquad \text{but} \qquad -((p \vee -p) \wedge q) \nvdash_{\mathcal{LP}} -q.$$

$\mathcal{K}$ is not selfextensional because

$$q \dashv\vdash_{\mathcal{K}} (p \wedge -p) \vee q \qquad \text{but} \qquad -q \nvdash_{\mathcal{K}} -((p \wedge -p) \vee q)\,.$$ □

The reader familiar with the Frege hierarchy will immediately observe that $\mathcal{BD}$, $\mathcal{KO}$, and $\mathcal{CL}$ are in fact fully self-extensional.

The following example showing that $\mathcal{BD}$ (and by the same token $\mathcal{KO}$) is not selfextensional is due to Font [25, Thm 2.11].

Proposition 8.5 (Fregean Super-Belnap Logics).
The only Fregean super-Belnap logic is $\mathcal{CL}$.

Proof. It suffices to show that the logics $\mathcal{BD}$ and $\mathcal{KO}$ are not Fregean. But $p, q \vdash_{\mathcal{BD}} p \wedge q$ and $p, p \wedge q \vdash_{\mathcal{BD}} q$ while $p, -(p \wedge q) \nvdash_{\mathcal{KO}} -q$. □

8.2 Algebraic counterparts and strong versions

We now characterize precisely those super-Belnap logics $\mathcal{L}$ whose algebraic counterpart $\mathrm{Alg}\,\mathcal{L}$ is a quasivariety. It turns out that $\mathrm{Alg}\,\mathcal{L}$ is either one of the varieties DMA, KA, BA or it is not even a universal class.

In the following theorem, the quasivariety of *non-idempotent* Kleene algebras, i.e. those Kleene algebras which contain no fixpoint of De Morgan negation, will be denoted NIKA, following Pynko [62]. In other words, NIKA is the quasivariety of Kleene algebras axiomatized by $x \approx -x \implies x \approx y$.

Theorem 8.6 (Algebraic counterparts of super-Belnap logics).
Exactly one of the following holds for each super-Belnap logic $\mathcal{L}$:

(i) $\mathcal{L} \in [\mathcal{BD}, \mathcal{ETL}]$ *and* $\mathrm{Alg}\,\mathcal{L} = \mathsf{DMA}$*,*

(ii) $\mathcal{L} \in [\mathcal{KO}, \mathcal{K}]$ *and* $\mathrm{Alg}\,\mathcal{L} = \mathsf{KA}$*,*

(iii) $\mathcal{L} = \mathcal{LP}$ *and* $\mathrm{Alg}\,\mathcal{L} = \mathsf{KA}$*,*

(iv) $\mathcal{L} = \mathcal{LP} \vee \mathcal{ECQ}$ *and* $\mathrm{Alg}\,\mathcal{L} \subsetneq \mathsf{NIKA}$ *is not closed under subalgebras,*

(v) $\mathcal{L} = \mathcal{CL}$ *and* $\mathrm{Alg}\,\mathcal{L} = \mathsf{BA}$*,*

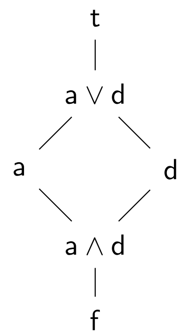
(vi) $\mathcal{L} \in [\mathcal{LP} \cap \mathcal{ECQ}_2, \mathcal{K}_-]$ *and* $\mathrm{Alg}\,\mathcal{L} \supsetneq \mathsf{KA}$ *is not closed under subalgebras.*

Proof. $\mathcal{L}$ belongs to exactly one of the listed sets by the results of Chapter 6, namely Theorem 6.8 and Proposition 6.9. It thus suffices to prove that $\mathcal{L} \in [\mathcal{BD}, \mathcal{ETL}]$ if and only if $\mathrm{Alg}\,\mathcal{L} = \mathsf{DMA}$ etc.

(i) We have $\mathbb{B}_\mathbf{2}, \mathbb{K}_\mathbf{3}, \mathbb{ETL}_\mathbf{4} \in \mathrm{Mod}^*\,\mathcal{ETL}$, thus $\mathbf{B_2}, \mathbf{K_3}, \mathbf{DM_4} \in \mathrm{Alg}\,\mathcal{BD}$. But $\mathrm{Alg}\,\mathcal{ETL}$ is closed under subdirect products and each De Morgan algebra is a subdirect product of these three (Theorem 2.7), therefore $\mathsf{DMA} \subseteq \mathrm{Alg}\,\mathcal{ETL} \subseteq \mathrm{Alg}\,\mathcal{L} \subseteq \mathrm{Alg}\,\mathcal{BD}$. Conversely, each algebra in $\mathrm{Alg}\,\mathcal{BD}$ is a subdirect product of algebras in $\mathrm{Alg}^*\,\mathcal{BD}$, which we know to be De Morgan algebras (Proposition 3.18). Thus $\mathrm{Alg}\,\mathcal{BD} \subseteq \mathsf{DMA}$.

(ii, iii) We have $\mathbb{B}_\mathbf{2}, \mathbb{P}_\mathbf{3} \in \mathrm{Mod}^*\,\mathcal{LP}$ and $\mathbb{B}_\mathbf{2}, \mathbb{K}_\mathbf{3} \in \mathrm{Mod}^*\,\mathcal{K}$, therefore $\mathbf{B_2}, \mathbf{K_3} \in \mathrm{Alg}\,\mathcal{L}$ for $\mathcal{L} = \mathcal{LP}$ as well as for $\mathcal{L} \in [\mathcal{KO}, \mathcal{K}]$. But $\mathrm{Alg}\,\mathcal{L}$ is closed

Figure 8.2: The algebra $\mathbf{DM_6}$



under subdirect products and each Kleene algebra is a subdirect product of $\mathbf{B_2}$ and $\mathbf{K_3}$ (Theorem 2.7), therefore $\mathsf{KA} \subseteq \operatorname{Alg} \mathcal{L}$ for each of these logics $\mathcal{L}$. Conversely, for $\mathcal{L} = \mathcal{LP}$ as well as for $\mathcal{L} \in [\mathcal{KO}, \mathcal{K}]$ we have $\operatorname{Alg} \mathcal{L} \subseteq \operatorname{Alg} \mathcal{KO}$, and each algebra in $\operatorname{Alg} \mathcal{KO}$ is a subdirect product of algebras in $\operatorname{Alg}^* \mathcal{L}$, which are Kleene algebras (Proposition 3.20).

(iv) The reduced models of $\mathcal{LP}$ are precisely the matrices $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a Kleene algebra and $F = \{a \in \mathbf{A} \mid -a \leq a\}$ (Proposition 3.20). But such a matrix is a model of $\mathcal{ECQ}$ if and only if there is no $a \in \mathbf{A}$ such that $a = -a$. Each algebra in $\operatorname{Alg} \mathcal{LP} \vee \mathcal{ECQ}$ is a subdirect product of such algebras, in particular it is a non-idempotent Kleene algebra.

To show that $\operatorname{Alg} \mathcal{LP} \vee \mathcal{ECQ}$ is not closed under subalgebras, let $\mathbf{K_4}$ be the four-element De Morgan chain $\mathsf{f} < \mathsf{a} < \mathsf{b} < \mathsf{t}$. Then $\mathbf{K_4} \notin \operatorname{Alg} \mathcal{LP} \vee \mathcal{ECQ}$, since $\{\mathsf{b}, \mathsf{t}\}$ is the only non-trivial filter of $\mathcal{LP} \vee \mathcal{ECQ}$ on $\mathbf{K_4}$. But $\mathbf{K_4} \leq \mathbf{K_3} \times \mathbf{B_2}$ and $\mathbf{K_3} \times \mathbf{B_2}$ is the algebraic reduct of $\mathbb{P}_\mathbf{3} \times \mathbb{B}_\mathbf{2} \in \operatorname{Mod}^* \mathcal{LP} \vee \mathcal{ECQ}$.

(v) We have $\mathbb{B}_\mathbf{2} \in \mathcal{CL}$, therefore $\mathbf{B_2} \in \operatorname{Alg} \mathcal{CL}$. But each Boolean algebra is a subdirect power of $\mathbf{B_2}$, therefore $\mathsf{BA} \subseteq \operatorname{Alg} \mathcal{CL}$. Conversely, each $\mathbf{A} \in \mathcal{CL}$ is a subdirect product of algebras in $\operatorname{Alg}^* \mathcal{CL}$, which are Boolean algebras (Proposition 3.20).

(vi) The completenes theorem for $\mathcal{K}_-$ (Proposition 5.17) states that $\mathcal{K}_- = \operatorname{Log} \langle \mathbf{DM_8}, \{\mathsf{t}\} \rangle$, where the algebra $\mathbf{DM_8}$ is shown in Figure 5.1. Let $\mathbf{DM_6}$ be the subalgebra of $\mathbf{DM_8}$ obtained by removing the elements b and c shown in Figure 8.2. If $\mathcal{L} \leq \mathcal{K}_-$, then $\mathbf{DM_8} \in \operatorname{Alg} \mathcal{K}_- \subseteq \operatorname{Alg} \mathcal{L}$. However, $\mathbf{DM_6} \notin \operatorname{Alg} \mathcal{LP} \cap \mathcal{ECQ}_2 \supseteq \operatorname{Alg} \mathcal{L}$ if $\mathcal{LP} \cap \mathcal{ECQ}_2 \leq \mathcal{L}$, since the only non-trivial filters of $\mathcal{LP} \cap \mathcal{ECQ}_2$ on $\mathbf{DM_6}$ are $\{\mathsf{t}\}$ and $[\mathsf{a} \wedge \mathsf{b}, \mathsf{t}]$, and $\langle \mathsf{a} \wedge \mathsf{d}, \mathsf{a} \vee \mathsf{d} \rangle \in \mathbf{\Omega}^{\mathbf{DM_6}} \{\mathsf{t}\} = \mathbf{\Omega}^{\mathbf{DM_6}} [\mathsf{a} \wedge \mathsf{d}, \mathsf{t}]$.

Moreover, if $\mathcal{L} \in [\mathcal{LP} \cap \mathcal{ECQ}_2, \mathcal{K}_-]$, then $\operatorname{Alg} \mathcal{L} \supseteq \operatorname{Alg} \mathcal{K} = \mathsf{KA}$. $\square$

Let us now also describe the strong versions of super-Belnap logics in the intervals $[\mathcal{BD}, \mathcal{ETL}]$ and $[\mathcal{KO}_-, \mathcal{CL}]$. Readers unfamiliar with this notion may skip the following lemma and theorem without any loss of continuity.

Recall the definition of Leibniz filters and finitizable Leibniz congruences from the preliminaries (Section 1.2), and observe that $\mathcal{BD}$ has finitizable Leibniz congruences by Proposition 3.17.

Lemma 8.7.
Let $\mathcal{L}$ be a finitary logic with finitizable Leibniz congruences and suppose that there is a first-order formula with equality in one free variable $\varphi(x)$ such that for each finitely generated model $\langle \mathbf{A}, F \rangle$ of $\mathcal{L}$ we have

$$F^*_{\mathcal{L}} = \{a \in \mathbf{A} \mid \langle \mathbf{A}, \boldsymbol{\Omega}^{\mathbf{A}} F \rangle \vDash \varphi(a)\}$$

where the equality sign in $\varphi(x)$ is interpreted by the Leibniz congruence $\boldsymbol{\Omega}^{\mathbf{A}} F$. Then for each finitary rule $\Gamma \vdash \varphi$ we have

$$\Gamma \vdash_{\mathcal{L}^+} \varphi \iff \Gamma \vDash_{\mathbb{M}} \varphi \textit{ for each finitely generated Leibniz model } \mathbb{M} \textit{ of } \mathcal{L}.$$

Proof. Let $\mathcal{L}'$ be the finitary logic determined by the Leibniz $\mathcal{L}$-filters on finitely generated algebras, or equivalently by the minimal $\mathcal{L}$-filters of finitely generated algebras. That is, a finitary rule holds in $\mathcal{L}'$ if and only if it holds in each minimal $\mathcal{L}$-filter of each finitely generated algebra.

Let F be a minimal filter of $\mathcal{L}$ on an algebra $\mathbf{A}$. The matrix $\langle \mathbf{A}, F \rangle$ embeds into an ultraproduct $\langle \mathbf{B}, G \rangle$ of its finitely generated submatrices. It now suffices to show that there is an $\mathcal{L}'$-filter $H \subseteq G$ on $\mathbf{B}$: it then follows that $\langle \mathbf{A}, H \cap \mathbf{A} \rangle$ is a model of $\mathcal{L}$ by the finitarity of $\mathcal{L}$, hence by minimality of F we have $F = H \cap \mathbf{A}$ and $\langle \mathbf{A}, F \rangle$ is thus a model of $\mathcal{L}'$.

Each finitely generated submatrix of $\langle \mathbf{A}, F \rangle$ is of course a model of $\mathcal{L}$. If $\varphi(x)$ is a first order-formula with equality in one variable, let $\varphi[\langle \mathbf{C}, I \rangle] := \{x \in \mathbf{C} \mid \langle \mathbf{C}, \boldsymbol{\Omega}^{\mathbf{C}} I \rangle \vDash \varphi(a)\}$, where the equality sign is interpreted by the congruence $\boldsymbol{\Omega}^{\mathbf{C}} I$. By assumption there is some $\varphi(x)$ such that $\varphi[\langle \mathbf{C}, I \rangle] = F^*_{\mathcal{L}}$ for each finitely generated model $\langle \mathbf{C}, I \rangle$ of $\mathcal{L}$. But the claim that $\varphi[\langle \mathbf{C}, I \rangle]$ is a model of each finitary rule of $\mathcal{L}^+$ and a subset of I is expressible by a set of first-order formulas, therefore the ultraproduct $\langle \mathbf{B}, G \rangle$ also satisfies this claim. That is, there is some $H \subseteq G$, namely $H := \varphi[\langle \mathbf{B}, G \rangle]$, such that H is a model of $\mathcal{L}'$. $\square$

Theorem 8.8 (Strong versions of super-Belnap logics).

(i) If $\mathcal{L} \in [\mathcal{BD}, \mathcal{ETL}]$, then $\mathcal{L}^+ = \mathcal{ETL}$,

(ii) If $\mathcal{L} \in [\mathcal{KO}, \mathcal{K}]$, then $\mathcal{L}^+ = \mathcal{K}$,

(iii) If $\mathcal{L} \in [\mathcal{LP}, \mathcal{CL}]$ or $\mathcal{L} \in [\mathcal{ETL}, \mathcal{CL}]$, then $\mathcal{L}^+ = \mathcal{L}$,

(iv) If $\mathcal{L} \in [\mathcal{KO}_-, \mathcal{K}_-]$, then $\mathcal{L}^+ = \mathcal{K}_-$.

(v) If $\mathcal{L} \in \operatorname{Exp} \operatorname{Ext} \mathcal{BD}$, then $\mathcal{L}^+ = \mathcal{L} \vee \mathcal{ETL}$.

Proof. $\mathcal{L} \in [\mathcal{B}, \mathcal{B}^+]$ implies $\mathcal{L}^+ = \mathcal{B}^+$, therefore it suffices to show that $\mathcal{BD}^+ = \mathcal{ETL}$ and $\mathcal{KO}^+ = \mathcal{K}$ in (i) and (ii).

(i) $\operatorname{Alg} \mathcal{BD} = \mathsf{DMA}$ (Theorem 8.6) and the smallest $\mathcal{BD}$-filter on a De Morgan algebra is the filter $\{\mathsf{t}\}$ (Proposition 3.16). But $\{\mathsf{t}\}$ is an $\mathcal{ETL}$-filter on each De Morgan algebra. Conversely, $\mathcal{ETL}$ is complete with respect to a single such matrix, namely $\mathbb{ETL}_{\mathbf{4}}$.

(ii) $\operatorname{Alg} \mathcal{KO} = \mathsf{KA}$ (Theorem 8.6) and the smallest $\mathcal{KO}$-filter on a Kleene algebra is the filter $\{\mathsf{t}\}$ (Proposition 3.16). But $\{\mathsf{t}\}$ is an $\mathcal{ETL}$-filter and therefore a $\mathcal{K}$-filter, since $\mathcal{K} = \mathcal{KO} \vee \mathcal{ETL}$. Conversely, $\mathcal{K}$ is complete with respect to a single such matrix, namely $\mathbb{K}_{\mathbf{3}}$.

(iii) These logics are truth-equational (Theorem 8.2) and $\mathcal{L}^+ = \mathcal{L}$ for each truth-equational logic $\mathcal{L}$.

(iv) The logic $\mathcal{K}_-$ is determined by the matrix $\mathbb{M}_{\mathbf{8}}$ (see Figure 5.1) whose designated filter is a minimal $\mathcal{BD}$-filter. It follows that $(\mathcal{K}_-)^+ = \mathcal{K}_-$ and moreover $(\mathcal{KO}_-)^+ \leq \mathcal{K}_-$. To prove (v) it thus suffices to show that $\mathcal{K}_- \leq (\mathcal{KO}_-)^+$. But $\mathcal{KO}_-$ is finitary and it has finitizable Leibniz congruences (Proposition 3.17). Moreover, $(\mathcal{KO}_-)^+$ is finitary, since each extension of $\mathcal{KO}_-$ is finitary (Proposition 6.15). By the previous lemma it suffices to show for each finitely generated model $\langle \mathbf{A}, F \rangle$ that $F^*_{\mathcal{KO}_-} = [\mathsf{t}]_{\mathbf{\Omega}^{\mathbf{A}} F}$, because then $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{ETL}$. In other words, it suffices to prove for each finitely generated reduced model $\langle \mathbf{A}, F \rangle$ of $\mathcal{KO}_-$, i.e. for each finite reduced model $\langle \mathbf{A}, F \rangle$ of $\mathcal{KO}_-$, that $\langle \mathbf{A}, \{\mathsf{t}\} \rangle$ is a model of $\mathcal{KO}_-$.

Observe that if $\mathbb{M} \times \mathbb{N}$ is a model of $\mathcal{KO}_-$, then so are $\mathbb{M}$ and $\mathbb{N}$. If $\mathbb{M}$ or $\mathbb{N}$ is trivial, this holds trivially. Otherwise $\mathcal{KO}_- \leq \operatorname{Log} \mathbb{M} \times \mathbb{N} = (\operatorname{Log} \mathbb{M} \cap \operatorname{Log} \mathbb{N}) \cup \operatorname{Exp}_{\mathcal{BD}} \operatorname{Log} \mathbb{M} \cup \operatorname{Exp}_{\mathcal{BD}} \operatorname{Log} \mathbb{N} \leq (\operatorname{Log} \mathbb{M} \cap \operatorname{Log} \mathbb{N}) \cup \mathcal{ECQ}_\omega$ implies $\mathcal{KO}_- \leq \operatorname{Log} \mathbb{M} \cap \operatorname{Log} \mathbb{N}$. This is because $\mathcal{KO}_-$ is axiomatized by a set of rules none of which hold in $\mathcal{ECQ}_\omega$ (Proposition 5.18). (It suffices to verify that $(p \wedge -p) \vee q, -q \vee r \vee -r \nvdash_{\mathcal{ECQ}_\omega} r \vee -r$.)

The finite reduced models of $\mathcal{KO}_-$ have the form $\boldsymbol{\mu}_+(G) \times \boldsymbol{\mu}_-(H) \times \mathbb{B}_{\mathbf{2}}^k$ (Proposition 7.17). By the above observation it follows that $\boldsymbol{\mu}_+(G)$ and $\boldsymbol{\mu}_-(H)$ and of course $\mathbb{B}_{\mathbf{2}}^k$ are models of $\mathcal{KO}_-$. It now suffices to prove that $\boldsymbol{\mu}_+(H)$ is a model of $\mathcal{KO}_-$ whenever $\boldsymbol{\mu}_-$ is for non-empty H, because then $\boldsymbol{\mu}_+(G) \times \boldsymbol{\mu}_+(H) \times \mathbb{B}_{\mathbf{2}}^k$ is a model of $\mathcal{KO}_-$.

But the matrix $\boldsymbol{\mu}_-(H)$ is not a model of $\mathcal{ECQ}$, i.e. there is a valuation $v : \mathbf{Fm} \to \boldsymbol{\mu}_-(H)$ which validates $p_1 \wedge -p_1$. Taking $v(p) := \mathsf{f}$ implies that $v(q \vee -q)$ is designated because $\chi_1 \vee p, -p \vee q \vee -q \vdash_{\mathcal{SDS}_1} q \vee -q$. It follows that $\boldsymbol{\mu}_-(H)$ is a model of $\mathcal{LP}$ whenever it is a model of $\mathcal{SDS}_1$. This implies that each vertex of the graph H is a loop, otherwise the valuation $v : \mathbf{Fm} \to \boldsymbol{\mu}_-(H)$ such that $v(p)$ is the complement of the principal downset generated by an element ∂u such that u is not a loop in H witnesses the failure of the rule $\emptyset \vdash p \vee -p$. But then $\boldsymbol{\mu}_+(H)$ is a model of $\mathcal{K}_-$ by the description of the finite reduced models of $\mathcal{SDS}_\omega$ (Proposition 7.29).

(v) This is a special case of Proposition 4.19, thanks to the description of the reduced models of $\mathcal{ETL}$ (Proposition 3.20). ☐

8.3 Miscellaneous properties

Finally, we consider several other miscellaneous properties of super-Belnap logics: variable sharing, structural completeness, contraposition, and the proof by cases property. We also note that some super-Belnap logics satisfy a weaker form of the proof by cases property, although we do not characterize all such logics.

For us, a logic $\mathcal{L}$ will have the *variable sharing property* in case $\Gamma \vdash_{\mathcal{L}} \varphi$ holds only if either Γ or φ contain a constant or some atom occurs in both Γ and φ. Pynko [59, Thm 4.2] proved that $\mathcal{BD}$ is the only super-Belnap logic with the proof by cases property which satisfies the variable sharing property. We now improve on his result.

Proposition 8.9 (Variable sharing in super-Belnap logics). *$\mathcal{BD}$ is the only super-Belnap logic with the variable sharing property.*

Proof. If $\mathcal{BD} < \mathcal{L}$, then $\mathcal{LP} \cap \mathcal{ECQ} \leq \mathcal{L}$, hence $p \wedge -p \vdash_{\mathcal{L}} q \vee -q$. □

A logic $\mathcal{L}$ is called *structurally complete* if it is the largest logic with the same set of theorems, i.e. if each extension of $\mathcal{L}$ adds new theorems to $\mathcal{L}$.

Proposition 8.10 (Structurally complete super-Belnap logics). *The only structurally complete super-Belnap logics are $\mathcal{K}$ and $\mathcal{CL}$.*

Proof. For each non-trivial super-Belnap logic $\mathcal{L}$ either $\mathcal{L} \leq \mathcal{K}$ or $\mathcal{LP} \leq \mathcal{L} \leq \mathcal{CL}$. In the former case $\mathcal{L}$ has the same theorems as $\mathcal{BD}$ and in the latter case $\mathcal{L}$ has the same theorems as $\mathcal{CL}$ (Proposition 3.4). □

We say that a super-Belnap logic $\mathcal{L}$ has the *(weak) proof by cases property* if it satisfies (for $\Gamma = \emptyset$) the equivalence

$$\Gamma, \varphi \vee \psi \vdash_{\mathcal{L}} \chi \iff \Gamma, \varphi \vdash_{\mathcal{L}} \chi \text{ and } \Gamma, \psi \vdash_{\mathcal{L}} \chi.$$

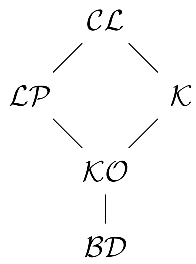
The super-Belnap logics with the proof by cases property are depicted in Figure 8.3. We shall see in Chapter 10 (Other frameworks) that these are precisely the super-Belnap logics which admit a multiple-conclusion version.

Proposition 8.11 (Proof by cases in super-Belnap logics). *The only super-Belnap logics which enjoy the (weak) proof by cases property are $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$.*

Proof. The logics $\mathcal{BD}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ enjoy the proof by cases property because the designated filters of the matrices $\mathbb{BD}_\mathbf{4}$, $\mathbb{P}_\mathbf{3}$, $\mathbb{K}_\mathbf{3}$, and $\mathbb{B}_\mathbf{2}$ are prime filters. The logic $\mathcal{KO}$ enjoys the proof by cases property because it is an intersection of two logics with this property.

Conversely, suppose that $\mathcal{L}$ is a proper extension of $\mathcal{BD}$ with the weak proof by cases property. Then $\mathcal{LP} \cap \mathcal{ECQ} \leq \mathcal{L}$, hence $p \wedge -p \vdash_{\mathcal{L}} q \vee -q$. But

Figure 8.3: Super-Belnap logics with the proof by cases property



then $(p \wedge -p) \vee r \vdash_{\mathcal{L}} (q \vee -q) \vee r$ by the weak proof by cases property, i.e. $\mathcal{KO} \leq \mathcal{L}$. Moreover, if $p \wedge -p \vdash_{\mathcal{L}} q$, then $(p \wedge -p) \vee q \vdash_{\mathcal{L}} q$, i.e. $\mathcal{K} \leq \mathcal{L}$. The claim now follows using the list of extensions of $\mathcal{KO}$ (Proposition 6.15). $\square$

Logics in Figure 8.3 are in fact also precisely the super-Belnap logics with what we might call a contrapositive companion. We call a pair of super-Belnap logics $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ a *contrapositive pair* if

$$\varphi \vdash_{\mathcal{L}_1} \psi \iff -\psi \vdash_{\mathcal{L}_2} -\varphi.$$

Proposition 8.12 (Contrapositive pairs of super-Belnap logics). *The only contrapositive pairs of super-Belnap logics are the pairs $\langle \mathcal{BD}, \mathcal{BD} \rangle$, $\langle \mathcal{KO}, \mathcal{KO} \rangle$, $\langle \mathcal{CL}, \mathcal{CL} \rangle$, $\langle \mathcal{LP}, \mathcal{K} \rangle$, and $\langle \mathcal{K}, \mathcal{LP} \rangle$.*

Proof. We already know that these are contrapositive pairs (Theorem 3.3). Conversely, let $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ be a contrapositive pair. Then $\mathcal{L}_1$ (and by the same token $\mathcal{L}_2$) has the proof by cases property: if $\varphi \vdash_{\mathcal{L}_1} \chi$ and $\psi \vdash_{\mathcal{L}_1} \chi$, then $-\chi \vdash_{\mathcal{L}_2} -\varphi$ and $-\chi \vdash_{\mathcal{L}_2} -\psi$, therefore $-\chi \vdash_{\mathcal{L}_2} -\varphi \wedge -\psi$ and by contraposition $-(-\varphi \wedge -\psi) \vdash_{\mathcal{L}_1} --\chi$, i.e. $\varphi \vee \psi \vdash_{\mathcal{L}_1} \chi$. But then Proposition 8.11 implies that there are no other contrapositive pairs. $\square$

Finally, we consider weaker variants of the proof by cases property. We shall say that a super-Belnap logic $\mathcal{L}$ has the *n-proof by cases property* for $n \geq 1$ if for each disjunction $\bigvee_{i \in I} \varphi_i$ we have

$$\Gamma, \bigvee_{i \in I} \varphi_i \vdash_{\mathcal{L}} \psi \iff \Gamma, \bigvee_{j \in J} \varphi_j \vdash_{\mathcal{L}} \psi \text{ for each } J \subseteq I \text{ with } |J| \leq n.$$

The condition $|J| \leq n$ may be replaced here by $|J| = n$. We say that $\mathcal{L}$ has the *proper* $(n+1)$-proof by cases property if it has the $(n+1)$-proof by cases property but not the n-proof by cases property.

The 1-proof by cases property is nothing but the ordinary proof by cases property. However, there are super-Belnap logics which have the proper n-proof by cases property for $n \geq 2$.

Just like the ordinary proof by cases property can be established for a super-Belnap logic by showing that the logic in question is complete with respect to a De Morgan matrix with a prime lattice filter, the n-proof by cases property can be established by showing completeness with respect to a De Morgan matrix with an n-prime filter.

Here, a lattice filter F on $\mathbf{L}$ will be called *n-prime* if $\bigvee_{i \in I} a_i \in F$ implies that $\bigvee_{j \in J} a_j \in F$ for some $J \subseteq I$ with $|J| \leq n$ (or equivalently, with $|J| = n$) for each finite set of elements $a_i \in \mathbf{A}$ with $i \in I$. A De Morgan matrix $\langle \mathbf{A}, F \rangle$ will be called n-prime if F is an n-prime filter on $\mathbf{A}$. Again, being 1-prime just means being prime in the ordinary sense.

We omit the easy proofs of the following observations.

Fact 8.13. *If $\mathbb{M}$ and $\mathbb{N}$ are respectively m-prime and n-prime De Morgan matrices, then $\mathbb{M} \times \mathbb{N}$ is an $(m+n)$-prime De Morgan matrix.*

Fact 8.14. *If $\mathbb{M}$ is n-prime, then $\operatorname{Log} \mathbb{M}$ has the n-proof by cases property.*

Fact 8.15. *If $\mathcal{L}$ has the m-proof by cases property, then it has the n-proof by cases property for $m \leq n$.*

Fact 8.16. *If the logics $\mathcal{L}_i$ have the n-proof by cases property for all $i \in I$, then so does $\bigcap_{i \in I} \mathcal{L}_i$.*

We now give examples of super-Belnap logics with the 2- and 3-proof by cases property. The 2- or 3-proof by cases property for their intersections can be inferred by the above observations.

Proposition 8.17.
$\mathcal{ETL}$, $\mathcal{K}_{-}$, $\mathcal{LP} \vee \mathcal{ECQ}$, and $\mathcal{ECQ}_{\omega}$ have the proper 2-proof by cases property.

Proof. We know that these logics do not satisfy the proof by cases property (Proposition 8.11). However, $\mathcal{ETL}$ and $\mathcal{K}_{-}$ are complete with respect to the matrices $\mathbb{ETL}_{\mathbf{4}}$ and $\mathbb{M}_{\mathbf{8}}$, which are easily seen to be 2-prime, while $\mathcal{LP} \vee \mathcal{ECQ}$ and $\mathcal{ECQ}_{\omega}$ are complete with respect to $\mathbb{P}_{\mathbf{3}} \times \mathbb{B}_{\mathbf{2}}$ and $\mathbb{BD}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$. But these matrices are 2-prime because the matrices $\mathbb{B}_{\mathbf{2}}$, $\mathbb{P}_{\mathbf{3}}$, and $\mathbb{BD}_{\mathbf{4}}$ are 1-prime. $\square$

Proposition 8.18.
The logics $\mathcal{ECQ}$ and $\mathcal{ETL}_{\omega}$ have the proper 3-proof by cases property but not the 2-proof by cases property.

Proof. $\mathcal{ECQ} = \operatorname{Log} \mathbb{BD}_{\mathbf{4}} \times \mathbb{ETL}_{\mathbf{4}}$ and $\mathcal{ETL}_{\omega} = \operatorname{Log} \mathbb{ETL}_{\mathbf{4}} \times \mathbb{B}_{\mathbf{2}}$ have the 3-proof by cases property by because the matrices $\mathbb{BD}_{\mathbf{4}}$ and $\mathbb{B}_{\mathbf{2}}$ are 1-prime and the matrix $\mathbb{ETL}_{\mathbf{4}}$ is 2-prime.

To prove that $\mathcal{ECQ}$ does not enjoy the 2-proof by cases property, let $\alpha := p \wedge -p \wedge (r \vee -r)$, $\beta := q \wedge -q \wedge (r \vee -r)$, and $\gamma := r \wedge -r$. Recall that by our description of consequence in explosive extensions (Proposition 4.6) $\Gamma \vdash_{\mathcal{ECQ}} \varphi$ if and only if either $\Gamma \vdash_{\mathcal{BD}} \varphi$ or $\Gamma \vdash_{\mathcal{ECQ}} \emptyset$, and $\Gamma \vdash_{\mathcal{ECQ}} \emptyset$ if and only if $\Gamma \vdash_{\mathcal{ETL}} \emptyset$ because $\mathrm{Exp}_{\mathcal{BD}}\,\mathcal{ETL} = \mathcal{ECQ}$. Then $\alpha \vee \beta \vee \gamma \nvdash_{\mathcal{ECQ}} \alpha \vee \beta$ because $\gamma \nvdash_{\mathcal{BD}} \alpha \vee \beta$ and $\alpha \vee \beta \vee \gamma \nvdash_{\mathcal{ETL}} \emptyset$, as witnessed by the valuations $v, w : \mathbf{Fm} \to \mathbf{DM_4}$ with $v(p) := \mathsf{n}$, $v(q) := \mathsf{n}$, $v(r) := \mathsf{b}$ and $w(p) := \mathsf{n}$, $w(q) := \mathsf{b}$, $w(r) := \mathsf{t}$. On the other hand, $\alpha \vee \beta \vdash_{\mathcal{ECQ}} \alpha \vee \beta$ and $\beta \vee \gamma \vdash_{\mathcal{ECQ}} \emptyset$ and $\gamma \vee \alpha \vdash_{\mathcal{ECQ}} \emptyset$ because $\beta \vee \gamma \vdash_{\mathcal{BD}} -\beta \wedge -\gamma$ and $\gamma \vee \alpha \vdash_{\mathcal{BD}} -\gamma \wedge -\alpha$.

To prove that $\mathcal{ETL}_\omega$ does not enjoy the 2-proof by cases property, let $\alpha := p \wedge -p$, $\beta := q \wedge -q$, and $\gamma := r \wedge (p \vee -p)$. Then $\alpha \vee \beta \vee \gamma \nvdash_{\mathcal{ETL}_\omega} \beta \vee \gamma$, as witnessed by the valuation $v : \mathbf{Fm} \to \mathbb{ETL}_\mathbf{4} \times \mathbb{B}_\mathbf{2}$ with $v(p) := \langle \mathsf{n}, \mathsf{f} \rangle$, $v(q) := \langle \mathsf{b}, \mathsf{f} \rangle$, $v(r) := \langle \mathsf{f}, \mathsf{t} \rangle$. On the other hand, clearly $\beta \vee \gamma \vdash_{\mathcal{ETL}_\omega} \beta \vee \gamma$ and $\alpha \vee \beta \vdash_{\mathcal{ETL}_\omega} \emptyset$. Moreover, it may be verified e.g. semantically that $\gamma \vee \alpha \vdash_{\mathcal{ETL}} \gamma$. □

We do not have a characterization of all super-Belnap logics which enjoy the n-proof by cases property even for $n = 2$. In particular, we do not know whether there are only finitely many such logics.

Chapter 9

Sequent calculi for super-Belnap logics

The current chapter deals with the Gentzen-style proof theory of super-Belnap logics. Our main result is a normal form theorem which generalizes the cut elimination theorem of classical logic to other super-Belnap logics and to proofs from arbitrary sets of sequential premises. We then use this normal form theorem to establish the interpolation property for many of the super-Belnap logics introduced so far. In addition to proving new interpolation results, we also provide a new proof of the recent refinement of the interpolation theorem for classical logic due to Milne [47, 48].

The main insight on which the Gentzen-style proof theory of super-Belnap logics as developed here rests is Pynko's observation [61] that a Gentzen calculus for $\mathcal{LP}$ (for $\mathcal{BD}$) may be obtained by adding elimination rules, i.e. the inverses of introduction rules, to a standard Gentzen calculus for classical logic and dropping the Cut rule (and the Identity axiom). That is, the logic $\mathcal{LP}$ ($\mathcal{BD}$) provides a semantics for the calculus obtained from the standard Gentzen calculus for classical logic by adding elimination rules and removing the Cut rule (and the Identity axiom). Our plan will be to extend this observation to arbitrary super-Belnap logics and formulate an analogue of the cut elimination theorem for classical logic for proofs from arbitrary sets of sequential premises in Gentzen calculi for super-Belnap logics.

In particular, we show that each super-Belnap logic corresponds to an extension of Pynko's calculus by a set of structural rules (rules which do not contain logical connectives). Super-Belnap logics may therefore be viewed as a class of logics analogical but orthogonal to the class of substructural logics. Substructural logics are obtained by keeping the logical rules of the classical Gentzen calculus fixed (as well as the Identity and Cut rules, although this is usually not mentioned explicitly) and tinkering with the structural rules of Exchange, Weakening, and Contraction. The situation with super-Belnap logics is dual: it is the rules of Exchange, Weakening, and Contraction which are kept fixed and the rules of Identity and Cut which are free to vary. In other words, we are justified in calling super-Belnap logics *subreflexive and subtransitive logics* by analogy with substructural logics.

Some of the results proved here may be of interest even to the classical logician. We recall Pynko's insight that the Logic of Paradox $\mathcal{LP}$ may be used to prove the admissibility of Cut in the Gentzen calculus for classical logic, and dualize it to prove the antiadmissibility of Identity. The idea of using (non-deterministic) three-valued semantics to prove the admissibility of Cut is, of course, not new — it dates back at least to the work of Schütte [70]. Later, it was used by Girard [29] to provide a three-valued semantics for a standard Gentzen calculus for classical logic without Cut. Dually, Hösli and Jäger [34] provided a three-valued semantics for a Gentzen calculus for classical logic without Identity. These ideas were then combined and extended by Lahav and Avron [42], who provided a uniform way of defining a non-deterministic four-valued Kripke semantics for a wide range of Gentzen calculi without Cut or Identity or both.

The difference between these approaches and the approach of Pynko [61], which we build on here, is that the latter is concerned with providing a semantics for a calculus which includes elimination rules. Their presence will allow us to define an appropriate normal form for proofs from a non-empty set of sequential premises. Note that the connection betwen deterministic semantics and the invertibility of logical rules was already pointed out in the two-valued case by Avron, Ciabattoni, and Zamansky [6].

Before we proceed in developing a particular kind of Gentzen-style proof theory for super-Belnap logics, it is worth recalling that there are basically two distinct approaches to providing a Gentzen calculus for a given logic, in particular for $\mathcal{BD}$. In the first approach, the logic $\mathcal{BD}$ is the logic of provable sequents. That is, $\Gamma \vdash_{\mathcal{BD}} \varphi$ if and only if the sequent $\Gamma \rhd \varphi$ is provable. This is the approach taken by Pynko [59] and Font [25].[1] On the other hand, we may also draw a connection between the consequence relation of $\mathcal{BD}$ and the derivability relation between sequents and sets of sequents, as Pynko [61] does. In the simplest form, this correspondence says that $\Gamma \vdash_{\mathcal{BD}} \varphi$ if and only if the sequent $\emptyset \rhd \varphi$ is provable from the sequents

[1]Pynko's calculus uses multiple-conclusion sequents and contains introduction rules for negated conjunction and disjunction but no structural rules (apart from Identity). The structural rules of Cut, Exchange, Weakening, and Contraction are then shown by Pynko to be admissible in this calculus. Font also briefly considers a single-conclusion version of this calculus with the structural rules present. However, the main calculus which he studies differs from Pynko's in replacing the introduction rules for negated conjunction and disjunction by Contraposition and Cut. Font's calculus, while being strongly adequate for $\mathcal{BD}$ in the sense of the theory of Font and Jansana [26], is in fact more of a calculus for the quasiequational theory of De Morgan lattices: De Morgan lattices form the equivalent algebraic semantics of Font's calculus, whereas $\mathcal{BD}$ has no equivalent algebraic semantics. A cosmetic difference between these calculi and the ones presented here is that we consider the truth constants $\mathtt{t}$ and $\mathtt{f}$ to be part of the signature of $\mathcal{BD}$, while Pynko and Font do not.

$\emptyset \rhd \gamma$ for $\gamma \in \Gamma$. It is well known that these two relations coincide e.g. in some standard Gentzen calculi for classical and intuitionistic logic, but when it comes to $\mathcal{BD}$, the two approaches require us to adopt rather different Gentzen calculi. In particular, the calculi of Pynko [59] and Font [25] contain the Identity axiom but neither the standard introduction rules for negation nor the elimination rules, while the calculus of Pynko [61] contains both the standard introduction rules for negation and the elimination rules while leaving out Identity and Cut. Here we opt for the latter approach.

The current chapter is structured as follows. We first introduce Gentzen relations as the sequential counterparts of logics (i.e. Hilbert relations) and provide the necessary preliminaries regarding equivalences between Gentzen and Hilbert relations. We then introduce a Gentzen calculus axiomatizing the Gentzen counterpart of $\mathcal{BD}$, denoted G$\mathcal{BD}$, and remark that each extension of G$\mathcal{BD}$ can be axiomatized by adding a set of structural rules, i.e. rules without logical connectives, to this calculus. The notion of a structurally atomic analytic–synthetic proof, related to the notion of a normal proof in natural deduction, is then introduced. Such proofs will be the counterparts of cut-free proofs in Gentzen calculi for super-Belnap logics. It is shown that each proof in the Gentzen calculi for $\mathcal{BD}$, $\mathcal{K}$, $\mathcal{LP}$, and $\mathcal{CL}$ may be transformed into such a proof. More generally, given an axiomatization of a super-Belnap logic we show how to transform it into a Gentzen calculus for which this normalization theorem holds. Finally, we use the normalization theorem for super-Belnap logics to show that each super-Belnap logic which is (in its Gentzen form) axiomatized by a set of what we call generalized cut rules enjoys a strong form of the interpolation property. In particular, this covers the logics $\mathcal{K}$, $\mathcal{ETL}$, and $\mathcal{SDS}_n$. We also provide a new proof of Milne's recent result [47, 48] that interpolation in classical logic can, as it were, be split between $\mathcal{K}$ and $\mathcal{LP}$.

9.1 Hilbert and Gentzen relations

Let us first recall the relevant parts of the theory of correspondences between Hilbert and Gentzen systems due to Pynko [60] and Raftery [65].[2] Raftery takes sequents to be pairs of finite sequences, but for the sake of simplicity we shall adopt a different definition. Nonetheless, his theory will apply straightforwardly to our sequents as well.

By a *sequent* we shall mean a pair of finite multisets of formulas, written as $\Gamma \rhd \Delta$. A sequent is *atomic* if all formulas in Γ and Δ are atoms. The *empty sequent* is the sequent $\emptyset \rhd \emptyset$. Note that our definition of a sequent

[2]To be more precise, Pynko [60] considers more general systems than Gentzen systems but assumes finitarity, while Raftery [65] restricts to Gentzen systems but does not assume finitarity. Since we are only interested in Gentzen systems (including possibly infinitary ones), we use the paper of Raftery as our main reference.

obviates the need to explicitly introduce the structural rule of Exchange. An *atomic sequent* is a sequent in which all formulas are atoms.

Recall that for us a logic, which might also be called a *Hilbert relation*, is a relation obtaining between sets formulas and formulas which satisfies reflexivity, monotonicity, structurality, and cut. A *Gentzen relation* differs from a Hilbert relation merely in taking sequents rather than formulas to be the objects between which the consequence relation obtains. More precisely, a Gentzen relation is a relation between sets of sequents and sequents which satisfies natural analogues of reflexivity, monotonicity, structurality, and cut. A Gentzen relation $\mathrm{G}\mathcal{L}$ is *finitary* if $S \vdash_{\mathrm{G}\mathcal{L}} \Gamma \rhd \Delta$ implies that $S' \vdash_{\mathrm{G}\mathcal{L}} \Gamma \rhd \Delta$ for some finite $S' \subseteq S$.

A Gentzen calculus, like a Hilbert calculus, is just a set of rules allowing us to derive sequents from sets of sequents. The rules simply pairs of sets of sequents interpreted as premises and sequents interpreted as conclusions. A Gentzen calculus axiomatizes a Gentzen relation if it is the least relation which contains all of the rules of the calculus. Equivalently, a Gentzen calculus $\mathbf{G}\mathcal{L}$ axiomatizes a Gentzen relation $\mathrm{G}\mathcal{L}$ if consequence in $\mathrm{G}\mathcal{L}$ coincides with provability in $\mathbf{G}\mathcal{L}$, where the definition of a proof in $\mathbf{G}\mathcal{L}$ is entirely analogous to the definition of a proof in a Hilbert relation. That is, proofs in $\mathbf{G}\mathcal{L}$ are nothing but well-founded trees, i.e. trees where all branches are finite, labelled in a suitable way by instances of the rules of $\mathbf{G}\mathcal{L}$.

Equivalences between Hilbert and Gentzen relations are set up by pairs of *definable transformers*. A definable transformer *from sequents to formulas* is a map $\boldsymbol{\tau}$ which assigns to each sequent $\Gamma \rhd \Delta$ a set of formulas $\boldsymbol{\tau}(\Gamma \rhd \Delta)$ such that $\sigma[\boldsymbol{\tau}(\Gamma \rhd \Delta)] = \boldsymbol{\tau}(\sigma(\Gamma \rhd \Delta))$. Likewise, a definable transformer *from formulas to sequents* is a map $\boldsymbol{\rho}$ which assigns to each formula φ a set of sequents $\boldsymbol{\rho}(\varphi)$ such that $\sigma[\boldsymbol{\rho}(\varphi)] = \boldsymbol{\rho}(\sigma(\varphi))$. Here we are using the notation $\sigma(\Gamma \rhd \Delta) := \sigma[\Gamma] \rhd \sigma[\Delta]$. The action of definable transformers extends to sets of sequents and sets of formulas in the expected way, i.e. it commutes with unions.

A Gentzen relation $\mathrm{G}\mathcal{L}$ and a Hilbert relation $\mathcal{L}$ are *equivalent* if there are definable transformers $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ such that

$$S \vdash_{\mathrm{G}\mathcal{L}} \Gamma \rhd \Delta \text{ if and only if } \boldsymbol{\tau}[S] \vdash_{\mathcal{L}} \boldsymbol{\tau}(\Gamma \rhd \Delta),$$
$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if and only if } \boldsymbol{\rho}[\Gamma] \vdash_{\mathrm{G}\mathcal{L}} \boldsymbol{\rho}(\varphi),$$

and

$$\varphi \dashv\vdash_{\mathcal{L}} \boldsymbol{\tau}\boldsymbol{\rho}(\varphi),$$
$$\Gamma \rhd \Delta \dashv\vdash_{\mathrm{G}\mathcal{L}} \boldsymbol{\rho}\boldsymbol{\tau}(\Gamma \rhd \Delta).$$

In fact, it only suffices to require the first and third conditions (or the second and fourth). A particularly natural case of such equivalences occurs when the transformer from formulas to sequents takes the form $\boldsymbol{\rho}(\varphi) := \emptyset \rhd \varphi$. In that case the relations are said to be *simply equivalent* via $\boldsymbol{\tau}$.

Theorem 9.1 [65, **Prop. 7.4**]
(Equivalence between Hilbert and Gentzen relations).
Let $\mathcal{B}$ and $\mathrm{G}\mathcal{B}$ be (finitary) Hilbert and Gentzen relations equivalent via $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. Then $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ induce an isomorphism between the (finitary) Hilbert relations extending $\mathcal{B}$ and the (finitary) Gentzen relations extending $\mathrm{G}\mathcal{B}$.

In particular, if $\mathcal{L}$ is the extension of the base logic $\mathcal{B}$ by the rules $\Gamma_i \vdash \varphi_i$ for $i \in I$, then the corresponding extension of $\mathrm{G}\mathcal{B}$ is the extension of $\mathrm{G}\mathcal{B}$ by the rules $\boldsymbol{\rho}[\Gamma_i] \vdash \boldsymbol{\rho}(\varphi_i)$ for $i \in I$.

Theorems and antitheorems may be defined for Gentzen relations in much the same way as for Hilbert relations. A sequent $\Gamma \rhd \Delta$ is a *theorem* of $\mathrm{G}\mathcal{L}$ if $\emptyset \vdash_{\mathrm{G}\mathcal{L}} \Gamma \rhd \Delta$. The syntactic definition of an antitheorem may be somewhat simplified, since we shall only deal with Gentzen relations which validate the rule of Weakening. In that case $S \vdash_{\mathrm{G}\mathcal{L}} \emptyset \rhd \emptyset$ will imply $S \vdash_{\mathrm{G}\mathcal{L}} \Gamma \rhd \Delta$ for each sequent $\Gamma \rhd \Delta$. For such Gentzen relations, we may therefore say that a set of sequents S is an *antitheorem* of G, symbolically $S \vdash_{\mathrm{G}\mathcal{L}} \emptyset$, if $S \vdash_{\mathrm{G}\mathcal{L}} \emptyset \rhd \emptyset$.

Finally, the notion of an admissible rule extends naturally to Gentzen rules. We shall also use the dual notion of an antiadmissible rule.

Definition 9.2 (Admissibility and antiadmissibility).
An inference rule is *admissible* (*antiadmissible*) in a logic if adding it does not yield any new theorems (antitheorems).

9.2 Sequent calculi for super-Belnap logics

We now introduce a Gentzen relation $\mathrm{G}\mathcal{BD}$ which is simply equivalent (in the sense explained in the previous section) to the Hilbert relation $\mathcal{BD}$ by means of a Gentzen calculus $\mathbf{G}\mathcal{BD}$. This will enable us to use proof-theoretic techniques to study super-Belnap logics, but also to use semantic techniques to study Gentzen calculi.

The Gentzen calculus $\mathbf{G}\mathcal{BD}$ defined by the rules of Figure 9.1 may be described simply as a standard Gentzen calculus for classical logic without the rules of Cut and Identity but with the inverse of each introduction rule for each logical connective or constant.[3] For example, the notation

$$\frac{\Gamma \rhd \Delta, \varphi \qquad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}$$

means that the calculus contains the three rules

[3] What we call Cut, Weakening, and Contraction are strictly speaking rule schemas or sets of structural rules. For example, Contraction is a set of rules which includes the rules $p, p \rhd q \vdash p \rhd q$ and $p, q, q \rhd r \vdash p, q \rhd r$ etc. However, no confusion will arise from talking simply about the rule of Cut, Weakening, or Contraction.

Figure 9.1: The sequent calculus $\mathbf{G}\mathcal{BD}$ for $\mathcal{BD}$

Logical rules

$$\frac{\Gamma \rhd \Delta, \varphi \qquad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi} \qquad \frac{\varphi, \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}$$

$$\frac{\varphi, \Gamma \rhd \Delta \qquad \psi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta} \qquad \frac{\Gamma \rhd \Delta, \varphi, \psi}{\Gamma \rhd \Delta, \varphi \vee \psi}$$

$$\frac{\varphi, \Gamma \rhd \Delta}{\Gamma \rhd \Delta, -\varphi} \qquad \frac{\Gamma \rhd \Delta, \varphi}{-\varphi, \Gamma \rhd \Delta}$$

$$\emptyset \rhd \mathsf{t} \qquad \frac{\mathsf{t}, \Gamma \rhd \Delta}{\Gamma \rhd \Delta} \qquad \frac{\Gamma \rhd \Delta, \mathsf{f}}{\Gamma \rhd \Delta} \qquad \mathsf{f} \rhd \emptyset$$

Structural rules

$$\frac{\Gamma \rhd \Delta}{\varphi, \Gamma \rhd \Delta} \qquad \frac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta, \varphi} \qquad \frac{\varphi, \varphi, \Gamma \rhd \Delta}{\varphi, \Gamma \rhd \Delta} \qquad \frac{\Gamma \rhd \Delta, \varphi, \varphi}{\Gamma \rhd \Delta, \varphi}$$

$$\frac{\Gamma \rhd \Delta, \varphi \qquad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi} \qquad \frac{\Gamma \rhd \Delta, \varphi \wedge \psi}{\Gamma \rhd \Delta, \varphi} \qquad \frac{\Gamma \rhd \Delta, \varphi \wedge \psi}{\Gamma \rhd \Delta, \psi}$$

the first one being called the right introduction rule for conjunction and the second and third being called the right elimination rules for conjunction. The sequents $\emptyset \rhd \mathsf{t}$ and $\mathsf{f} \rhd \emptyset$ are the only two axioms of this Gentzen calculus. The Gentzen relation axiomatized by $\mathbf{G}\mathcal{BD}$ will be called $\mathrm{G}\mathcal{BD}$. Whenever we talk about derivability without specifying a calculus in this section, we will mean derivability in $\mathbf{G}\mathcal{BD}$.

The equivalence of $\mathrm{G}\mathcal{BD}$ and $\mathcal{BD}$ was proved by Pynko [61].

Theorem 9.3 [61] **(Gentzen formulation of $\mathcal{BD}$).**
The Gentzen relation $\mathrm{G}\mathcal{BD}$ is simply equivalent to the Hilbert relation $\mathcal{BD}$ via the transformer $\boldsymbol{\tau} : \Gamma \rhd \Delta \mapsto \{-\bigwedge \Gamma \vee \bigvee \Delta\}$.

By the theory outlined in the previous section, we know that the transformers $\boldsymbol{\tau}$ and $\boldsymbol{\rho} : \varphi \mapsto \{\emptyset \rhd \varphi\}$ establish an equivalence between the extensions of $\mathcal{BD}$ and the extensions of $\mathrm{G}\mathcal{BD}$. The Gentzen counterpart of a super-Belnap logic $\mathcal{L}$ will be denoted $\mathrm{G}\mathcal{L}$.

It now takes but a moment's reflection to see that each extension of $\mathrm{G}\mathcal{BD}$ may in fact be axiomatized by a set of *structural rules*, i.e. rules which do not contain any occurrence of a logical connective or constant.

Proposition 9.4 (Composition and decomposition of sequents). *Each sequent is equivalent in the Gentzen relation* $\mathrm{G}\mathcal{BD}$ *to a finite set of atomic sequents. Each finite set of sequents is equivalent in the Gentzen relation* $\mathrm{G}\mathcal{BD}$ *to a single sequent of the form* $\emptyset \rhd \varphi$.

Proof. Both of these claims may be proved by a straightforward induction over the complexity of the sequent or the finite set of sequents. □

Proposition 9.5 (Structural axiomatization of extensions of $\mathrm{G}\mathcal{BD}$). *Each (finitary) extension of* $\mathrm{G}\mathcal{BD}$ *may be axiomatized as an extension of the calculus* $\mathbf{G}\mathcal{BD}$ *by a set of (finitary) structural rules.*

Proof. Let $\{\Gamma_i \rhd \Delta_i \mid i \in I\} \vdash \Gamma \rhd \Delta$ be a Gentzen rule. By the previous proposition there is for each $i \in I$ a finite set of atomic sequents S_i equivalent to $\Gamma_i \rhd \Delta_i$ and there is a finite set of atomic sequents S equivalent to $\Gamma \rhd \Delta$. Then the rule in question is equivalent to the finite set of rules $\bigcup_{i \in I} S_i \vdash \Lambda \rhd \Pi$ for $\Lambda \rhd \Pi \in S$. Moreover, these rules are finitary if the rule in question is finitary. □

Extensions of the calculus $\mathbf{G}\mathcal{BD}$ by a set of structural rules will be called *super-Belnap (Gentzen) calculi.* Let us now introduce Gentzen calculi for the most important super-Belnap logics. For the logics $\mathcal{L}$ mentioned below, the calculus obtained by adding the appropriate rule to $\mathbf{G}\mathcal{BD}$ will be denoted $\mathbf{G}\mathcal{L}$. The calculi for $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ were already considered by Pynko [61]. See Figure 9.2 for the definition of the structural rules mentioned below.

Proposition 9.6 [61] (Calculi for basic super-Belnap logics).

(i) $\mathrm{G}\mathcal{LP}$ *extends* $\mathrm{G}\mathcal{BD}$ *by Identity.*

(ii) $\mathrm{G}\mathcal{K}$ *extends* $\mathrm{G}\mathcal{BD}$ *by Cut.*

(iii) $\mathrm{G}\mathcal{CL}$ *extends* $\mathrm{G}\mathcal{BD}$ *by Identity and Cut.*

(iv) $\mathrm{G}\mathcal{ECQ}$ *extends* $\mathrm{G}\mathcal{BD}$ *by Explosive Cut.*

(v) $\mathrm{G}\mathcal{ETL}$ *extends* $\mathrm{G}\mathcal{BD}$ *by Limited Cut.*

In addition, the reader may easily check that e.g. the rule $p \wedge -p \vdash q \vee -q$ axiomatizing the logic $\mathcal{LP} \cap \mathcal{ECQ}$ corresponds to the Gentzen rule (schema)

$$\frac{\emptyset \rhd \varphi \qquad \varphi \rhd \emptyset}{\psi \rhd \psi}$$

Figure 9.2: Some additional structural rules

Identity

$$\varphi \rhd \varphi$$

Cut

$$\frac{\Gamma \rhd \Delta, \varphi \qquad \varphi, \Gamma' \rhd \Delta}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$$

Limited Cut

$$\frac{\emptyset \rhd \varphi \qquad \varphi, \Gamma \rhd \Delta}{\Gamma \rhd \Delta} \qquad \frac{\Gamma \rhd \Delta, \varphi \qquad \varphi \rhd \emptyset}{\Gamma \rhd \Delta}$$

Explosive Cut

$$\frac{\emptyset \rhd \varphi \qquad \varphi \rhd \emptyset}{\emptyset \rhd \emptyset}$$

while the rule $(p \wedge -p) \vee r \vdash (q \vee -q) \vee r$ axiomatizing the logic $\mathcal{KO}$ corresponds to the Gentzen rule (schema)

$$\frac{\Gamma \rhd \Delta, \varphi \qquad \varphi, \Gamma \rhd \Delta}{\psi, \Gamma \rhd \Delta, \psi}$$

which may be seen as a combination of Identity and Cut.

Fact 9.7. *Cut is a derivable rule from Identity and Limited Cut.*

Proof. Recall that $\mathcal{CL} = \mathcal{LP} \vee \mathcal{K} = \mathcal{LP} \vee \mathcal{ETL}$. □

We now focus on the calculi for $\mathcal{CL}$, $\mathcal{LP}$, and $\mathcal{K}$. It turns out that for $\mathcal{CL}$ we may pick and choose any combination of introduction and elimination rules, provided we include at least one of these for each connective. For $\mathcal{LP}$ we may do without the elimination rules provided that we are only interested in which sequents are derivable from the empty set of sequents, and dually for $\mathcal{K}$ we may do without the introduction rules provided that we are only interested in sets of sequents from which the empty sequent is derivable.

Proposition 9.8 (Introduction–elimination interderivability). *In the presence of Identity, Cut, Contraction, and Weakening, the left (right) introduction and elimination rules for each connective are interderivable.*

Proof. To simulate the left introduction rule by the left elimination rule, we use the following strategy:

$$\dfrac{\dfrac{\dfrac{\varphi \wedge \psi \rhd \varphi \wedge \psi}{\varphi \wedge \psi \rhd \psi} \qquad \dfrac{\dfrac{\varphi \wedge \psi \rhd \varphi \wedge \psi}{\varphi \wedge \psi \rhd \varphi} \qquad \varphi, \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \psi, \Gamma \rhd \Delta}}{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \rhd \Delta}}{\varphi \wedge \psi, \Gamma \rhd \Delta}$$

The left elimination rule may then be simulated by the following proof:

$$\dfrac{\dfrac{\dfrac{\varphi \rhd \varphi}{\varphi, \psi \rhd \varphi} \qquad \dfrac{\psi \rhd \psi}{\varphi, \psi \rhd \psi}}{\varphi, \psi \rhd \varphi \wedge \psi} \qquad \varphi \wedge \psi, \Gamma \rhd \Delta}{\varphi, \psi, \Gamma \rhd \Delta}$$

To simulate the right introduction rule for conjunction by the left elimination rule, we use the following strategy:

$$\dfrac{\dfrac{\Gamma \rhd \Delta, \psi \qquad \dfrac{\Gamma \rhd \Delta, \varphi \qquad \dfrac{\varphi \wedge \psi \rhd \varphi \wedge \psi}{\varphi, \psi \rhd \varphi \wedge \psi}}{\psi, \Gamma \rhd \Delta, \varphi \wedge \psi}}{\Gamma, \Gamma \rhd \Delta, \Delta, \varphi \wedge \psi}}{\Gamma \rhd \Delta, \varphi \wedge \psi}$$

The right elimination rules may then be simulated by the following proofs:

$$\dfrac{\Gamma \rhd \Delta, \varphi \wedge \psi \qquad \dfrac{\dfrac{\varphi \rhd \varphi}{\varphi, \psi \rhd \varphi}}{\varphi \wedge \psi \rhd \varphi}}{\Gamma \rhd \Delta, \varphi} \qquad \dfrac{\Gamma \rhd \Delta, \varphi \wedge \psi \qquad \dfrac{\dfrac{\psi \rhd \psi}{\varphi, \psi \rhd \psi}}{\varphi \wedge \psi \rhd \psi}}{\Gamma \rhd \Delta, \psi}$$

The argument for disjunction is dual and the arguments for negation and for the truth constants are simpler, therefore we omit them. □

We needed both Identity and Cut to establish the interderivability of the introduction and elimination rules. However, if we are only interested in the admissibility (antiadmissibility) of the elimination (introduction) rules, the presence of Cut (Identity) is not needed. This claim (for admissibility) is called the Inversion Lemma by Troelstra and Schwichtenberg [74], and it constitutes a step in the standard proof of cut elimination for classical logic.

Proposition 9.9 [74, Prop 3.5.4] (Admissibility of elimination rules). *Each elimination rule is admissible in the calculus which contains Identity, Weakening, Contraction, and the introduction rules for all connectives.*

This proposition explains why the elimination rules are invisible from the standard point of view, which is only concerned with sequents provable from an empty set of premises.

Proposition 9.10 (Antiadmissibility of introduction rules). *Each introduction rule is antiadmissible in the calculus which contains Cut, Weakening, Contraction, and the elimination rules for all connectives.*

Proof. Let us say that a sequent $\Gamma \rhd \Delta$ *explodes directly* relative to a *multiset* of sequents S if there is a proof of $\emptyset \rhd \emptyset$ from $S \cup \{\Gamma \rhd \Delta\}$ which does not contain any introduction rules and in which each sequent is used precisely the specified number of times as a premise of the proof. Explicit reference to the set of side assumptions S will be suppressed in the following. It now suffices to prove by induction over the height h of the premise $\Gamma \rhd \Delta$ in the proof of $\emptyset \rhd \emptyset$ that

- if $\varphi \wedge \psi, \Gamma \rhd \Delta$ explodes directly, so does $\varphi, \psi, \Gamma \rhd \Delta$
- if $\Gamma \rhd \Delta, \varphi \wedge \psi$ explodes directly, so does $\{\Gamma \rhd \Delta, \varphi\} \cup \{\Gamma \rhd \Delta, \psi\}$
- if $\varphi \vee \psi, \Gamma \rhd \Delta$ explodes directly, so does $\{\varphi, \Gamma \rhd \Delta\} \cup \{\varphi, \Gamma \rhd \Delta\}$
- if $\Gamma \rhd \Delta, \varphi \vee \psi$ explodes directly, so does $\Gamma \rhd \Delta, \varphi, \psi$
- if $-\varphi, \Gamma \rhd \Delta$ explodes directly, so does $\Gamma \rhd \Delta, \varphi$
- if $\Gamma \rhd \Delta, -\varphi$ explodes directly, so does $\varphi, \Gamma \rhd \Delta$
- if $\mathsf{t}, \Gamma \rhd \Delta$ explodes directly, so does $\Gamma \rhd \Delta$
- if $\emptyset \rhd \mathsf{t}$ explodes directly, so does $\emptyset$
- if $\mathsf{f} \rhd \emptyset$ explodes directly, so does $\emptyset$
- if $\Gamma \rhd \Delta, \mathsf{f}$ explodes directly, so does $\Gamma \rhd \Delta$

We only deal with the first item, the rest of them are entirely analogical. The sequent $\varphi \wedge \psi, \Gamma \rhd \Delta$ cannot be the last sequent of a proof of $\emptyset \rhd \emptyset$, hence the base case holds trivially. Now suppose that an instance of $\varphi \wedge \psi, \Gamma \rhd \Delta$ explodes directly via a proof where this sequent has height $h + 1$. If the rule which follows this instance of $\varphi \wedge \psi, \Gamma \rhd \Delta$ is any rule other than an elimination rule applied to $\varphi \wedge \psi$, we may simply use the inductive hypothesis for h. If, on the other hand, this instance of $\varphi \wedge \psi, \Gamma \rhd \Delta$ occurs as a premise of the left conjunction elimination rule, then clearly $\varphi, \psi, \Gamma \rhd \Delta$ explodes directly (with respect to the same multiset of sequents S). $\square$

The above proof is height-preserving in the same sense as the original Inversion Lemma. Notice the slight twist involving multisets forced on us by the fact that a proof has only one conclusion but it may have many premises.

Having proved the Inversion Lemma and its dual, we may now prove the admissibility of Cut and the antiadmissibility of Identity in the standard calculus for classical logic using a semantic argument. This is because for us these are not mere fragments of some calculus, but rather calculi in their own right with a perfectly good semantics provided by the logics $\mathcal{LP}$ and $\mathcal{K}$.

The admissibility of Cut and the antiadmissibility of Identity are now immediate consequences of the previous three propositions. We emphasize again that this route to proving the admissibility of Cut in the Gentzen calculus for classical logic was already taken by Pynko [61]. We therefore only provide a proof of the latter assertion.

Note that the restriction of the calculus $\mathbf{G}\mathcal{K}$ to atomic sequents is precisely the resolution calculus for classical logic, thus the antiadmissibility of Identity is merely a more sophisticated version of the trivial observation that we never need to apply resolution to clauses of the form $p \vee -p$.

Theorem 9.11 (Admissibility of Cut).
Cut rule is admissible in the Gentzen calculus which contains Identity, Weakening, Contraction, and the introduction rules for all connectives.

Theorem 9.12 (Antiadmissibility of Identity).
Identity is antiadmissible in the Gentzen calculus which contains Cut, Weakening, Contraction, and the elimination rules for all connectives.

Proof. Suppose that the empty sequent is derivable from the set of sequents S in the Gentzen calculus which contains Identity, Cut, Weakening, Contraction, and the elimination rules. Then it is derivable from S in $\mathbf{G}\mathcal{CL}$. But we know that $\mathcal{CL}$ and $\mathcal{K}$ have the same antitheorems, since $\mathrm{Exp}_{\mathcal{BD}}\,\mathcal{CL} \leq \mathcal{K}$. By the equivalence between Hilbert and Gentzen versions of super-Belnap logics it follows that the empty sequent is derivable from S in $\mathbf{G}\mathcal{K}$. Proposition 9.10 then implies that the empty sequent is in fact derivable from S using only Cut, Weakening, Contraction, and the elimination rules. □

These mostly semantic proofs of course do not yield a procedure for eliminating Cut or Identity from a given proof. Note that the proof is not entirely semantic, as it relies on the syntactic proof of the Inversion Lemma. For a purely semantic proof of the admissibility of Cut, the non-deterministic framework of Lahav and Avron [42] is more appropriate.

Fact 9.13. *Cut is admissible in each super-Belnap calculus.*

Proof. Adding the Cut rule to a super-Belnap calculus for $\mathrm{G}\mathcal{L}$ yields a calculus for the Gentzen counterpart of $\mathcal{L} \vee \mathcal{K}$. But either $\mathcal{L} \leq \mathcal{K}$ or $\mathcal{LP} \leq \mathcal{K}$ (Proposition 6.5), and in both cases $\mathcal{L} \vee \mathcal{K}$ has the same theorems as $\mathcal{L}$ (Proposition 3.4). □

When it comes to antiadmissibility, there is no such single theorem to cover all situations. Let us therefore only consider one simple example.

Fact 9.14. *Limited Cut is antiadmissible in the calculus* $\mathbf{G}\mathcal{ECQ}$.

Proof. $\mathrm{G}\mathcal{ETL}$ is the extension of $\mathrm{G}\mathcal{ECQ}$ by Limited Cut and $\mathcal{ETL}$ has the same antitheorems as $\mathcal{ECQ}$, since $\mathrm{Exp}_{\mathcal{BD}}\,\mathcal{ETL} = \mathcal{ECQ}$ (Proposition 5.4). $\square$

It is also not the case that Identity is antiadmissible in all super-Belnap calculi. This is because e.g. $\mathcal{LP} \vee \mathcal{ECQ}$ has more antitheorems than $\mathcal{ECQ}$.

9.3 Normalization in super-Belnap calculi

The Gentzen-style proof theory of classical logic has mainly been concerned with which sequents are provable, meaning provable from an empty set of premises. Accordingly, its "Hauptsatz" states that with an empty set of premises we may restrict to proofs which do not contain Cut.

In the current context of super-Belnap calculi, we are mainly interested in proving sequents from other sequents. We would therefore like to formulate an appropriate generalization of this result which would cover proofs from non-empty sets of premises. Since in classical logic we may be interested in proving sequents from other sequents as well, such a generalization may be of interest even to the classical logician. In particular, we shall use it in the following section to provide an alternative syntactic proof of (a certain refinement of) the Craig interpolation theorem for classical logic.

We propose to generalize the notion of a cut-free proof to proofs from a non-empty set of premises by decomposing this notion into a conjunction of two distinct conditions: structural atomicity and analyticity–syntheticity. The former notion concerns only the applications of structural rules in the proof, whereas the latter notion concerns only logical rules.

Definition 9.15 (Structurally Atomic Proofs).
A proof is *structurally atomic* if both the premises and conclusions of all occurrences of structural rules in the proof are atomic sequents.

Definition 9.16 (Analytic–Synthetic Proofs).
A proof is *analytic–synthetic* if in each branch of the proof all instances of elimination rules precede all instances of introduction rules.

Structurally atomic analytic–synthetic proofs can be divided into three parts: elimination rules at the top, atomic instances of structural rules in the middle, and introduction rules at the bottom (each of these parts may of course be empty). The importance of structural atomicity is precisely that it yields this tripartite structure in conjunction with *local* analyticity–syntheticity, as the following (easy but crucial) lemma states. Moreover, transforming a structurally atomic proof into a proof which is also locally analytic–synthetic is easy with the help of elimination rules, as we shall see shortly.

Lemma 9.17 (Structurally atomic analytic–synthetic proofs).
A structurally atomic proof is analytic–synthetic if and only if no elimination rule in it immediately follows an introduction rule.

Proof. Elimination rules may not immediately follow and introduction rules may not immediately precede any structural rule in a structurally atomic proof. Therefore if in some branch of the proof an instance of an introduction rule precedes an instance of an elimination rule, then there must be a pair of rules in between these two which consists of an introduction rule followed by an elimination rule. □

If we restrict in the case of classical logic to an empty set of premises, structurally atomic analytic–synthetic proofs are essentially ordinary cut-free proofs. Clearly no elimination rules may occur in such proofs, and all instances of Cut are restricted to atomic sequents. The following proposition, whose proof is immediate, is now all it takes to transfom structurally atomic analytic–synthetic proofs from an empty set of premises in $\mathbf{G}\mathcal{CL}$ into cut-free proofs in the standard sense of the term. Thus, although elimination rules have no place in the standard Gentzen calculi for classical logic, cut-free proofs essentially arise naturally as the intersection of two classes of proofs defined in terms of elimination rules and atomic sequents.

(By an *atomic instance* of a rule, we mean an instance where all of the premises as well as the conclusion are atomic sequents.)

Proposition 9.18 (Atomic theorems of $\mathcal{CL}$).
The following are equivalent for atomic sequents $\Gamma \rhd \Delta$:

(i) $\Gamma \rhd \Delta$ *has the form* $p, \Gamma' \rhd \Delta', p$.

(ii) $\Gamma \rhd \Delta$ *is derivable using atomic instances of Identity and Weakening.*

(iii) $\Gamma \rhd \Delta$ *is derivable using atomic instances of Identity, Cut, Weakening, and Contraction.*

Proving the dual observation is slightly less straightforward.

Proposition 9.19 (Atomic antitheorems of $\mathcal{CL}$).
The following are equivalent for sets of atomic sequents S:

(i) $\emptyset \rhd \emptyset$ *is derivable from S using atomic instances of Contraction followed by atomic instances of Cut.*

(ii) $\emptyset \rhd \emptyset$ *is derivable from S using atomic instances of Contraction, Weakening, Identity, and Cut.*

Proof. Suppose that (ii) holds. We first show that Weakening is not needed. Each atomic instance of Weakening by p on the right (on the left) may be permuted below each immediately following atomic instance of Cut where p is not the cut formula on the right (on the left) of the appropriate premise of Cut and below each immediately following atomic instance of Contraction where p is not the contracted formula on the right (on the left). On the other hand, an atomic instance of Weakening by p on the right (on the left) followed by an atomic instance of Cut where p is the cut formula on the right (on the left) of the appropriate premise of Cut may be replaced by several atomic instances of Weakening. That is, the proof segment

$$\dfrac{\dfrac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta, p} \qquad p, \Gamma' \rhd \Delta'}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$$

may be replaced by the proof segment

$$\dfrac{\Gamma \rhd \Delta}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$$

where we have condensed several instances of Weakening into one step, and likewise for Weakening on the left. Similarly, a proof segment consisting of an atomic instance of Weakening by p on the right (on the left) followed by an atomic instance of Contraction where p is the contracted formula on the right (on the left) may simply be omitted from the proof.

Weakening cannot be the last rule in a proof of the empty sequent, therefore every instance of Weakening (starting with the bottommost ones) in a proof of the empty sequent from S which only uses atomic versions of Identity, Cut, Weakening, and Contraction may be permuted downward until it is removed from the proof. This yields a proof which only uses atomic instances of Identity, Cut, and Contraction.

Each instance of Identity in such a proof may only be followed by an instance of Cut, which is then clearly redundant (its conclusion coincides with the other premise). Eliminating such redundant Cuts yields a proof which only uses atomic versions of Cut and Contraction. Finally, each instance of Contraction may be permuted above each instance of Cut, yielding a proof which consists of atomic instances of Contraction followed by atomic instances of Cut. □

One superficial difference between cut-free proofs and structurally atomic analytic–synthetic proofs is that strictly speaking the latter need not satisfy the subformula property as defined below.

Definition 9.20 (Subformula property).
A proof has the *subformula property* if each formula of the proof is a subformula of some formula either in the premises or in the conclusion.

For example, we may apply Weakening by a variable p which occurs neither in the premises nor in the conclusion to each side of an atomic sequent $\Gamma \rhd \Delta$, and then Cut on p. Of course, in the particular case of proofs from an empty set of premises or proofs of the empty sequent in $\mathbf{G}\mathcal{CL}$, such detours may be avoided in structurally atomic analytic–synthetic proofs by Propositions 9.18 and 9.19. Moreover, in the absence of structural rules other than Weakening and Contraction the subformula property holds for each structurally atomic analytic–synthetic proof.

Fact 9.21. *Each structurally atomic analytic–synthetic proof in* $\mathbf{G}\mathcal{BD}$ *has the subformula property.*

Proof. It suffices to observe that no subformulas disappear in Weakening and Contraction going from the premise to the conclusion. $\square$

In addition to transforming structurally atomic proofs into structurally atomic analytic–synthetic proofs, we shall therefore also require that the resulting proofs satisfy the subformula property.

Theorem 9.22 (Normalization Theorem).
If a sequent has a structurally atomic proof from a set of sequents S in a super-Belnap calculus, then it has a structurally atomic analytic–synthetic proof from S with the subformula property.

Proof. We first show that each structurally atomic analytic–synthetic proof may be transformed into a proof which moreover satisfies the subformula property. Suppose first that the premises do not contain any atom (as a subformula). Each variable-free formula of $\mathcal{BD}$ is equivalent in $\mathcal{BD}$ either to $\mathtt{t}$ or to $\mathtt{f}$, therefore each premise is equivalent in $\mathrm{G}\mathcal{BD}$ either to the empty sequent or to the empty set of sequents. By induction over the complexity of $\Gamma \rhd \Delta$ we may show that if $\Gamma \rhd \Delta$ implies $\emptyset \rhd \emptyset$ in $\mathrm{G}\mathcal{BD}$, then the $\emptyset \rhd \emptyset$ is provable from $\Gamma \rhd \Delta$ using only elimination rules. On the other hand, if all premises are equivalent in $\mathrm{G}\mathcal{BD}$ to the empty set of sequents, then the conclusion of the proof is provable from $\emptyset$ in $\mathbf{G}\mathcal{BD}$. But then we may show by induction over the complexity of the conclusion $\Gamma \rhd \Delta$ that if $\Gamma \rhd \Delta$ is a theorem of $\mathrm{G}\mathcal{BD}$, then it is provable in $\mathbf{G}\mathcal{BD}$ using only introduction rules and Weakening.

Otherwise, let q be an atom which occurs either in the premises of in the conclusion of the proof. Observe that each atom p which occurs neither in the premises nor in the conclusion of a structurally atomic analytic–synthetic proof may only occur in atomic sequents, and the only rules which may apply to sequents containing p are structural ones. Replacing all occurrences of such atoms p by q now yields a structurally atomic analytic–synthetic proof which moreover has the subformula property.

We now show that each structurally atomic proof may be transformed into one which is moreover analytic–synthetic. By Lemma 9.17, it suffices to produce a structurally atomic proof in which no elimination rule immediately follows an introduction rule. We shall first deal with finite proofs.

Let us call an instance of an elimination rule *problematic* if it immediately follows an instance of an introduction rule. The *depth* of a given occurrence of a rule will be the length of the longest branch of the subproof which ends this rule. It suffices to show that a finite structurally atomic proof which contains exactly one problematic rule of depth d and it is the final rule of the proof may be reduced to a finite structurally atomic proof which either contains no problematic rules or it contains exactly one problematic rule and it has height lower than d.

There are only two cases: either the formula being broken down by the problematic rule is a side formula of the introduction rule above or it is the principal formula of the introduction rule above. In the former case, it is straightforward to permute the problematic rule above the introduction rule, thereby either decreasing its depth or making it unproblematic. In the latter case, the problematic rule may be eliminated directly. We again only consider the case of conjunction, the case of disjunction being dual and the cases of negation and the truth constants being simpler. The required reductions are straightforward: the proof segment

$$\dfrac{\dfrac{\varphi, \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi \rhd \Delta}}{\varphi, \psi \rhd \Delta}$$

is replaced simply by

$$\varphi, \psi, \Gamma \rhd \Delta$$

while the proof segments

$$\dfrac{\dfrac{\Gamma \rhd \Delta, \varphi \qquad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}}{\Gamma \rhd \Delta, \varphi} \qquad\qquad \dfrac{\dfrac{\Gamma \rhd \Delta, \varphi \qquad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}}{\Gamma \rhd \Delta, \psi}$$

are replaced simply by

$$\Gamma \rhd \Delta, \varphi \qquad\qquad \Gamma \rhd \Delta, \psi$$

It is moreover clear that these reductions preserve structural atomicity.

(The rest of the proof is devoted to the technical details involved in handling infinitary rules, which the reader may wish to skip.)

It remains to deal with proofs which are not finite. We do so by breaking them into finite parts. By a *non-structural segment* of a proof, we shall mean a maximal subproof which does not contain any structural rules (a subproof being a suitably labelled subtree of a proof). That is, a non-structural segment is a subproof which (i) does not contain any structural rules, (ii) its root is either the conclusion of the whole proof or the premise of a structural rule, and (iii) its terminal nodes are either premises of the whole proof or conclusions of a structural rule. Each sequent in the proof belongs to some non-structural segment (possibly with only one node) and each non-structural segment is a finite proof, since it is finitely branching (by virtue of not containing any structural rules) and it does not contain an infinite branch (by virtue of being a subtree of a well-founded tree). Each non-structural segment may be assigned a finite *structural height*, defined as the number of occurrences of structural rules which occur below its conclusion.

We now transform the original proof into an analytic–synthetic proof in ω stages while preserving structural atomicity. In stage 0, we transform the non-structural segment of structural height 0 into an analytic–synthetic proof and append the appropriate subproofs of the original proof above the premises of this non-structural segment which are not premises of the original proof. In stage $n+1$, we transform each non-structural segment of structural height $n+1$ of the proof obtained after stage n into an analytic–synthetic proof. Note that after stage n, each non-structural segment of structural height $m > n$ is in fact a non-structural segment of structural height m in the original proof. In stages $m > n$, the non-structural segments of height at most n are left unchanged. The limit case of this process is a suitably labelled tree in which each non-structural segment of structural height n is precisely as it was after stage n.

Suppose that this tree has an infinite branch. In particular, this branch contains infinitely many instances of structural rules. Now observe that two instances of structural rules are only connected by a branch in the limit stage if they were already connected at some finite stage, and they are already connected before stage $n+1$ if they were already connected before stage n. There is therefore a branch in the original proof containing infinitely many instances of structural rules. Since this cannot be the case, the limit stage in fact yields a well-founded tree and therefore a proof. □

In order to transform each proof in a given calculus into a structurally atomic analytic–synthetic one, it therefore suffices to reduce every instance of a structural rule of the calculus into a proof which only contains logical rules and atomic instances of the structural rules of the calculus. Rules which admit such reductions will be said to enjoy the expansion property, which is essentially the syntactic propagation property of Terui [73].

Definition 9.23 (Expansion property).
A set of structural rules R satisfies the *expansion property* if the conclusion of each instance of a rule $\rho \in R$ has a proof from the corresponding instances of premises of ρ which only uses the logical rules and atomic instances of rules in R.

A super-Belnap calculus satisfies the expansion property if the set of its structural rules does. A rule ρ satisfies the expansion property if $\{\rho\}$ does.

Proposition 9.24.
The structural rules of a super-Belnap calculus satisfy the expansion property if and only if for each proof of a sequent $\Gamma \rhd \Delta$ from a set of sequents S in the calculus, there is a structurally atomic proof of $\Gamma \rhd \Delta$ from S.

Proof. In the left-to-right direction, it suffices to replace each non-atomic instance of a structural rule with a suitable structurally atomic proof. Conversely, let ρ be a structural rule of the calculus. By the Normalization Theorem (Theorem 9.22) there is a structurally atomic analytic–synthetic proof of the (atomic) conclusion of ρ from the (atomic) premises of ρ. But such a proof cannot contain any logical rules. $\square$

The structural rules of $\mathbf{G}\mathcal{CL}$ satisfy the expansion property.

Proposition 9.25 (Expansion property for the standard rules).
Identity, Weakening, and Contraction satisfy the expansion property. So does the set of rules $\{$Cut, Contraction$\}$.

Proof. We prove this by induction over the complexity of the main formula of the rule, i.e. the formula denoted φ in Figures 9.1 and 9.2.

The proof for Identity is well known. For Weakening, the proof segment

$$\frac{\Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}$$

is replaced by

$$\cfrac{\cfrac{\cfrac{\Gamma \rhd \Delta}{\varphi, \Gamma \rhd \Delta}}{\varphi, \psi, \Gamma \rhd \Delta}}{\varphi \wedge \psi, \Gamma \rhd \Delta}$$

while the proof segment

$$\frac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta, \varphi \wedge \psi}$$

is replaced by

$$\cfrac{\cfrac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta, \varphi} \qquad \cfrac{\Gamma \rhd \Delta}{\Gamma \rhd \Delta, \psi}}{\Gamma \rhd \Delta, \varphi \wedge \psi}$$

For Contraction, the proof segment

$$\frac{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}$$

is replaced by

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \rhd \Delta}{\varphi, \psi, \varphi \wedge \psi, \Gamma \rhd \Delta}}{\varphi, \psi, \varphi, \psi, \Gamma \rhd \Delta}}{\varphi, \psi, \psi, \Gamma \rhd \Delta}}{\varphi, \psi, \Gamma \rhd \Delta}}{\varphi \wedge \psi, \Gamma \rhd \Delta}$$

while the proof segment

$$\cfrac{\Gamma \rhd \Delta, \varphi \wedge \psi, \varphi \wedge \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}$$

is replaced by

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \rhd \Delta, \varphi \wedge \psi, \varphi \wedge \psi}{\Gamma \rhd \Delta, \varphi, \varphi \wedge \psi}}{\Gamma \rhd \Delta, \varphi, \varphi}}{\Gamma \rhd \Delta, \varphi} \qquad \cfrac{\cfrac{\cfrac{\Gamma \rhd \Delta, \varphi \wedge \psi, \varphi \wedge \psi}{\Gamma \rhd \Delta, \psi, \varphi \wedge \psi}}{\Gamma \rhd \Delta, \psi, \psi}}{\Gamma \rhd \Delta, \psi}}{\Gamma \rhd \Delta, \varphi \wedge \psi}$$

Finally, in the case of Cut the proof segment

$$\cfrac{\Gamma \rhd \Delta, \varphi \wedge \psi \qquad \varphi \wedge \psi, \Gamma' \rhd \Delta'}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$$

is replaced by

$$\cfrac{\cfrac{\cfrac{\Gamma \rhd \Delta, \varphi \wedge \psi}{\Gamma \rhd \Delta, \psi} \qquad \cfrac{\cfrac{\Gamma \rhd \Delta, \varphi \wedge \psi}{\Gamma \rhd \Delta, \varphi} \qquad \cfrac{\varphi \wedge \psi, \Gamma' \rhd \Delta'}{\varphi, \psi, \Gamma' \rhd \Delta'}}{\psi, \Gamma, \Gamma' \rhd \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma' \rhd \Delta, \Delta, \Delta'}}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$$

where the last step condenses several instances of Contraction.

The other connectives are again either dual or simpler to handle. □

Corollary 9.26. *If a sequent has a proof from a given set of sequents in one of the calculi* $\mathbf{G}\mathcal{BD}$*,* $\mathbf{G}\mathcal{LP}$*,* $\mathbf{G}\mathcal{K}$*, or* $\mathbf{G}\mathcal{CL}$*, then it has a structurally atomic analytic–synthetic proof with the subformula property.*

Proof. This follows from the Normalization Theorem (Theorem 9.22) and Proposition 9.25. □

The above procedure for reducing structural rules to atomic structural rules and then transforming the resulting proof into a proof which is moreover analytic–synthetic subsumes a cut elimination procedure for classical propositional logic in view of Proposition 9.18.

Several remarks are now in order concerning this procedure. Firstly, it illustrates again that the admissibility of Cut is related to the problem of reducing arbitrary instances of structural rules to atomic instances, as observed already by Terui [73]. Secondly, it shows that admitting elimination rules in a Gentzen calculus may be useful even if we are only interested in sequents provable from an empty set of premises. Although the final result of this transformation applied to such a proof does not contain any instances of elimination rules, the intermediate reductions do. And thirdly, it is worth noting explicitly that the notion of a structurally atomic analytic–synthetic proof does not *a priori* single out any particular structural rule for special treatment, unlike the notion of a cut-free proof.

9.4 Expanding structural rules

In the previous section, we saw that each structurally atomic proof can easily be transformed into a proof in a certain normal form, using the fact that we have both introduction and elimination rules in the basic calculus $\mathbf{G}\mathcal{BD}$. In order to exploit this normal form, the real work therefore consists in providing a calculus in which arbitrary proofs can be transformed into structurally atomic proofs, i.e. calculi in which non-atomic instances of structural rules are derivable from atomic instances of structural rules.

We saw that for Identity and Cut, no extra work was needed. Things need not always go as smoothly, though. For example, the Explosive Cut rule cannot be replaced by its atomic version. It would be tempting to reduce the proof segment

$$\frac{\emptyset \rhd \varphi \wedge \psi \qquad \varphi \wedge \psi \rhd \emptyset}{\emptyset \rhd \emptyset}$$

to the following would-be proof segment

$$\frac{\dfrac{\emptyset \rhd \varphi \wedge \psi}{\emptyset \rhd \psi} \qquad \dfrac{\dfrac{\emptyset \rhd \varphi \wedge \psi}{\emptyset \rhd \varphi} \qquad \dfrac{\varphi \wedge \psi \rhd \emptyset}{\varphi, \psi \rhd \emptyset}}{\psi \rhd \emptyset}}{\emptyset \rhd \emptyset}$$

but of course the right-hand premise of the final inference was not obtained by an Explosive Cut. The reader may verify that if φ and ψ are distinct atoms, then the above instance of Explosive Cut is in fact not derivable in any way in $\mathbf{G}\mathcal{BD}$ using only atomic instances of the rule.

To handle such situations, we need to show how to expand a given set of structural rules into one which satisfies the expansion property. This will be the goal of the current section.

In order to define such expansions, we first introduce some auxiliary notions. We say that a sequent is *elimination-derivable (introduction-derivable)* from a set of sequents S if it is derivable from S using only elimination rules (only introduction rules). We denote this relation $S \vdash_{\text{elim}} \Gamma \rhd \Delta$ ($S \vdash_{\text{intro}} \Gamma \rhd \Delta$). The set of all atomic sequents elimination-derivable from the sequent $\Gamma \rhd \Delta$ will be denoted $\mathrm{At}(\Gamma \rhd \Delta)$.

Lemma 9.27 (Atomic decomposition of sequents).

(i) $\mathrm{At}(\varphi \wedge \psi, \Gamma \rhd \Delta) = \mathrm{At}(\varphi, \psi, \Gamma \rhd \Delta)$.

(ii) $\mathrm{At}(\Gamma \rhd \Delta, \varphi \wedge \psi) = \mathrm{At}(\Gamma \rhd \Delta, \varphi) \cup \mathrm{At}(\Gamma \rhd \Delta, \psi)$.

(iii) $\mathrm{At}(\varphi \vee \psi, \Gamma \rhd \Delta) = \mathrm{At}(\varphi, \Gamma \rhd \Delta) \cup \mathrm{At}(\psi, \Gamma \rhd \Delta)$.

(iv) $\mathrm{At}(\Gamma \rhd \Delta, \varphi \vee \psi) = \mathrm{At}(\Gamma \rhd \Delta, \varphi, \psi)$.

(v) $\mathrm{At}(-\varphi, \Gamma \rhd \Delta) = \mathrm{At}(\Gamma \rhd \Delta, \varphi)$.

(vi) $\mathrm{At}(\Gamma \rhd \Delta, -\varphi) = \mathrm{At}(\varphi, \Gamma \rhd \Delta)$.

(vii) $\mathrm{At}(\mathsf{t}, \Gamma \rhd \Delta) = \mathrm{At}(\Gamma \rhd \Delta)$.

(viii) $\mathrm{At}(\Gamma \rhd \Delta, \mathsf{t})$.

(ix) $\mathrm{At}(\Gamma \rhd \Delta, \mathsf{f}) = \mathrm{At}(\Gamma \rhd \Delta)$.

(x) $\mathrm{At}(\mathsf{f}, \Gamma \rhd \Delta) = \emptyset$.

Proof. The right-to-left inclusions are trivial. To prove the converse inclusions, it suffices to replace $\emptyset \rhd \emptyset$ by a given atomic sequent in the proof of the antiadmissibility of introduction rules (Proposition 9.10). □

Lemma 9.28 (Elimination-derivability).
$\mathrm{At}(\sigma(\Gamma \rhd \Delta)) \subseteq \mathrm{At}(\sigma[\mathrm{At}(\Gamma \rhd \Delta)])$.

Proof. We prove the claim by induction over the complexity of the sequent $\Gamma \rhd \Delta$, i.e. the number of connectives in it. If $\Gamma \rhd \Delta$ is atomic, the claim holds trivially by the definition of $\mathrm{At}(\sigma(\Gamma \rhd \Delta))$. Now consider sequents of the form $\Gamma \rhd \Delta, \varphi \wedge \psi$. By the previous lemma

$$\mathrm{At}(\sigma(\Gamma \rhd \Delta, \varphi \wedge \psi)) = \mathrm{At}(\sigma(\Gamma \rhd \Delta, \varphi)) \cup \mathrm{At}(\sigma(\Gamma \rhd \Delta, \psi)).$$

By the induction hypothesis

$$\begin{aligned} \mathrm{At}(\sigma(\Gamma \rhd \Delta, \varphi)) &\subseteq \mathrm{At}(\sigma[\mathrm{At}(\Gamma \rhd \Delta, \varphi)]), \\ \mathrm{At}(\sigma(\Gamma \rhd \Delta, \psi)) &\subseteq \mathrm{At}(\sigma[\mathrm{At}(\Gamma \rhd \Delta, \psi)]). \end{aligned}$$

Moreover, $\mathrm{At}(\Gamma \rhd \Delta, \varphi) \cup \mathrm{At}(\Gamma \rhd \Delta, \psi) \subseteq \mathrm{At}(\Gamma \rhd \Delta, \varphi \wedge \psi)$, therefore $\mathrm{At}(\sigma(\Gamma \rhd \Delta, \varphi \wedge \psi)) \subseteq \mathrm{At}(\sigma[\mathrm{At}(\Gamma \rhd \Delta, \varphi \wedge \psi)])$.

The remaining cases are either analogous or simpler. □

Lemma 9.29 (Introduction-derivability).
$\mathrm{At}(\sigma[\Gamma \rhd \Delta]) \vdash_{intro} \sigma[\mathrm{At}(\Gamma \rhd \Delta)]$.

Proof. We prove the claim by induction over the complexity of $\Gamma \rhd \Delta$. If $\Gamma \rhd \Delta$ is atomic, we are to show that $\mathrm{At}(\Lambda \rhd \Pi) \vdash_{\mathrm{intro}} \Lambda \rhd \Pi$ for each sequent $\Lambda \rhd \Pi$. This can be proved by a straightforward induction on the complexity of $\Lambda \rhd \Pi$. Now consider sequents of the form $\Gamma \rhd \Delta, \varphi \wedge \psi$. By Lemma 9.27

$$\mathrm{At}(\sigma[\Gamma \rhd \Delta, \varphi \wedge \psi]) = \mathrm{At}(\sigma[\Gamma \rhd \Delta, \varphi]) \cup \mathrm{At}(\sigma[\Gamma \rhd \Delta, \psi]).$$

But by the induction hypothesis

$$\mathrm{At}(\sigma[\Gamma \rhd \Delta, \varphi]) \vdash_{\mathrm{intro}} \sigma[\mathrm{At}(\Gamma \rhd \Delta, \varphi)],$$
$$\mathrm{At}(\sigma[\Gamma \rhd \Delta, \psi]) \vdash_{\mathrm{intro}} \sigma[\mathrm{At}(\Gamma \rhd \Delta, \psi)],$$

and again by Lemma 9.27

$$\sigma[\mathrm{At}(\Gamma \rhd \Delta, \varphi \wedge \psi)] = \sigma[\mathrm{At}(\Gamma \rhd \Delta, \varphi)] \cup \sigma[\mathrm{At}(\Gamma \rhd \Delta, \psi)].$$

Thus $\mathrm{At}(\sigma[\Gamma \rhd \Delta, \varphi \wedge \psi]) \vdash_{\mathrm{intro}} \sigma[\mathrm{At}(\Gamma \rhd \Delta)]$.

The remaining cases are again either analogous or simpler. □

Fact 9.30. *Each sequent $\Gamma \rhd \Delta$ is equivalent to $\mathrm{At}(\Gamma \rhd \Delta)$.*

Proof. Consider an atomic sequent $\Lambda \rhd \Pi$ and a substitution σ such that $\sigma[\Lambda \rhd \Pi] = \Gamma \rhd \Delta$. The claim now holds by Lemma 9.29. □

Given a structural rule ρ, we now provide a syntactically defined set of rules which contains ρ, satisfies the expansion property, and moreover each of the rules is valid in each logic which validates ρ.

Definition 9.31.
Let $\{\Gamma_i \rhd \Delta_i \mid i \in I\} \vdash \Gamma \rhd \Delta$ be a structural rule and σ be a substitution. Then a *σ-expansion* of this structural rule is a structural rule of the form $\bigcup_{i \in I} \mathrm{At}(\sigma(\Gamma_i \rhd \Delta_i)) \vdash \Lambda \rhd \Pi$ for $\Lambda \rhd \Pi$ in $\mathrm{At}(\sigma(\Gamma \rhd \Delta))$.

For example, the left-hand version of the Limited Cut rule in Figure 9.2 is a schema standing for a set of structural rules which contains the rule

$$\frac{\emptyset \rhd p \qquad p, r \rhd s}{r \rhd s}$$

whose σ-expansion for $\sigma(p) = p \wedge q$ and $\sigma(r) = r$ and $\sigma(s) = s$ is the rule

$$\frac{\emptyset \rhd p \qquad \emptyset \rhd q \qquad p, q, r \rhd s}{r \rhd s}$$

interpreted as

$$\frac{\emptyset \rhd \varphi \qquad \emptyset \rhd \psi \qquad \varphi, \psi, \Gamma \rhd \Delta}{\Gamma \rhd \Delta}$$

The following observations are now immediate.

Fact 9.32. *If a structural rule is valid in a super-Belnap Gentzen relation, then so are all of its σ-expansions. Conversely, if the σ_{id}-expansion of the rule is valid, where σ_{id} is the identity substitution, then so is the rule itself.*

Proof. The claim follows from structurality because each sequent $\Gamma \rhd \Delta$ is equivalent to $\mathrm{At}(\Gamma \rhd \Delta)$. □

We will in fact be interested in expansions of a particular kind. In the following definition, positive and negative occurrences of atoms are defined inductively as expected, e.g. the atom p occurs negatively and the atom q occurs positively in the formula $-p \vee q$.

Definition 9.33 (Balanced and separating substitutions).
A formula is *balanced* if each atom occurs only positively or only negatively in it. A substitution σ is *balanced* if the formula $\sigma(p)$ is balanced for each atom p, *separating* if $\sigma(p)$ and $\sigma(q)$ do not share any variables for distinct atoms p and q, and *atomic* if $\sigma(p)$ is an atom for each atom p.

A *balanced (separating) expansion* of a structural rule is a σ-expansion of the rule for some balanced (separating) substitution σ.

Lemma 9.34.
Each substitution σ is the composition $\sigma_{sa} \circ \sigma_{bs}$ of a balanced separating substitution and a surjective atomic substitution.

Proof. Let σ_p and τ_p for each atom p be atomic substitutions such that $(\tau_p \circ \sigma_p)(q) = q$ for each atom q and moreover the ranges of σ_p and σ_q are disjoint for distinct atoms p and q. Let γ_p and δ_p be atomic substitutions such that $(\delta_p \circ \gamma_p)(q) = q$ for each q and the ranges of γ_p and γ_q are distinct for distinct atoms p and q and moreover the ranges of γ_p and σ_q are disjoint for all atoms p and q. Suppose also that there are κ atoms which do not lie in the range of any of the functions σ_p or γ_p, where κ is the cardinality of the set of all atoms. Such substitutions always exist, since each set of cardinality κ may be decomposed into κ disjoint subsets of cardinality κ.

Now given a substitution σ we define a separating substitution σ_{s} so that $\sigma_{\mathrm{s}}(p) = (\sigma_p \circ \sigma)(p)$. When then modify σ_{s} to obtain a balanced separating substitution σ_{bs} by changing each negative occurrence of a variable q in $\sigma_{\mathrm{s}}(p)$ to $\gamma_p(q)$. The substitution σ_{sa} may now be defined so that $\sigma_{\mathrm{sa}}(q) = \tau_p(q)$ whenever q is in the range of σ_p and $\sigma_{\mathrm{sa}}(q) = (\tau_p \circ \delta_p)(q)$ whenever q is in the (disjoint) range of γ_p. Moreover, we may define $\sigma_{\mathrm{sa}}(q)$ for q outside the ranges of these functions so that σ_{sa} is a surjective atomic substitution. □

Proposition 9.35 (Balanced separating expansions).
The set of all balanced separating expansions of a structural rule satisfies the expansion property and is equivalent to the original rule.

Proof. Let $\{\Gamma_i \rhd \Delta_i \mid i \in I\} \vdash \Gamma \rhd \Delta$ be a given structural rule. We have already observed that if this rule is valid in $\mathrm{G}\mathcal{L}$, so are all of its expansions, and conversely the rule holds whenever its σ_{id}-expansion does. Since σ_{id} is a balanced separating substitution, it suffices to prove the expansion property.

Let us write $S \vdash_{\mathrm{at}} \Gamma \rhd \Delta$ to abbreviate the claim that $\Gamma \rhd \Delta$ has a proof from S which in addition to the rules of $\mathbf{G}\mathcal{BD}$ only uses atomic instances of a balanced separating expansions of the given rule. We are to show that

$$\bigcup_{i \in I} \tau[\mathrm{At}(\sigma(\Gamma_i \rhd \Delta_i))] \vdash_{\mathrm{at}} \tau[\mathrm{At}(\sigma(\Gamma \rhd \Delta))]$$

for each instance given by τ of each σ-expansion of the given rule, where σ is a balanced separating expansion.

By the previous lemma there is a balanced separating substitution τ_{bs} and a surjective atomic substitution τ_{sa} such that $\tau = \tau_{\mathrm{sa}} \circ \tau_{\mathrm{bs}}$. Observe that $\tau_{\mathrm{bs}} \circ \sigma$ is also a balanced separating expansion.

By Lemma 9.28 we have

$$\tau_{\mathrm{bs}}[\mathrm{At}(\sigma(\Gamma_i \rhd \Delta_i))] \vdash_{\mathrm{elim}} \mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma_i \rhd \Delta_i)),$$

therefore the structurality of the relation $\vdash_{\mathrm{elim}}$ yields that

$$\tau[\mathrm{At}(\sigma(\Gamma_i \rhd \Delta_i))] \vdash_{\mathrm{elim}} \tau_{\mathrm{sa}}[\mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma_i \rhd \Delta_i))].$$

But by the definition of the relation $\vdash_{\mathrm{at}}$ we have

$$\bigcup_{i \in I} \mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma_i \rhd \Delta_i)) \vdash_{\mathrm{at}} \mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma \rhd \Delta)),$$

and because τ_{sa} is an atomic substitution also

$$\tau_{\mathrm{sa}}[\bigcup_{i \in I} \mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma_i \rhd \Delta_i))] \vdash_{\mathrm{at}} \tau_{\mathrm{sa}}[\mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma \rhd \Delta))].$$

Finally, Lemma 9.29 yields that

$$\tau_{\mathrm{sa}}[\mathrm{At}((\tau_{\mathrm{bs}} \circ \sigma)(\Gamma \rhd \Delta))] \vdash_{\mathrm{intro}} \tau[\mathrm{At}(\sigma(\Gamma \rhd \Delta))],$$

since $\tau = \tau_{\mathrm{sa}} \circ \tau_{\mathrm{bs}}$. Composing the above consequences now yields a proof of $\tau[\mathrm{At}(\sigma(\Gamma \rhd \Delta))]$ from $\bigcup_{i \in I} \tau[\mathrm{At}(\sigma(\Gamma_i \rhd \Delta_i))]$ which in addition to the rules of $\mathbf{G}\mathcal{BD}$ only uses atomic instances of balanced separating expansions of the given rule. □

To transform a given super-Belnap calculus into an equivalent calculus which satisfies the expansion property, it therefore suffices to extend the calculus by all balanced separating expansions of all structural rules to the calculus. Although this may seem like a brute force solution, we shall see in the following section that it will in fact allow us to establish interpolation theorems for all super-Belnap logics axiomatized by what we call generalized cut rules, including $\mathcal{K}$, $\mathcal{ETL}$, and $\mathcal{SDS}_n$.

9.5 Interpolation in super-Belnap logics

We now apply the results of the previous section to obtain new interpolation theorems as well as new proofs of known interpolation theorems for super-Belnap logics. Our main contribution is to show that each super-Belnap logic axiomatized by a so-called generalized cut rule enjoys a strong form of interpolation. In particular, this holds for the logics $\mathcal{K}$, $\mathcal{ETL}$, and $\mathcal{SDS}_n$. We also provide new proofs of the refinement of the interpolation theorem for $\mathcal{LP}$, $\mathcal{CL}$, and $\mathcal{K}$ obtained recently by Milne [47, 48].

We say that a logic has the *(simple) Craig interpolation property*, or briefly *has interpolation*, if $\varphi \vdash_{\mathcal{L}} \psi$ implies the existence of a formula χ called the *interpolant* of φ and ψ such that $\varphi \vdash_{\mathcal{L}} \chi$ and $\chi \vdash_{\mathcal{L}} \varphi$ and each atom which occurs in χ occurs in both φ and ψ. We prove by a simple argument that the logics $\mathcal{BD}$, $\mathcal{K}$, $\mathcal{ETL}$ and some others enjoy interpolation, in fact in a somewhat stronger form.

Let $\mathcal{L}$, $\mathcal{L}_1$, and $\mathcal{L}_2$ be logics such that $\mathcal{L}_1, \mathcal{L}_2 \leq \mathcal{L}$. We say that $\mathcal{L}$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-*interpolation* if $\varphi \vdash_{\mathcal{L}} \psi$ implies the existence of an interpolant χ such that $\varphi \vdash_{\mathcal{L}_1} \chi$ and $\chi \vdash_{\mathcal{L}_2} \psi$ and each atom which occurs in χ occurs in both φ and ψ. In that case $\mathcal{L} = \mathcal{L}_1 \vee \mathcal{L}_2$. Clearly $(\mathcal{L}, \mathcal{L})$-interpolation for $\mathcal{L}$ amounts precisely to interpolation for $\mathcal{L}$.

Fact 9.36. *Let $\mathcal{L}$ be an extension $\mathcal{B}$ such that $\mathsf{f} \vdash_{\mathcal{B}} \emptyset$ for some constant formula f. If $\mathcal{L}$ has interpolation or $(\mathcal{L}, \mathcal{B})$-interpolation, then so do all of its explosive extensions.*

Proof. Let $\mathcal{L}_{\mathrm{exp}}$ be an explosive extension of $\mathcal{L}$. If $\varphi \vdash_{\mathcal{L}_{\mathrm{exp}}} \psi$, then either $\varphi \vdash_{\mathcal{L}} \psi$ or $\varphi \vdash_{\mathcal{L}_{\mathrm{exp}}} \emptyset$. In the former case the existence of the interpolant is guaranteed by the assumption that $\mathcal{L}$ has interpolation or $(\mathcal{L}, \mathcal{B})$-interpolation, in the latter case we may take f as the interpolant. □

Let us now review what is known about interpolation in super-Belnap logics. Interpolation for the Dunn–Belnap logic $\mathcal{BD}$ was proved early on by Anderson and Belnap [3, p. 161]. Interpolation for the strong three-valued Kleene logic $\mathcal{K}$ was proved much later using by Bendová [10]. Milne offered an alternative proof in [47, 48].[4] In the same paper, Milne also observed that

[4]The papers [10] and [47, 48] in fact consider the constant-free fragment of $\mathcal{K}$, therefore

interpolation for $\mathcal{LP}$ is equivalent to interpolation for $\mathcal{K}$ in view of the fact that $(\mathcal{K}, \mathcal{LP})$ is a contrapositive pair (in the sense of Theorem 3.3). Moreover, he proved that classical logic $\mathcal{CL}$ enjoys $(\mathcal{K}, \mathcal{LP})$-interpolation, $\mathcal{K}$ enjoys $(\mathcal{K}, \mathcal{BD})$-interpolation, and $\mathcal{LP}$ enjoys $(\mathcal{BD}, \mathcal{LP})$-interpolation (in fact, he proved these claims for the first-order versions of these logics). Bendová also observed in her paper that the logic $\mathcal{KO}$ does not enjoy interpolation: the rule $(p \wedge -p) \vee r \vdash (q \vee -q) \vee r$ is valid in $\mathcal{KO}$ but lacks an interpolant. The same example shows that $\mathcal{KO} \vee \mathcal{ECQ}$ does not enjoy interpolation either.

As far as we are aware, this exhausts the present state of knowledge about interpolation in super-Belnap logics. To the best of our knowledge, interpolation has so far not been studied in other super-Belnap logics, which have only been introduced quite recently.

Proposition 9.37 (Interpolation in super-Belnap logics).
If a super-Belnap logic $\mathcal{L}$ enjoys interpolation, then either $\mathcal{L} = \mathcal{BD}$ or $\mathcal{L} = \mathcal{LP}$ or $\mathcal{ECQ} \leq \mathcal{L}$.

Proof. Each proper extension $\mathcal{L}$ of $\mathcal{BD}$ satisfies the rule $p, -p \vdash_{\mathcal{L}} q \vee -q$ (Proposition 6.4). Only a variable-free formula may be an interpolant of this rule, and all such formulas are equivalent in $\mathcal{BD}$ to either $\mathtt{f}$ or $\mathtt{t}$. Therefore either $p, -p \vdash_{\mathcal{L}} \mathtt{f}$ or $\mathtt{t} \vdash_{\mathcal{L}} q \vee -q$, i.e. either $\mathcal{LP} \leq \mathcal{L}$ or $\mathcal{ECQ} \leq \mathcal{L}$. But $\mathcal{LP} < \mathcal{L}$ already implies $\mathcal{ECQ} \leq \mathcal{LP}$ (Proposition 6.9). □

Taking into account that there is a continuum of finitary explosive extensions of $\mathcal{BD}$ as well as a continuum of finitary logics in the interval $[\mathcal{BD}, \mathcal{LP}]$ (see Section 7.4), we obtain the following fact.

Fact 9.38. *There is a continuum of finitary super-Belnap logics with interpolation, as well as a continuum of such logics without interpolation.*

The interpolation properties defined above extend naturally to Gentzen relations. To obtain the appropriate definitions for Gentzen relations, it suffices to replace the formulas φ, ψ, and χ by sequents.

Fact 9.39. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be super-Belnap logic. Then $\mathcal{L}_0$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-interpolation if and only if $\mathrm{G}\mathcal{L}_0$ enjoys $(\mathrm{G}\mathcal{L}_1, \mathrm{G}\mathcal{L}_2)$-interpolation.*

Proof. This holds because a variable occurs in $\boldsymbol{\tau}(\Gamma \rhd \Delta)$ if and only if it occurs in $\Gamma \rhd \Delta$, and it occurs in $\boldsymbol{\rho}(\varphi)$ if and only if it occurs in φ. □

they have to formulate the interpolation property with more care. However, ordinary interpolation for $\mathcal{K}$ is a straightforward consequence of their interpolation results. We consider this to be yet another reason to include the truth constants in the signature of super-Belnap logics. Likewise, Anderson and Belnap in fact consider the corresponding fragment of $\mathcal{BD}$, although in their case no adjustments are needed in the definition of the interpolation property.

Note that in order to establish interpolation for a Gentzen version of a super-Belnap logic it suffices by Proposition 9.4 to find a *set* of sequents which jointly plays the role of the interpolant. We now provide a broad sufficient condition for a super-Belnap logic $\mathcal{L}$ to enjoy $(\mathcal{L}, \mathcal{BD})$-interpolation and therefore ordinary interpolation.

Definition 9.40 (Cut formulas and side formulas).
A *cut formula* of a structural rule is a formula which only occurs in the premises of the rule. A *side formula* of a structural rule is a formula which occurs only on the left-hand sides or only on the right-hand sides of sequents in the rule.

A cut formula (a side formula) of an instance of a structural rule is the appropriate instance of the atomic cut formula (the atomic side formula). It may happen, although this case is not very interesting, that a formula is both a cut formula and a side formula of a structural rule.

Definition 9.41 (Generalized cut rules).
A *generalized cut rule* is a structural rule such that each formula which occurs in the rule is either a cut formula or a side formula.

For example, Limited Cut and Explosive Cut are generalized cut rules, whereas Identity and the rule for $\mathcal{KO}$ combining Identity and Cut are not.

Definition 9.42 (Introducing new variables).
A rule *does not introduce new variables* if all variables which occur in the conclusion also occur in some of the premises.

In particular, a generalized cut rule does not introduce new variables, and neither do any of the elimination rules. As far as interpolation goes, Weakening is the only problematic rule which may introduce new variables.

Lemma 9.43.
If a structural rule is a generalized cut rule, then so are all of its balanced separating expansions.

Proof. Let $\rho := \{\Gamma_i \rhd \Delta_i \mid i \in I\} \vdash \Gamma \rhd \Delta$ be a generalized cut rule and let σ be a balanced separating substitution. It is easy to prove that an atom which only occurs positively in a sequent $\Lambda \rhd \Pi$ (i.e. only occurs positively in formulas in Π and only occurs negatively in formulas in Λ) will only occur on the right-hand side of each sequent in $\mathrm{At}(\Lambda \rhd \Pi)$, and likewise an atom which only occurs negatively in $\Lambda \rhd \Pi$ will only occur on the left-hand side of each sequent in $\mathrm{At}(\Lambda \rhd \Pi)$. Therefore if p is a side formula in ρ, then each atom of $\sigma(p)$ will be a side formula of the σ-expansion of ρ, using the fact that σ is balanced and separating. It is also easy to observe that $\mathrm{At}(\Gamma \rhd \Delta)$ and $\Gamma \rhd \Delta$ contain exactly the same atoms. Therefore if p is a cut formula of ρ, then each atom of $\sigma(p)$ will be a cut formula of the σ-expansion of ρ, using again the fact that σ is separating. □

Theorem 9.44
(General Interpolation Theorem for super-Belnap logics).
Each super-Belnap logic $\mathcal{L}$ such that G$\mathcal{L}$ *extends* G$\mathcal{BD}$ *by a set of generalized cut rules enjoys* $(\mathcal{L}, \mathcal{BD})$*-interpolation.*

Proof. If G$\mathcal{L}$ is the extension of G$\mathcal{BD}$, by a set of generalized cut rules, consider the calculus obtained by adding all balanced separating expansions of these generalized cut rules to **G**$\mathcal{BD}$. By Proposition 9.35 this yields a calculus for G$\mathcal{L}$ which satisfies the expansion property.

Suppose that a sequent $\Gamma \rhd \Delta$ is provable from S in this calculus. By Normalization Theorem (Theorem 9.22) it has a structurally atomic analytic–synthetic proof. Moreover, Weakening is the only structural rule of this calculus which introduces new variables.

Let us call a node in this proof *critical* if all inferences above the node are elimination rules or structural rules and all inferences below are introduction rules. Each branch of the proof either intersects a critical node or terminates in a logical axiom. If each critical node only contains variables which occur in some premise of the proof, then the set of critical nodes forms an interpolant between S and $\Gamma \rhd \Delta$, since each variable which occurs in a critical sequent also occurs in the conclusion.

To prove the theorem, it therefore suffices to show that if p does not occur in S and an atomic sequent $\Lambda \rhd \Pi, p$ or $p, \Lambda \rhd \Pi$ has a structurally atomic analytic–synthetic proof from S, then so does $\Lambda \rhd \Pi$. This is because each critical node $\Lambda \rhd \Pi$ may then be transformed into a sequent $\Lambda' \rhd \Pi'$ which only contains variables which occur in S by finitely many applications of this transformation, and moreover the sequent $\Lambda \rhd \Pi$ is derivable from $\Lambda' \rhd \Pi'$ using finitely many atomic instances of Weakening. This transformation may be performed on all critical nodes simultaneously, since no branch of the proof contains two such nodes.

Thus, consider an atom p which does not occur in S and a sequent $\Lambda \rhd \Pi, p$ or $p, \Lambda \rhd \Pi$ which has a structurally atomic analytic–synthetic proof from S in the calculus. The tree of all ancestors of this instance of p is defined in the obvious way. The leaves of this tree must be the results of Weakening, since no other rule in the proof above $\Lambda \rhd \Pi$ introduces new variables. Crucially, the only structural rules of the calculus are Weakening, Contraction, and generalized cut rules, therefore the appropriate ancestor of p is a side formula of each structural rule where it occurs in the premises. It is now immediate that we may delete all the ancestors of this instance of p from the subproof, and obtain a proof of $\Lambda \rhd \Pi$. □

This theorem yields a new proof of known interpolation theorems, as well as a proof of new interpolation theorems covering most of the logics introduced so far.

Proposition 9.45 [3, 47, 48] **(Interpolation for $\mathcal{BD}$, $\mathcal{K}$, and $\mathcal{LP}$).**

(i) $\mathcal{BD}$ enjoys interpolation.

(ii) $\mathcal{K}$ enjoys $(\mathcal{K}, \mathcal{BD})$-interpolation.

(iii) $\mathcal{LP}$ enjoys $(\mathcal{BD}, \mathcal{LP})$ interpolation.

Proof. (i) and (ii) follow immediately from the previous theorem, while (iii) follows from (ii) because $\langle \mathcal{K}, \mathcal{LP} \rangle$ is a contrapositive pair (Theorem 3.3). $\square$

Proposition 9.46 (Interpolation for $\mathcal{ETL}$, $\mathcal{SDS}_n$, and $\mathcal{LP} \vee \mathcal{ECQ}$).

(i) $\mathcal{ETL}$ enjoys $(\mathcal{ETL}, \mathcal{BD})$-interpolation.

(ii) $\mathcal{SDS}_n$ enjoys $(\mathcal{SDS}_n, \mathcal{BD})$-interpolation.

(iii) $\mathcal{LP} \vee \mathcal{ECQ}$ enjoys $(\mathcal{LP} \vee \mathcal{ECQ}, \mathcal{LP})$-interpolation.

Proof. (i) and (ii) follow immediately from the previous theorem, while (iiii) holds because $\mathcal{LP}$ enjoys $(\mathcal{LP}, \mathcal{LP})$-interpolation and $\mathcal{LP} \vee \mathcal{ECQ}$ is an explosive extension of $\mathcal{LP}$. $\square$

Moreover, a slight modification of the proof of the General Interpolation Theorem for super-Belnap logics (Theorem 9.44) yields a syntactic proof of Milne's non-classical refinement of the Craig interpolation theorem for classical logic. The reader is encouraged to compare this proof with the standard syntactic proof of interpolation for classical logic based on the cut elimination theorem, found e.g. in [74, Section 4.4.2]

Proposition 9.47 ([47, 48]).
The logic $\mathcal{CL}$ enjoys $(\mathcal{K}, \mathcal{LP})$-interpolation.

Proof. Recall that the calculus $\mathbf{G}\mathcal{CL}$ extends $\mathbf{G}\mathcal{BD}$ by Identity and Cut. We wish to separate each structurally atomic analytic–synthetic proof in this calculus into a part which contains no instances of Cut and a part which contains no instances of Identity. To this end, suppose that a branch of the proof contains an instance of an Identity rule followed at some point by a Cut, and suppose that no other instances of Cut occur between these two rules. Then only Weakening and Contraction may occur between these two rules, therefore one of the premises of the Cut has the form $p, \Gamma \rhd \Delta, p$. If the cut formula of such an instance of Cut is p, this instance of Cut may be replaced by Weakening. If it is some other formula, then the conclusion of the cut has the form $p, \Gamma' \rhd \Delta', p$ and thus may be derived using Identity and Weakening only. Repeated applications of this transformation yield a structurally atomic analytic–synthetic proof in which there is no branch containing both Identity and Cut.

Define the *separating set* of this proof as the set of all sequents in the proof such that only elimination rules and structural rules other than Identity occur above them and only introduction rules occur below. Each branch of the proof either intersects the separating set or ends with an instance of Identity (or one of the axioms $\emptyset \rhd \mathsf{t}$ or $\mathsf{f} \rhd \emptyset$). Moreover, as in the proof of the General Interpolation Theorem for super-Belnap logics (Theorem 9.44), each variable in the separating set must occur both in the premises and in the conclusion of the proof. The separating set therefore again jointly plays the role of the $(\mathcal{K}, \mathcal{LP})$-interpolant. □

It is easy to see that for $\mathcal{CL}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{ETL}$ the interpolation theorems above are optimal in a natural sense. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be super-Belnap logics.

Proposition 9.48.

(i) If $\mathcal{CL}$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-interpolation, then $\mathcal{K} \leq \mathcal{L}_1$ and $\mathcal{LP} \leq \mathcal{L}_2$.

(ii) If $\mathcal{LP}$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-interpolation, then $\mathcal{L}_2 = \mathcal{LP}$.

(iii) If $\mathcal{K}$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-interpolation, then $\mathcal{L}_1 = \mathcal{K}$.

(iv) If $\mathcal{ETL}$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-interpolation, then $\mathcal{L}_1 = \mathcal{ETL}$.

Proof. The logics $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-trivial in each case. Observe that each formula which does not contain any variable other than q is equivalent in $\mathcal{BD}$ to $q \wedge -$, q, $-q$, $q \vee -q$, t, or f.

Since $(p \wedge -p) \vee q \vdash_{\mathcal{CL}} q$, we have $(p \wedge -p) \vee q \vdash_{\mathcal{L}_1} \chi$ and $\chi \vdash_{\mathcal{L}_2} q$ for some χ which does not contain any variable other than q. Then clearly $\chi \dashv\vdash_{\mathcal{BD}} q$, hence $(p \wedge -p) \vee q \vdash_{\mathcal{L}_1} q$ and $\mathcal{K} \leq \mathcal{L}_1$. Likewise, since $\emptyset \vdash_{\mathcal{CL}} p \vee -p$, we have $\emptyset \vdash_{\mathcal{L}_1} \chi$ and $\chi \vdash_{\mathcal{L}_2} p \vee -p$ for some χ which does not contain any variables. Then clearly $\chi \dashv\vdash_{\mathcal{BD}} \mathsf{t}$, hence $\mathsf{t} \vdash_{\mathcal{L}_2} p \vee -p$ and $\mathcal{LP} \leq \mathcal{L}_2$. The same argument shows that $\mathcal{LP} \leq \mathcal{L}_2$ in the case of $\mathcal{LP}$.

Finally, since $(p \wedge -p) \vee q \vdash_{\mathcal{K}} q$, we have $(p \wedge -p) \vee q \vdash_{\mathcal{K}} q$, we have $(p \wedge -p) \vee q \vdash_{\mathcal{L}_1} \chi$ and $\chi \vdash_{\mathcal{L}_2} q$ for some χ which does not contain any variable other than q. Then clearly $\chi \dashv\vdash_{\mathcal{BD}} q$, hence $(p \wedge -p) \vee q \vdash_{\mathcal{L}_1} q$ and $\mathcal{K} \leq \mathcal{L}_1$. The argument for $\mathcal{ETL}$ is entirely analogous. □

Chapter 10

Other frameworks

In this chapter we consider some modifications of the notion of a super-Belnap logic, namely constant-free super-Belnap logics, multiple-conclusion super-Belnap logics, and variants of $\mathcal{BD}$ with more than one predicate (e.g. containing both a truth and an exact truth predicate).

The super-Belnap landscape only changes marginally if we drop the constants $\mathtt{t}$ and $\mathtt{f}$ from the signature: this yields exactly four logics which are not constant-free reducts of super-Belnap logics, and they are in fact all trivial variants of ordinary super-Belnap logics. By contrast, moving to the multiple-conclusion framework essentially amounts to restricting to super-Belnap logics with the proof by cases property, i.e. to the logics $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$. Finally, we axiomatize the logics (in a generalized sense of the word) obtained by considering the exact truth predicate or the non-falsity predicate in addition to the truth predicate on $\mathbf{DM_4}$ and $\mathbf{K_3}$.

10.1 Constant-free logics

Let us first focus on *constant-free super-Belnap logics*, i.e. extensions of the constant-free fragment of $\mathcal{BD}$. We prove that moving to the constant-free framework yields exactly four new logics. Moreover, these four logics are trivial variants of the logics $\mathcal{LP}$, $\mathcal{LP} \vee \mathcal{ECQ}$, $\mathcal{CL}$, and $\mathcal{TRIV}$. The logic $\mathcal{LP}_-$ is the extension of constant-free $\mathcal{BD}$ by the rule $p \vdash q \vee -q$, then there are the logics $\mathcal{LP}_- \vee \mathcal{ECQ}$ and $\mathcal{CL}_- := \mathcal{LP}_- \vee \mathcal{ETL}$, and finally the almost trivial logic $\mathcal{TRIV}_-$ axiomatized by the rule $p \vdash q$.

The consequence relations of these logics only differ from those of their counterparts with constants in not having theorems. That is, $\Gamma \vdash_{\mathcal{LP}_-} \varphi$ if and only if Γ is non-empty and $\Gamma \vdash_{\mathcal{LP}} \varphi$, and similarly for the other logics.

Theorem 10.1 (Constant-free super-Belnap logics).
There are precisely four constant-free super-Belnap logics which are not constant-free fragments of super-Belnap logics, namely $\mathcal{LP}_-$, $\mathcal{LP}_- \vee \mathcal{ECQ}$, $\mathcal{CL}_-$, and $\mathcal{TRIV}_-$.

Proof. The reader should recall the distinction between De Morgan lattices and De Morgan algebras introduced in Chapter 2 (De Morgan algebras).

Let $\mathcal{L}$ be a constant-free super-Belnap logic. If $\mathcal{LP}_{-} \leq \mathcal{L}$, then each model of $\mathcal{L}$ which is not almost trivial is in fact a model of $\mathcal{LP}$. Therefore each extension of $\mathcal{LP}_{-}$ is either an extension of $\mathcal{LP}$ or it is the intersection of an extension of $\mathcal{LP}$ and $\mathcal{TRIV}_{-}$. But Pynko [64, Thm 4.13] has shown that $\mathcal{L}$ is the constant-free fragment of $\mathcal{LP}$, $\mathcal{LP} \vee \mathcal{ECQ}$, $\mathcal{CL}$, or $\mathcal{TRIV}$. The intersections of these logics with $\mathcal{TRIV}_{-}$ are precisely $\mathcal{LP}_{-}$, $\mathcal{LP}_{-} \vee \mathcal{ECQ}$, $\mathcal{CL}_{-}$, and $\mathcal{TRIV}_{-}$.

If $\mathcal{LP}_{-} \nleq \mathcal{L}$, then there is a reduced model $\langle \mathbf{A}, F \rangle$ of $\mathcal{L}$ such that $b \in F$ and $a \vee -a \notin F$ for some $a, b \in \mathbf{A}$. The algebra $\mathbf{A}$ is a De Morgan lattice and F is a lattice filter on $\mathbf{A}$ by [25, Thm 3.14]. But then the four-element De Morgan subchain of $\mathbf{A}$ with only the top element designated is a submatrix of $\langle \mathbf{A}, F \rangle$ with the universe $\{a \wedge -a \wedge -b, a \wedge -a, a \vee -a, a \vee -a \vee b\}$, and the Leibniz reduction of this submatrix is $\mathbb{K}_{\mathbf{3}}$. Therefore $\mathcal{L} \leq \mathcal{K}$. (More precisely, here $\mathcal{K}$ of course denotes the constant-free fragment of $\mathcal{K}$.)

If $\mathcal{L} \leq \mathcal{K}$, then $\mathcal{L}$ is complete with respect the subclass of its models which are neither trivial nor almost trivial, since the undesignated singleton is a submatrix of $\mathbb{K}_{\mathbf{3}}$. It now suffices to show that each model $\langle \mathbf{A}, F \rangle$ of $\mathcal{L}$ with $\mathbf{A} \in \mathsf{DML}$ which is neither trivial nor almost trivial embeds into some model $\langle \mathbf{B}, G \rangle$ of $\mathcal{L}$ with $\mathbf{B} \in \mathsf{DMA}$. To do so, we use the simplest possible construction: let $\mathbf{B}$ be the De Morgan algebra obtained by adding a new (designated) top and (undesignated) bottom element to $\mathbf{A}$. We now need to show that $\langle \mathbf{B}, G \rangle$ is a model of $\mathcal{L}$.

Suppose therefore that $\Gamma \vdash \varphi$ fails in $\langle \mathbf{B}, G \rangle$ as witnessed by some valuation $v : \mathbf{Fm} \to \mathbf{B}$. We need to show that it also fails in $\langle \mathbf{A}, F \rangle$. We may assume without loss of generality that $\Gamma \cup \{\varphi\}$ only consists of disjunctive clauses. If there are no atoms p with $v(p) \in \{\mathsf{t}, \mathsf{f}\}$, then we are done. If there are atoms q with $v(q) = \mathsf{f}$, we substitute them by their negations. Then we apply a substitution which unifies all atoms p with $v(p) = \mathsf{t}$. We may therefore assume without loss of generality that $v(p) = \mathsf{t}$ for exactly one atom, say for $p = p_0$, and $v(p) \in \mathbf{A}$ otherwise.

The rule Γ then has the form $\Delta, p_0 \vee \Pi, -p_0 \vee \Sigma \vdash -p_0 \vee \psi$ or the form $\Delta, p_0 \vee \Pi, -p_0 \vee \Sigma \vdash \psi$ for some Δ, Π, Σ, ψ such that p_0 does not occur in Σ and ψ. Let $a := v(\psi)$ and $b \in F$ and consider the valuation $w : \mathbf{Fm} \to \mathbf{A}$ such that $w(p_0) := -a \vee b$ and $w(p) := v(p)$ otherwise. Then w witnesses the failure of the rule $\Gamma \vdash \varphi$ in $\langle \mathbf{A}, F \rangle$: $w[p_0 \vee \Pi] \subseteq F$ because $w(p_0) \in F$, $w[-p_0 \vee \Sigma] \subseteq F$ because $w[\Sigma] = v[\Sigma] = v[p_0 \vee \Sigma] \subseteq F$, and either $w(\psi) = v(\psi) \notin F$ or $w(-p_0 \vee \psi) = -(-a \vee b) \vee a = (a \wedge -b) \vee a = a \notin F$. $\square$

In the constant-free framework, the lattice of super-Belnap logics therefore has two co-atoms, namely classical logic and the almost trivial logic.

Knowing that essentially no new logics appear in the constant-free framework, we briefly review (without proof) the changes that need to be made to the results of Chapter 8. In the constant-free framework, $\mathcal{CL}$ is still the only protoalgebraic super-Belnap logic. The almost trivial logic needs to be added to the list of selfextensional logics, Fregean logics, and logics with the proof by cases property, and to replace $\mathcal{K}$ in the list of structurally complete logics. It also yields one more contrapositive pair.

More substantially, the only truth-equational constant-free super-Belnap logics are the extensions of $\mathcal{LP}$ (this is because the almost trivial logic is not truth-equational), therefore the only non-trivial assertional constant-free super-Belnap logic is $\mathcal{CL}$. The Leibniz hierarchy therefore trivializes even more in the constant-free framework. The absence of theorems also trivializes the study of strong versions of super-Belnap logics outside Ext $\mathcal{LP}$.

10.2 Multiple-conclusion logics

Moving to a multiple-conclusion setting has more profound consequences. Recall that in the framework of Shoesmith and Smiley [71] a *multiple-conclusion consequence logic* is a relation between a pair of sets of formulas, written $\Gamma \vdash \Delta$, which satisfies the following:

$$
\begin{aligned}
&p \vdash p && \text{(reflexivity)}\\
&\text{if } \Gamma \vdash \Delta, \text{ then } \Gamma, \Gamma' \vdash \Delta, \Delta' && \text{(monotonicity)}\\
&\text{if } \Pi, \Gamma \vdash \Delta, \Sigma \text{ for all partitions } \langle \Pi, \Sigma \rangle \text{ of } \Lambda, \text{ then } \Gamma \vdash \Delta && \text{(cut)}\\
&\text{if } \Gamma \vdash \Delta, \text{ then } \sigma[\Gamma] \vdash \sigma[\Delta] \text{ for each substitution } \sigma && \text{(structurality)}
\end{aligned}
$$

Here a *partition* is a decomposition of a set into two complementary subsets.

The multiple-conclusion logic determined by a class of matrices K is defined as follows: $\Gamma \vdash \Delta$ if and only if for each $\langle \mathbf{A}, F \rangle \in \mathsf{K}$ and each valuation $v : \mathbf{Fm} \to \mathbf{A}$ we have either $v[\Gamma] \nsubseteq F$ or $v[\Delta] \nsubseteq \mathbf{A} \setminus F$.

By the multiple-conclusion versions of $\mathcal{BD}$, $\mathcal{LP}$, $\mathcal{K}$, $\mathcal{CL}$, and $\mathcal{KO}$, we shall mean the multiple-conclusion logics defined semantically via the matrices $\mathbb{BD}_4$, $\mathbb{P}_3$, $\mathbb{K}_3$, $\mathbb{B}_2$, and the set of matrices $\{\mathbb{P}_3, \mathbb{K}_3\}$.[1] See [7] for a discussion of the multiple-conclusion version of $\mathcal{LP}$.

The multiple-conclusion versions of these logics are finitary in the sense that $\Gamma \vdash \Delta$ implies that $\Gamma' \vdash \Delta'$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. They are related to the single-conclusion versions as follows:

$$\Gamma \vdash \Delta \text{ if and only if } \Gamma \vdash \bigvee \Delta \text{ for finite } \Delta.$$

The logic $\mathcal{BD}_{mc}$ is axiomatized by the following rules:

$$
\begin{array}{ccc}
p, q \vdash p \wedge q & p \wedge q \vdash p & p \wedge q \vdash q\\
p \vee q \vdash p, q & p \vdash p \vee q & q \vdash p \vee q
\end{array}
$$

[1] The multiple-conclusion version of the trivial logic is the logic of the empty class of matrices, axiomatized by the rule $\emptyset \vdash \emptyset$. It is *not* the multiple-conclusion logic of the trivial matrix, which is axiomatized by the rule $\emptyset \vdash p$.

$$-p, -q \vdash -(p \vee q) \qquad -(p \vee q) \vdash -p \qquad -(p \vee q) \vdash -q$$

$$-(p \wedge q) \vdash -p, -q \qquad -p \vdash -(p \wedge q) \qquad -q \vdash -(p \wedge q)$$

$$p \vdash --p \qquad --p \vdash p \qquad \emptyset \vdash \mathsf{t} \qquad \mathsf{f} \vdash \emptyset$$

Note that these are just the translations of the rules of the Hilbert calculus for $\mathcal{BD}$ into a multiple-conclusion format.

The logic $\mathcal{LP}_{mc}$ extends $\mathcal{BD}_{mc}$ by the rule $\emptyset \vdash p, -p$, $\mathcal{K}_{mc}$ by the rule $p, -p \vdash \emptyset$, $\mathcal{KO}_{mc}$ by the rule $p, -p \vdash q, -q$, and $\mathcal{CL} = \mathcal{LP}_{mc} \vee \mathcal{K}_{mc}$.

The reader will recall that logics $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ are in fact precisely those super-Belnap logics which satisfy the proof by cases property (see Figure 8.3). This should come as no suprise, since in the multiple-conclusion framewok the proof by cases property can be established by cutting twice on the rule $p \vee q \vdash p, q$.

Theorem 10.2 (Multiple-conclusion super-Belnap logics). *The non-trivial multiple-conclusion extensions of multiple-conclusion $\mathcal{BD}$ are precisely the multiple-conclusion versions of $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, $\mathcal{CL}$.*

Proof. We show that every multiple-conclusion rule $\Gamma \vdash \Delta$ is equivalent over $\mathcal{BD}$ either to $p \vdash p$ or to one of the rules

$$\emptyset \vdash \emptyset \qquad p, -p \vdash \emptyset$$

$$p, -p \vdash q, -q \qquad \emptyset \vdash q, -q$$

Since every formula is equivalent over $\mathcal{BD}$ to a formula in the conjunctive normal form and a formula in the disjunctive normal form, we may assume by appeal to the cut rules that all formulas in Γ and Δ are either atoms or negated atoms. If a literal occurs on both sides, the rule is equivalent to $p \vdash p$. Otherwise, if p occurs on one side and $-p$ does not occur on the same side (or if $-p$ occurs on one side and p does not occur on the same side), then substituting t or f for p yields an equivalent rule which in effect erases all instances of p and $-p$. Suppose therefore if p (or $-p$) occurs on one side of the rule, then so does $-p$ (or p). But then neither p nor $-p$ occurs on the other side of the rule. Thus, for each atom p the rule has one of the forms

$$p, -p, \Gamma \vdash \Delta \qquad \text{or } \Gamma \qquad \vdash \Delta, p, -p \text{ or } \Gamma \vdash \Delta,$$

where neither p nor $-p$ occurs in Γ, Δ.

Substituting, say, p for all atoms which occur (negated or non-negated) on the left and, say, q for all atoms which occur (negated or non-negated) on the right then yields an equivalent rule. Up to equivalence, this means that we have precisely the options listed above. □

It is perhaps worth noting that the above argument does not depend essentially on the presence of the constants t, f. Dropping them from the language would merely complicate the picture by forcing us to distinguish (i) between the rules $\emptyset \vdash \emptyset$, $p \vdash \emptyset$, $\emptyset \vdash q$, and $p \vdash q$, (ii) between the rules $\emptyset \vdash p, -p$ and $q \vdash p, -p$, and (iii) between the rules $p, -p \vdash \emptyset$ and $p, -p \vdash q$. It would not, however, yield any substantially new logic.

10.3 More than one predicate

Finally, we study logics obtained by adding a new unary predicate to $\mathcal{BD}$. We consider the exact truth predicate, interpreted on $\mathbf{DM_4}$ by the filter $\{\mathsf{t}\}$, and the non-falsity predicate, interpreted on $\mathbf{DM_4}$ by the filter $\{\mathsf{t}, \mathsf{n}\}$. This will yield the logics $\mathcal{BD}_E$ and $\mathcal{BD}_{NF}$ determined by the matrices (structures) $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}\} \rangle$ and $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}, \mathsf{n}\} \rangle$.

To clarify what we mean by a logic with two predicates, let us recall that a logic is nothing but the (possibly infinitary) universal strict Horn theory of a class of matrices in a language with a single unary predicate and no equality. That is, a rule $\Gamma \vdash \varphi$ is in fact a universally quantified disjunction of the formulas $\mathrm{True}(\varphi)$ and $\neg\,\mathrm{True}(\gamma)$ for $\gamma \in \Gamma$.

Adding a new predicate to a logic, say the exact truth predicate ExTrue, simply means admitting disjuncts of the forms $\mathrm{ExTrue}(\varphi)$ and $\neg\,\mathrm{ExTrue}(\gamma)$. Such rules will have one of the forms

$$\mathrm{True}(\gamma_1), \dots, \mathrm{ExTrue}(\delta_1), \dots \vdash \mathrm{True}(\varphi),$$
$$\mathrm{True}(\gamma_1), \dots, \mathrm{ExTrue}(\delta_1), \dots \vdash \mathrm{ExTrue}(\varphi).$$

We shall not develop the theory of abstract algebraic logic for logics with more than one predicate here. The interested reader may consult [16, 22]. For us, the only important observation will be that the Leibniz congruence of a matrix with two predicates $\langle \mathbf{A}, T, E \rangle$ is the congruence $\mathbf{\Omega}^{\mathbf{A}} T \cap \mathbf{\Omega}^{\mathbf{A}} E$. In particular, factoring a matrix by this congruence yields a reduced matrix which determines the same logic as the original one.

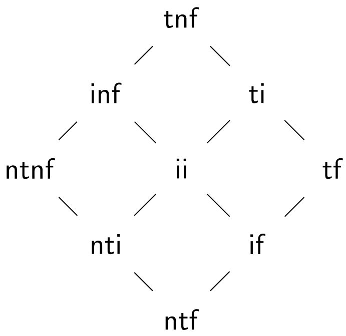
Let us now introduce the logic $\mathcal{BD}_E$ and show that it coincides with the logic of $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}\} \rangle$. This logic is axiomatized by taking an axiomatization of $\mathcal{BD}$ for the predicate True and an axiomatization of $\mathcal{ETL}$ for the predicate ExTrue and adding the rules

$$\mathrm{ExTrue}(p) \vdash \mathrm{True}(p),$$
$$\mathrm{ExTrue}(p), \mathrm{True}(-p \vee q) \vdash \mathrm{True}(q),$$
$$\mathrm{True}(p), \mathrm{True}(q), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q).$$

Proposition 10.3 (Reduced models of $\mathcal{BD}_E$).
Each reduced model of $\mathcal{BD}_E$ has this form $\langle \mathbf{A}, F, \{\mathsf{t}\} \rangle$, where $\langle \mathbf{A}, F \rangle$ is a De Morgan matrix.

Proof. The algebra $\mathbf{A} = \mathbf{A}/(\mathbf{\Omega}^{\mathbf{A}} T \cap \mathbf{\Omega}^{\mathbf{A}} E)$ is a De Morgan algebra by virtue of being a subdirect product of the De Morgan algebras $\mathbf{A}/\mathbf{\Omega}^{\mathbf{A}} T$

Figure 10.1: The algebra $\mathbf{DM_9}$



and $\mathbf{A}/\mathbf{\Omega}^{\mathbf{A}}E$. The sets T and E are lattice filters on $\mathbf{A}$ because the rules $T(p), T(q) \vdash T(p \wedge q)$ and $E(p), E(q) \vdash E(p \wedge q)$ are both valid in $\mathcal{BD}_E$. It remains to show that $E = \{\mathsf{t}\}$. We do so as in the proof of Proposition 3.20.

It suffices to show that for each $a, b \in E$ with $a < b$ the congruence $\theta := \mathrm{Cg}^{\mathbf{A}}\langle a, b\rangle$ is compatible with both T and E. Let $\langle x, y\rangle \in \theta$. Then by the equational description of the principal congruences of De Morgan lattices (Theorem 2.10) we have $(x \wedge a) \vee -a = (y \wedge a) \vee -a$. Because $E(p) \vdash_{\mathcal{BD}_E} T(p)$, we have $a \in T$. But then $x \in T$ implies that $x \wedge a \in T$, hence $(y \wedge a) \vee -a = (x \wedge a) \vee -a \in T$. But $E(p), T(-p \vee q) \vdash_{\mathcal{BD}_E} T(q)$, therefore $y \wedge a \in T$ and $y \in T$. Likewise, $x \in E$ implies that $x \wedge a \in E$, hence $(y \wedge a) \vee -a) = (x \wedge a) \vee -a \in E$. But $E(p), E(-p \vee q) \vdash_{\mathcal{BD}_E} E(q)$, therefore $y \wedge a \in E$ and $y \in E$. □

As in the case of the logic $\mathcal{ETL}$, the above proposition does not in fact depend on having the constants t and f in the signature. Note that the rule $\mathrm{True}(p), \mathrm{True}(q), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q)$ was not used in the proof.

Proposition 10.4 (Completeness for $\mathcal{BD}_E$).
$\mathcal{BD}_E$ is complete with respect to the matrix $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}\}\rangle$.

Proof. It is easy to verify that rules axiomatizing $\mathcal{BD}_E$ hold in this matrix. Conversely, suppose that a rule fails in $\mathcal{BD}_E$. By the previous proposition it fails in some matrix of the form $\langle \mathbf{A}, T, E\rangle$, where $\mathbf{A}$ is a De Morgan algebra and T and E are lattice filters on $\mathbf{A}$.

The rule $\mathrm{ExTrue}(p), \mathrm{True}(-p \vee q) \vdash \mathrm{True}(q)$ states that for each $x \notin T$ the ideal generated by adding x to $-[E]$ is disjoint from T. This ideal extends by the Filter–Ideal Separation Lemma (Lemma 1.2) to a prime ideal I_x disjoint from T. Let $T_x := \mathbf{A} \setminus I_x$ and $M_x := -[I_x]$.

By the same rule the ideal generated by adding $y \in T \setminus E$ to $-[T]$ is disjoint from E. The ideal extends by the Filter–Ideal Separation Lemma to a prime ideal J_y disjoint from E. Let $U_y := -[J_y]$ and $N_y := \mathbf{A} \setminus J_y$.

Then

$$T = \bigcap_{x \in \mathbf{A} \setminus T} T_x \cap \bigcap_{y \in T \setminus E} U_y \quad \text{and} \quad E = \bigcap_{x \in \mathbf{A} \setminus T} (T_x \cap M_x) \cap \bigcap_{y \in T \setminus E} (U_y \cap N_y).$$

It follows that if a rule fails in the matrix $\langle \mathbf{A}, T, E \rangle$, then it fails in one of the matrices $\langle \mathbf{A}, T_x, T_x \cap M_x \rangle$ or $\langle \mathbf{A}, U_y, U_y \cap N_y \rangle$.

We claim that these matrices are models of $\mathcal{BD}_E$. The matrices $\langle \mathbf{A}, T_x \rangle$, $\langle \mathbf{A}, U_y \rangle$, $\langle \mathbf{A}, T_x \cap M_x \rangle$, and $\langle \mathbf{A}, U_y \cap N_y \rangle$ are models of $\mathcal{BD}$ because T_x, U_y, M_x, and N_y are lattice filters on a De Morgan algebra. The rule $\mathrm{ExTrue}(p) \vdash \mathrm{True}(p)$ holds trivially in $\langle \mathbf{A}, T_x, T_x \cap M_x \rangle$ and $\langle \mathbf{A}, U_y, U_y \cap N_y \rangle$. The rule

$$\mathrm{ExTrue}(p), \mathrm{True}(-p \vee q) \vdash \mathrm{True}(q)$$

is valid in these matrices because T_x is prime and disjoint from $-[M_x]$ and U_y is prime and disjoint from $-[N_y]$. The rule

$$\mathrm{True}(p), \mathrm{True}(q), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q)$$

is valid because M_x is prime and disjoint from $-[T_x]$ and N_y is prime and disjoint from $-[U_y]$. Finally, the rule

$$\mathrm{ExTrue}(p), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q)$$

is valid because T_x and U_y are prime and M_x and N_y are prime and disjoint from $-[T_x]$ and $-[U_y]$, respectively. In verifying the validity of this last rule the validity of $\mathrm{True}(-p), \mathrm{True}(q), \mathrm{ExTrue}(p \vee q) \vdash \mathrm{ExTrue}(q)$ is used.

Now supose that a rule fails in a matrix of the form $\langle \mathbf{A}, T, T \cap N \rangle$, where T and N are prime filters and T is disjoint from $-N$. Observe that $\theta := \boldsymbol{\Omega}^{\mathbf{A}} T \cap \boldsymbol{\Omega}^{\mathbf{A}} N \subseteq \boldsymbol{\Omega}^{\mathbf{A}} T \cap \boldsymbol{\Omega}^{\mathbf{A}} (T \cap N)$. Factoring $\langle \mathbf{A}, T, T \cap N \rangle$ by θ, we may assume without loss of generality that θ is the identity congruence on $\mathbf{A}$. But then by the description of Leibniz congruences of $\mathcal{BD}$-filters (Proposition 3.17) $\langle a, b \rangle \in \theta$ if and only if

$$\begin{aligned} a \in T &\iff b \in T, & \qquad a \in N &\iff a \in N, \\ -a \in T &\iff -b \in T, & \qquad -a \in N &\iff -a \in N. \end{aligned}$$

Each element x of the matrix $\langle \mathbf{A}, T, E \rangle$ is thus uniquely determined if we know whether $x \in T$, $-x \in T$, $x \in N$, and $-x \in N$. Moreover, the algebraic structure of $\mathbf{A}$ is fully determined by the facts that T and E are prime filters. We can in particular infer that $\langle \mathbf{A}, T, E \rangle$ is a submatrix of the matrix $\langle \mathbf{DM_9}, \{\mathsf{tf}, \mathsf{ti}, \mathsf{tnf}\}, \{\mathsf{tnf}\} \rangle$, where $\mathbf{DM_9}$ is the algebra shown in Figure 10.1. (As usual, De Morgan negation is given by reflection across the horizontal axis of symmetry. In particular, De Morgan negation on $\mathbf{DM_9}$ has three fixpoints.) But $\Delta_{\mathbf{DM_9}} = \mathrm{Cg}^{\mathbf{DM_9}} \langle \mathsf{inf}, \mathsf{tnf} \rangle \cap \mathrm{Cg}^{\mathbf{DM_9}} \langle \mathsf{ti}, \mathsf{tnf} \rangle$, therefore the matrix in question is a subdirect power of the matrix $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}\} \rangle$. It follows that each rule which fails in $\mathcal{BD}_E$ fails in $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}\} \rangle$. $\square$

We also introduce the logic $\mathcal{KLP}$ and show that it coincides with the logic determined by the matrix $\langle \mathbf{K_3}, \{\mathsf{t}, \mathsf{i}\}, \{\mathsf{t}\} \rangle$. The logic $\mathcal{KLP}$ is the extension of $\mathcal{BD}_E$ by the rule

$$\emptyset \vdash \mathrm{True}(p \vee -p).$$

In this axiomatization the rule

$$\mathrm{True}(p), \mathrm{True}(q), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q)$$

may in fact be replaced by the simpler rule

$$\mathrm{True}(p), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q).$$

Proposition 10.5 (Completeness for $\mathcal{KLP}$).
$\mathcal{KLP}$ is complete with respect to the matrix $\langle \mathbf{K_3}, \{\mathsf{t}, \mathsf{i}\}, \{\mathsf{t}\} \rangle$.

Proof. It is easy to verify that the rules axiomatizing $\mathcal{KLP}$ hold in this matrix. Conversely, suppose that a rule fails in $\mathcal{KLP}$. Then it fails in some reduced model $\langle \mathbf{A}, T, E \rangle$ of $\mathcal{KLP}$. We know that $\langle \mathbf{A}, T \rangle$ is a model of $\mathcal{LP}$ and we now show that $\langle \mathbf{A}, E \rangle$ is a model of $\mathcal{K}$. First observe that the rule

$$\mathrm{True}(p), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q)$$

is derivable in $\mathcal{KLP}$ from the rule

$$\mathrm{True}(p), \mathrm{True}(q), \mathrm{ExTrue}(-p \vee q) \vdash \mathrm{ExTrue}(q)$$

because $\mathrm{ExTrue}(-p \vee q), \mathrm{True}((p \wedge -q) \vee q) \vdash_{\mathcal{KLP}} \mathrm{True}(q)$ and $\mathrm{True}(p) \vdash_{\mathcal{KLP}} \mathrm{True}((p \wedge -q) \vee q)$. The rule

$$\mathrm{ExTrue}((p \wedge -p) \vee q) \vdash \mathrm{ExTrue}(q)$$

is now derivable because $\mathrm{True}(p \vee -p), \mathrm{ExTrue}((p \wedge -p) \vee q) \vdash_{\mathcal{KLP}} \mathrm{ExTrue}(q)$ and $\emptyset \vdash_{\mathcal{KLP}} \mathrm{True}(p \vee -p)$.

Because $\langle \mathbf{A}, T \rangle$ is a model of $\mathcal{LP}$ and $\langle \mathbf{A}, E \rangle$ is a model of $\mathcal{K}$, the algebras $\mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}} T$ and $\mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}} E$ are Kleene algebras (Proposition 3.20). It follows that $\mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}} T \cap \boldsymbol{\Omega}^{\mathbf{A}} E$ is a Kleene algebra. Thus each rule which fails in $\mathcal{KLP}$ fails in some reduced model $\langle \mathbf{A}, T, E \rangle$ of $\mathcal{KLP}$, where $\mathbf{A}$ is a Kleene algebra.

We define T_x, U_y, M_x, and N_y as in the previous proof. The matrices $\langle \mathbf{A}, T_x \rangle$ and $\langle \mathbf{A}, U_y \rangle$ are models of $\mathcal{LP}$ because $\langle \mathbf{A}, T \rangle$ is a model of $\mathcal{LP}$ and $T \subseteq T_x$ and $T \subseteq U_y$. Moreover, the matrices $\langle \mathbf{A}, T_x \cap M_x \rangle$ are $\langle \mathbf{A}, U_y \cap N_y \rangle$ are models of $\mathcal{K} = \mathcal{KO} \vee \mathcal{ETL}$ because they are models of $\mathcal{ETL}$ and $\mathbf{A}$ is a Kleene algebra. The interaction rules between the two predicates hold by the same argument as in the previous proof.

Each rule which fails in some models of $\mathcal{KLP}$ thus fails in some model $\langle \mathbf{A}, T, T \cap N \rangle$ such that T and N are prime filters with $T \cap -[N] = \emptyset$ and T is an $\mathcal{LP}$-filter on $\mathbf{A}$, while $T \cap N$ is a $\mathcal{K}$-filter on $\mathbf{A}$.

Since for each $a \in \mathbf{A}$ either $a \in T$ or $-a \in T$, the nine-element matrix over the algebra $\mathbf{DM_9}$ reduces to the five-element chain $\mathbf{K_5}$ with $\mathsf{f} < \mathsf{a} < \mathsf{b} < \mathsf{c} < \mathsf{t}$ equipped with $T := \{\mathsf{b}, \mathsf{c}, \mathsf{t}\}$ and $E := \{\mathsf{t}\}$. But the matrix $\langle \mathbf{K_5}, \{\mathsf{b}, \mathsf{c}, \mathsf{t}\}, \{\mathsf{t}\} \rangle$ is a subdirect power of $\langle \mathbf{K_3}, \{\mathsf{t}, \mathsf{i}\}, \{\mathsf{t}\} \rangle$. □

The logic $\mathcal{BD}_{NF}$ is axiomatized by taking an axiomatization of $\mathcal{BD}$ for the predicates True and NonFalse and adding the rules

$$\mathrm{True}(p), \mathrm{NonFalse}(-p \vee q) \vdash \mathrm{NonFalse}(q),$$
$$\mathrm{NonFalse}(p), \mathrm{True}(-p \vee q) \vdash \mathrm{True}(q).$$

Proposition 10.6 (Completeness for $\mathcal{BD}_{NF}$).
$\mathcal{BD}_{NF}$ is complete with respect to the matrix $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}, \mathsf{n}\} \rangle$.

Proof. It is easy to verify that the rules axiomatizing $\mathcal{BD}_{NF}$ hold in this matrix. Conversely, suppose that a rule fails in $\mathcal{BD}_{NF}$. By Proposition 10.3 it fails in some matrix of the form $\langle \mathbf{A}, T, N \rangle$, where $\mathbf{A}$ is a De Morgan algebra and T and N are lattice filters on $\mathbf{A}$.

The rule $\mathrm{True}(p), \mathrm{NonFalse}(-p \vee q) \vdash \mathrm{NonFalse}(q)$ states that for each $x \notin N$ the ideal generated by adding x to $-T$ is disjoint from N. This ideal extends by the Filter–Ideal Separation Lemma (Lemma 1.2) to a prime ideal I_x disjoint from T. Let $T_x := \mathbf{A} \setminus I_x$ and $M_x := -I_x$.

Likewise, the rule $\mathrm{NonFalse}(p), \mathrm{True}(-p \vee q) \vdash \mathrm{True}(q)$ states that for each $y \notin T$ the ideal generated by adding y to $-N$ is disjoint from T. This ideal extends by the Filter–Ideal Separation Lemma to a prime ideal J_y disjoint from T. Let $U_y := \mathbf{A} \setminus J_y$ and $N_y := -[J_y]$.

Then

$$T = \bigcap_{x \in \mathbf{A} \setminus T} T_x \cap \bigcap_{y \in \mathbf{A} \setminus N} U_y \qquad \text{and} \qquad N = \bigcap_{x \in \mathbf{A} \setminus T} M_x \cap \bigcap_{y \in \mathbf{A} \setminus N} N_y.$$

It follows that if a rule fails in the matrix $\langle \mathbf{A}, T, N \rangle$, then it fails in one of the matrices $\langle \mathbf{A}, T_x, M_x \rangle$ or $\langle \mathbf{A}, U_y, N_y \rangle$ for $x \in \mathbf{A} \setminus T$ and $y \in \mathbf{A} \setminus N$.

We claim that these matrices are models of $\mathcal{BD}_{NF}$. The matrices $\langle \mathbf{A}, T_x \rangle$ and $\langle \mathbf{A}, M_x \rangle$ are models of $\mathcal{BD}$ because T_x and M_x are lattice filters on a De Morgan algebra. The rule

$$\mathrm{True}(p), \mathrm{NonFalse}(-p \vee q) \vdash \mathrm{NonFalse}(q)$$

is valid in each of these matrices because M_x is prime disjoint from $-[T_x]$. A symmetric argument shows that the rule $\mathrm{NonFalse}(p), \mathrm{True}(-p \vee q) \vdash \mathrm{True}(q)$ is also valid. The argument for the matrices $\langle \mathbf{A}, U_y, J_y \rangle$ is entirely analogous.

It therefore suffices to show that each rule which fails in a matrix of the form $\langle \mathbf{A}, T, N \rangle$, where T and N are prime filters such that T is disjoint from

$-[N]$, also fails in the matrix $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}, \mathsf{n}\}\rangle$. By Proposition 3.17 we have $\langle a, b\rangle \in \mathbf{\Omega^A} T \cap \mathbf{\Omega^A} N$ if and only if

$$a \in T \iff b \in T, \qquad a \in N \iff a \in N,$$
$$-a \in T \iff -b \in T, \qquad -a \in N \iff -a \in N.$$

Each element x of the Leibniz reduct of the $\langle \mathbf{A}, T, N\rangle$ is thus uniquely determined if we know whether $x \in T$, $-x \in T$, $x \in N$, and $-x \in N$. Since $T \cap -[N] = \emptyset = N \cap -[T]$, the Leibniz reduct of the matrix $\langle \mathbf{A}, T, N\rangle$ has at most nine elements. The algebraic structure on these elements is uniquely determined by the fact that T and N are prime filters. We can infer that the Leibniz reduct is a submatrix of $\langle \mathbf{DM_9}, \{\mathsf{tf}, \mathsf{ti}, \mathsf{tnf}\}, \{\mathsf{ntnf}, \mathsf{inf}, \mathsf{tnf}\}\rangle$, where $\mathbf{DM_9}$ is the matrix shown in Figure 10.1. (As usual, De Morgan negation is given by reflection across the horizontal axis of symmetry. In particular, De Morgan negation on $\mathbf{DM_9}$ has three fixpoints.) But this matrix is a subdirect power of the matrix $\langle \mathbf{DM_4}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}, \mathsf{n}\}\rangle$. Each rule which fails in $\mathcal{BD}_{NF}$ therefore fails in this matrix. □

Note that axiomatizing the multiple-conclusion versions of these logics does not pose any technical challenge. This is because the filters $\{\mathsf{t}\}$ and $\{\mathsf{t}, \mathsf{n}\}$ are definable in terms of $\{\mathsf{t}, \mathsf{b}\}$ as follows:

$$x \in \{\mathsf{t}\} \iff x \in \{\mathsf{t}, \mathsf{b}\} \text{ and } -x \notin \{\mathsf{t}, \mathsf{b}\},$$
$$x \in \{\mathsf{t}, \mathsf{n}\} \iff -x \notin \{\mathsf{t}, \mathsf{b}\}.$$

To axiomatize the multiple-conclusion version of $\mathcal{BD}_E$, it therefore suffices to add to the multiple-conclusion version of $\mathcal{BD}$ rules which express these equivalence, i.e. the rules

$$\mathrm{ExTrue}(p) \vdash \mathrm{True}(p),$$
$$\mathrm{True}(-p), \mathrm{ExTrue}(p) \vdash \emptyset,$$
$$\mathrm{True}(p) \vdash \mathrm{ExTrue}(p), \mathrm{True}(-p).$$

Likewise, to axiomatize the multiple-conclusion version of $\mathcal{BD}_{NF}$, it suffices to add to the multiple-conclusion version of $\mathcal{BD}$ the rules

$$\emptyset \vdash \mathrm{True}(p), \mathrm{NonFalse}(-p) \qquad \text{and} \qquad \mathrm{True}(p), \mathrm{NonFalse}(-p) \vdash \emptyset.$$

Chapter 11

The truth operator Δ

In this final chapter of the thesis, we study the expansion of the Belnap–Dunn logic by the unary operator Δ which allows us to talk about the truth of a proposition separately from its falsity: Δp is exactly true if p is true (and possibly also false), and it is exactly false otherwise. This study continues a line of research initiated by Sano and Omori [69].

Our plan is to first study the algebras corresponding to this logic, which we call De Morgan algebras with Δ. We establish their basic properties, show that they form a variety generated by a four-element algebra $\mathbf{DM_4^\Delta}$, and prove a twist representation theorem. Then we provide an alternative presentation of these algebras, which trades the De Morgan negation $-x$ and the truth operator Δx for the negation operators $\overline{\Delta}x := -\Delta x$ and $\overline{\nabla}x := \Delta{-x}$. In the final section, we then study the link between these algebras and the corresponding extension of $\mathcal{BD}$ by Δ.

11.1 De Morgan algebras with Δ

The current section will be devoted to studying the variety of De Morgan algebras expanded by a truth operator Δ.

We first introduce this variety semantically. Let us define the algebra $\mathbf{DM_4^\Delta}$ as the expansion of $\mathbf{DM_4}$ by the unary operation Δx such that

$$\Delta a = \mathsf{t} \text{ if } a \in \{\mathsf{t}, \mathsf{b}\} \qquad \text{and} \qquad \Delta a = \mathsf{f} \text{ if } a \in \{\mathsf{f}, \mathsf{n}\}.$$

That is, Δa is exactly true if a is true, and it is exactly false if a is not true. We can also define the dual operator ∇x such that

$$\nabla a = \mathsf{t} \text{ if } a \in \{\mathsf{t}, \mathsf{n}\} \qquad \text{and} \qquad \nabla a = \mathsf{f} \text{ if } a \in \{\mathsf{f}, \mathsf{b}\}.$$

That is, ∇a is exactly false if a is false, and it is exactly true if a is not false. The two operators are interdefinable via the equalities

$$\nabla a = -\Delta{-a} \qquad \text{and} \qquad \Delta a = -\nabla{-a}.$$

Throughout this section, we shall take Δx as a primitive operation and ∇x as an abbreviation for $-\Delta{-x}$.

The algebra $\mathbf{DM_4^\Delta}$ has exactly three proper subalgebras: the algebra $\mathbf{K_3^\Delta}$ with the universe $\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, the algebra $\mathbf{P_3^\Delta}$ with the universe $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, and the algebra $\mathbf{B_2^\Delta}$ with the universe $\{\mathsf{f}, \mathsf{t}\}$. Note that $\mathbf{K_3^\Delta}$ and $\mathbf{P_3^\Delta}$ are not isomorphic, even though $\mathbf{K_3}$ and $\mathbf{P_3}$ are.

We can define several further natural operations in terms of the operators Δx and ∇x. The *weak* and *strong implication* will be particularly useful in our investigation. The weak implication is defined as

$$x \supset y := -\Delta x \vee \Delta y.$$

That is,

$$\begin{aligned} a \supset b &= \mathsf{t} \text{ if } a \in \{\mathsf{t}, \mathsf{b}\} \implies b \in \{\mathsf{t}, \mathsf{b}\}, \\ a \supset b &= \mathsf{f} \text{ if } a \in \{\mathsf{t}, \mathsf{b}\} \text{ and } b \notin \{\mathsf{t}, \mathsf{b}\}. \end{aligned}$$

The strong implication is defined in terms of weak implication as

$$\begin{aligned} x \to y &:= (x \supset y) \wedge (-y \supset -x) \\ &= (-\Delta x \vee \Delta y) \wedge (-\nabla x \vee \nabla y). \end{aligned}$$

We also define the strong bi-implication as

$$\begin{aligned} x \leftrightarrow y &:= (x \to y) \wedge (y \to x) \\ &= (\Delta x \wedge \nabla x \wedge \Delta y \wedge \nabla y) \vee (\Delta x \wedge -\nabla x \wedge \Delta y \wedge -\nabla y) \\ &\quad \vee (-\Delta x \wedge \nabla x \wedge -\Delta y \wedge \nabla y) \vee (-\Delta x \wedge -\nabla x \wedge -\Delta y \wedge -\nabla y). \end{aligned}$$

These two operations may in fact be described more succinctly:

$$\begin{aligned} &a \to b = \mathsf{t} \text{ if } a \leq b && \text{and} && a \to b = \mathsf{f} \text{ if } a \nleq b, \\ &a \leftrightarrow b = \mathsf{t} \text{ if } a = b && \text{and} && a \to b = \mathsf{f} \text{ if } a \neq b. \end{aligned}$$

The above operations allow us to express the so-called *quaternary discriminator* on $\mathbf{DM_4^\Delta}$, i.e. the quaternary function $d(x, y, z, w)$ such that

$$d(x, y, z, w) = \begin{cases} z \text{ if } x = y, \\ w \text{ if } x \neq y. \end{cases}$$

The quaternary discriminator is definable on $\mathbf{DM_4^\Delta}$ by the term

$$d(x, y, z, w) = (z \wedge (x \leftrightarrow y)) \vee (w \wedge -(x \leftrightarrow y)),$$

therefore the variety generated by $\mathbf{DM_4^\Delta}$ is a so-called *discriminator variety* (see e.g. [12]). Although we shall not make use of this fact in what follows, it is worth noting that this implies many pleasant properties for the variety generated by $\mathbf{DM_4^\Delta}$. Let us now axiomatize this variety.

Definition 11.1 (De Morgan algebras with Δ).
A *De Morgan algebra with* Δ is a De Morgan algebra equipped with a unary operator Δx which satisfies the following equations:

$$\Delta\Delta x \approx \Delta x$$

$$\Delta(x \wedge y) \approx \Delta x \wedge \Delta y \qquad \Delta \mathsf{t} \approx \mathsf{t}$$
$$\Delta(x \vee y) \approx \Delta x \vee \Delta y \qquad \Delta \mathsf{f} \approx \mathsf{f}$$

$$x \wedge \nabla y \leq y \vee \Delta x \qquad \Delta x \wedge -\Delta x \leq \mathsf{f}$$
$$x \wedge \Delta y \leq y \vee \nabla x \qquad \Delta x \wedge \nabla x \leq x$$

The variety of De Morgan algebras will be denoted $\mathsf{DMA}\Delta$.

Fact 11.2. $\mathbf{DM}_4^{\boldsymbol{\Delta}}$ *is a De Morgan algebra with* Δ*.*

Recall that two elements a and b of a bounded distributive lattice are called *(Boolean) complements* if $a \wedge b \leq \mathsf{f}$ and $\mathsf{t} \leq a \vee b$. A De Morgan algebra with Δ will be called *Boolean* if it is Boolean as a De Morgan algebra, i.e. if $-a$ is the Boolean complement of a for each element a.

Lemma 11.3 (Boolean cores).
The fixpoints of Δ *form a Boolean subalgebra of each De Morgan algebra with* Δ*, called the* Boolean core *of* $\mathbf{A}$ *and denoted* $\Delta\mathbf{A}$*.*

Proof. The first five axioms of De Morgan algebras with Δ state that the fixpoints of Δ are precisely the elements of the form Δa for some a and that the fixpoints of Δ form a bounded distributive sublattice. The axiom $\Delta x \wedge -\Delta x \leq \mathsf{f}$ then states that this sublattice is a Boolean subalgebra. $\square$

Recall that a *retraction* is a homomorphism $h : \mathbf{A} \to \mathbf{B}$ such that $\mathbf{B} \leq \mathbf{A}$ and $h(b) = b$ for $b \in \mathbf{B}$.

Lemma 11.4 (Retractions onto the Boolean core).
The maps $\Delta : \mathbf{A} \to \Delta\mathbf{A}$ *and* $\nabla : \mathbf{A} \to \Delta\mathbf{A}$ *are retractions of bounded distributive lattices which preserve all existing Boolean complements.*

Proof. The fact that Δ is retraction of bounded distributive lattices is stated directly by the axioms of $\mathsf{DMA}\Delta$. Because $\Delta\mathbf{A}$ is closed under De Morgan negation, ∇ is also a retraction of bounded distributive lattices.

Now suppose that a and b are Boolean complements. Then $\Delta a \wedge \Delta b \leq \Delta(a \wedge b) \leq \Delta \mathsf{f} \leq \mathsf{f}$ and $\mathsf{t} \leq \Delta \mathsf{t} \leq \Delta(a \vee b) \leq \Delta a \vee \Delta b$, i.e. Δa and Δb are Boolean complements. $\square$

Fact 11.5. $\Delta\nabla x \approx \nabla x$ *holds in De Morgan algebras with* Δ*.*

Proof. $\Delta{-a}$ and $-\Delta{-a}$ are Boolean complements, therefore so are $\Delta\Delta{-a} = \Delta{-a}$ and $\Delta{-\Delta{-a}}$. But then $\Delta\nabla a = \Delta{-\Delta{-a}} = -\Delta{-a} = \nabla a$. $\square$

We now establish a correspondence between certain prime filters on De Morgan algebras with Δ and homomorphisms into $\mathbf{DM_4^\Delta}$, based on the correspondence between prime filters on De Morgan algebras and homomorphisms into $\mathbf{DM_4}$.

Definition 11.6 (Δ-filters).
A Δ-*filter* on a De Morgan algebra with Δ is a filter F such that

$$a \in F \iff \Delta a \in F.$$

Lemma 11.7 (Characterization of Δ-filters).
The lattice of Δ-filters on $\mathbf{A}$ is isomorphic to the lattice filters on $\Delta\mathbf{A}$ via

$$F \mapsto \Delta\mathbf{A} \cap F \qquad \textit{and} \qquad G \mapsto \Delta^{-1}[G].$$

Proof. If G is a filter on $\Delta\mathbf{A}$, then $\Delta^{-1}[G]$ is a filter on $\mathbf{A}$ by Lemma 11.4. It is a Δ-filter, since

$$a \in \Delta^{-1}[G] \iff \Delta a \in G \iff \Delta\Delta a \in G \iff \Delta a \in \Delta^{-1}G.$$

Conversely, if F is a filter on $\mathbf{A}$, then $\Delta\mathbf{A} \cap F$ is a filter on $\Delta\mathbf{A}$.

The two maps are clearly order-preserving. Now let F be a Δ-filter on $\mathbf{A}$. Then

$$a \in \Delta^{-1}[\Delta\mathbf{A} \cap F] \iff \Delta a \in \Delta\mathbf{A} \cap F \iff \Delta a \in F \iff a \in F.$$

Conversely, let G be a filter on $\Delta\mathbf{A}$. Then

$$a \in \Delta\mathbf{A} \cap \Delta^{-1}[G] \iff a \in \Delta\mathbf{A} \text{ and } \Delta a \in G \iff a \in \Delta\mathbf{A} \cap G = G.$$

It follows that the two maps are mutually inverse. □

Lemma 11.8 (Filter–homomorphism correspondence in DMAΔs).
The bijection between homomorphisms $h : \mathbf{A} \to \mathbf{DM_4}$ of De Morgan algebras and prime filters on $\mathbf{A}$ restricts to a bijection between homomorphisms $h : \mathbf{A} \to \mathbf{DM_4^\Delta}$ of De Morgan algebras with Δ and prime Δ-filters on $\mathbf{A}$.

Proof. By the Filter–homomorphism correspondence in DMAs (Lemma 2.5), it suffices to prove that F_h is a Δ-filter if $h : \mathbf{A} \to \mathbf{DM_4^\Delta}$ is a homomorphism of De Morgan algebras with Δ and that h_F is a homomorphism of De Morgan algebras with Δ with F is a Δ-filter on $\mathbf{A}$.

Given a homomorphism $h : \mathbf{A} \to \mathbf{DM_4^\Delta}$ of De Morgan algebras with Δ,

$$a \in F_h \iff h(a) \in \{\mathsf{t}, \mathsf{b}\} \iff h(\Delta a) = \Delta h(a) \in \{\mathsf{t}, \mathsf{b}\} \iff \Delta a \in F_h.$$

Conversely, if F is a prime Δ-filter on $\mathbf{A}$, then $h_F(\Delta a) \in \{\mathsf{t}, \mathsf{f}\}$ because $\Delta a \vee -\Delta a \in F$ and $\Delta a \wedge -\Delta a \notin F$. The equality $\Delta h_F(a) = h_F(\Delta a)$ therefore follows from the equivalence

$$\begin{aligned}
\Delta h_F(a) \in \{\mathsf{t}, \mathsf{b}\} &\iff h_F(a) \in \{\mathsf{t}, \mathsf{b}\} \\
&\iff a \in F \\
&\iff \Delta a \in F \\
&\iff h_F(\Delta a) \in \{\mathsf{t}, \mathsf{b}\}.
\end{aligned}$$

□

The following lemma tells us that each failure of an inequality $a \nleq b$ is witnessed in the Boolean core. In particular, each element a of a De Morgan algebra with Δ is uniquely determined by the pair $\langle \Delta a, \nabla a \rangle$.

Lemma 11.9.
Let $a, b \in \mathbf{A} \in \mathsf{DMA\Delta}$. If $\Delta a \leq \Delta b$ and $\nabla a \leq \nabla b$, then $a \leq b$.

Proof. Suppose that $\Delta a \leq \Delta b$ and $\Delta {-b} \leq \Delta {-a}$. Then $\mathsf{t} \leq -\Delta a \vee \Delta b$ and $\nabla a \wedge -\nabla b \leq \mathsf{f}$, because Δa and $-\Delta a$ are Boolean complements, as are ∇b and $-\nabla b$. It now suffices to prove that $a \wedge (-\Delta a \vee \Delta b) \leq b \vee (\nabla a \wedge -\nabla b)$. In particular, it suffices to prove the following four inequalities:

$$a \wedge -\Delta a \leq b \vee -\nabla b \qquad\qquad a \wedge -\Delta a \leq \nabla a$$
$$a \wedge \Delta b \leq b \vee \nabla a \qquad\qquad \Delta b \leq b \vee -\nabla b$$

The axiomatization of $\mathsf{DMA\Delta}$ directly postulates that the two equalities on the left all hold in $\mathbf{A}$. To prove that $\Delta b \leq b \vee -\nabla b$ holds, recall that ∇b and $-\nabla b$ are complements by Lemma 11.3, hence the inequality is equivalent to $\Delta b \wedge \nabla b \leq b$, which is postulated to hold in $\mathbf{A}$. Finally, $a \wedge -\Delta a \leq \nabla a$ is by the same reasoning equivalent to $a \leq \Delta a \vee \nabla a$. But we have $\Delta {-a} \wedge \nabla {-a} \leq -a$, hence $a \leq --a \leq -\Delta {-a} \vee -\nabla {-a} \leq \nabla a \vee \Delta a$. □

Lemma 11.10 (Prime Δ-Filter Separation Lemma).
If $a \nleq b$ in some $\mathbf{A} \in \mathsf{DMA\Delta}$, then there is a prime Δ-filter F on $\mathbf{A}$ such that either $a \in F$ and $b \notin F$ or $-b \in F$ and $-a \notin F$.

Proof. Suppose that $a \nleq b$. By Lemma 11.9 either $\Delta a \nleq \Delta b$ or $\nabla a \nleq \nabla b$. Suppose first that $\Delta a \nleq \Delta b$. Pick a prime filter G on $\Delta\mathbf{A}$ such that $\Delta a \in G$ but $\Delta b \notin G$. Then by the characterization of Δ-filters (Lemma 11.7) $F := \Delta^{-1}[G]$ is a prime Δ-filter such that $a \in F$ but $b \notin F$. On the other hand, suppose that $\nabla a \nleq \nabla b$. Then $\Delta {-b} \nleq \Delta {-a}$, therefore by the previous argument there is a prime Δ-filter F such that $-b \in F$ but $-a \notin F$. □

This lemma will now allow us to axiomatize the varieties generated by $\mathbf{DM_4^\Delta}$ and its subalgebras.

Definition 11.11 (Boolean, Kleene, and Priest algebras with Δ).
A De Morgan algebra with Δ is:

(i) a *Boolean algebra with* Δ if it satisfies $\Delta x \approx x$,

(ii) a *Kleene algebra with* Δ if it satisfies $\Delta x \leq x$,

(iii) a *Priest algebra with* Δ if it satisfies $x \leq \Delta x$,

(iv) a *Kleene–Priest algebra with* Δ if it satisfies $x \wedge -x \leq y \vee -y$.

These classes of algebras will be denoted $\mathsf{BA\Delta}$, $\mathsf{KA\Delta}$, $\mathsf{PA\Delta}$, and $\mathsf{KPA\Delta}$.

Theorem 11.12 (Algebraic completeness for DMAΔs).

(i) $\mathsf{DMA}\Delta = \mathbb{SP}(\mathbf{DM_4^\Delta})$.

(ii) $\mathsf{KA}\Delta = \mathbb{SP}(\mathbf{K_3^\Delta})$.

(iii) $\mathsf{PA}\Delta = \mathbb{SP}(\mathbf{P_3^\Delta})$.

(iv) $\mathsf{KPA}\Delta = \mathbb{SP}(\mathbf{K_3^\Delta}, \mathbf{P_3^\Delta})$.

(v) $\mathsf{BA}\Delta = \mathbb{SP}(\mathbf{B_2^\Delta})$.

Proof. (i) Let $\mathbf{A}$ be a De Morgan algebra with Δ and suppose that $a \nleq b$ in $\mathbf{A}$. Then by the Prime Δ-Filter Separation Lemma (Lemma 11.10) there is a prime Δ-filter F such that either $a \in F$ and $b \notin F$ or $-b \in F$ and $-a \notin F$. The Filter–homomorphism correspondence in DMAΔs (Lemma 11.8) then yields a homomorphism $h : \mathbf{A} \to \mathbf{DM_4}$ such that either $h(a) \in \{\mathsf{t}, \mathsf{b}\}$ and $h(b) \notin \{\mathsf{t}, \mathsf{b}\}$ or $h(-b) \in \{\mathsf{t}, \mathsf{b}\}$ and $h(-a) \notin \{\mathsf{t}, \mathsf{b}\}$. Therefore $h(a) \nleq h(b)$ in $\mathbf{DM_4}$ in both cases. It follows that $\mathbf{A}$ embeds into a power of $\mathbf{DM_4^\Delta}$.

(ii) Let $\mathbf{A}$ be a Kleene algebra with Δ and $h : \mathbf{A} \to \mathbf{DM_4}$ be a homomorphism. Kleene algebras with Δ form a variety, therefore the image of $\mathbf{A}$ is a Kleene subalgebra of $\mathbf{DM_4^\Delta}$. But each Kleene subalgebra of $\mathbf{DM_4^\Delta}$ is isomorphic to a subalgebra of $\mathbf{K_3^\Delta}$.

(iii) Let $\mathbf{A}$ be a Priest algebra with Δ and $h : \mathbf{A} \to \mathbf{DM_4}$ be a homomorphism. Priest algebras with Δ form a variety, therefore the image of $\mathbf{A}$ is a Priest subalgebra of $\mathbf{DM_4^\Delta}$. But each Priest subalgebra of $\mathbf{DM_4^\Delta}$ is isomorphic to a subalgebra of $\mathbf{K_3^\Delta}$.

(iv) Let $\mathbf{A}$ be a Kleene–Priest algebra with Δ and $h : \mathbf{A} \to \mathbf{DM_4}$ be a homomorphism. Kleene–Priest algebras with Δ form a variety, therefore the image of $\mathbf{A}$ is a Kleene–Priest subalgebra of $\mathbf{DM_4^\Delta}$. But each Kleene–Priest subalgebra of $\mathbf{DM_4^\Delta}$ is isomorphic to a subalgebra of either $\mathbf{K_3^\Delta}$ or $\mathbf{P_3^\Delta}$.

(v) Let $\mathbf{A}$ be a Boolean algebra with Δ and $h : \mathbf{A} \to \mathbf{DM_4}$ be a homomorphism. Boolean algebras with Δ form a variety, therefore the image of $\mathbf{A}$ is a Boolean subalgebra of $\mathbf{DM_4^\Delta}$. But the only Boolean subalgebra of $\mathbf{DM_4^\Delta}$ is $\mathbf{B_2^\Delta}$. □

Theorem 11.13 (Subdirectly irreducible DMAs with Δ). *There are exactly four subdirectly irreducible De Morgan algebras with Δ: $\mathbf{B_2^\Delta}$, $\mathbf{K_3^\Delta}$, $\mathbf{P_3^\Delta}$, $\mathbf{DM_4^\Delta}$.*

Proof. Each subdirectly irreducible De Morgan algebra with Δ embeds into a power of $\mathbf{DM_4^\Delta}$, therefore it embeds into $\mathbf{DM_4^\Delta}$. But each subalgebra of $\mathbf{DM_4}$ is isomorphic to $\mathbf{B_2^\Delta}$, $\mathbf{K_3^\Delta}$, $\mathbf{P_3^\Delta}$, or $\mathbf{DM_4^\Delta}$, and these are all subdirectly irreducible. □

Corollary 11.14 (Varieties of De Morgan algebras with Δ). *There are only five non-trivial varieties of De Morgan algebras with Δ:* $\mathsf{BA}\Delta$*,* $\mathsf{KPA}\Delta$*,* $\mathsf{KA}\Delta$*,* $\mathsf{PA}\Delta$*, and* $\mathsf{DMA}\Delta$*.*

The algebraic completeness theorem for $\mathsf{DMA\Delta}$ (Theorem 11.12) will be helpful in showing that congruences on De Morgan algebras with Δ correspond precisely to Δ-filters.

Theorem 11.15 (Filter–congruence correspondence in DMAΔs). *The lattice of congruences* $\operatorname{Con}\mathbf{A}$ *on a De Morgan algebra with* Δ $\mathbf{A}$ *is isomorphic to the lattice of* Δ*-filters on* $\mathbf{A}$ *via the maps* $\theta \mapsto F_\theta$ *and* $F \mapsto \theta_F$ *such that*

$$F_\theta := \Delta^{-1}[\mathsf{t}]_\theta \qquad \text{and} \qquad \langle a, b \rangle \in \theta_F \iff a \leftrightarrow b \in F.$$

Proof. The set F_θ is a Δ-filter on $\mathbf{A}$ by the characterization of Δ-filters (Lemma 11.7). The relation θ_F is reflexive because $a \leftrightarrow a = \mathsf{t}$, transitive because $a \leftrightarrow b = b \leftrightarrow a$, and transitive because $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leq (a \leftrightarrow c)$. To quickly verify such inequalities, by the algebraic completeness theorem for $\mathsf{DMA\Delta}$ (Theorem 11.12) it suffices to verify them in $\mathbf{DM_4^{\Delta}}$, where

$$a \leftrightarrow b = \mathsf{t} \iff a = b \qquad \text{and} \qquad a \leftrightarrow b = \mathsf{f} \iff a \neq b.$$

Moreover, $\langle a, b \rangle \in \theta_F \implies \langle -a, -b \rangle \in \theta_F$ because $a \leftrightarrow b = -a \leftrightarrow -b$, and $\langle a, c \rangle, \langle b, d \rangle \in \theta_F \implies \langle a \wedge b, c \wedge d \rangle \in \theta_F$ because $(a \leftrightarrow c) \wedge (b \leftrightarrow d) \leq (a \wedge b) \leftrightarrow (c \wedge d)$. It follows that $\langle a, c \rangle, \langle b, d \rangle \in \theta_F \implies \langle a \vee b, c \vee d \rangle \in \theta_F$ because $x \vee y = -(-x \wedge -y)$. The relation θ_F is therefore a congruence.

The two maps are clearly monotone. Moreover,

$$a \in F_{\theta_F} \iff \langle \Delta a, \mathsf{t} \rangle \in \theta_F \iff \Delta a \leftrightarrow \mathsf{t} \in F \iff \Delta a \in F \iff a \in F,$$

because $\Delta a \leftrightarrow \mathsf{t} = \Delta a$. Conversely,

$$\langle a, b \rangle \in \theta_{F_\theta} \iff a \leftrightarrow b \in F_\theta \iff \langle \Delta(a \leftrightarrow b), \mathsf{t} \rangle \in \theta.$$

But the equivalence $\Delta(a \leftrightarrow b) = \mathsf{t} \iff a = b$ holds in $\mathbf{DM_4^{\Delta}}$, therefore by Theorem 11.12 also in $\mathbf{A}/\theta$. It follows that

$$\langle \Delta(a \leftrightarrow b), \mathsf{t} \rangle \in \theta \iff \langle a, b \rangle \in \theta. \qquad \square$$

We can now infer the following characterization of principal congruences.

Theorem 11.16 (Principal congruences of DMAΔs). *Take* $a, b, x, y \in \mathbf{A} \in \mathsf{DMA\Delta}$*. Then*

$$\langle x, y \rangle \in \operatorname{Cg}^{\mathbf{A}}\langle a, b \rangle \iff a \leftrightarrow b \leq x \leftrightarrow y.$$

If $a \leq b$ *and* $x \leq y$*, then*

$$\langle x, y \rangle \in \operatorname{Cg}^{\mathbf{A}}\langle a, b \rangle \iff \Delta y \wedge (b \to a) \leq \Delta x \text{ and } \nabla y \wedge (b \to a) \leq \nabla x.$$

Proof. The Filter–congruence correspondence in DMAΔs (Theorem 11.15) yields that

$$\langle x, y\rangle \in \mathrm{Cg}^{\mathbf{A}}\langle a, b\rangle \iff x \leftrightarrow y \in F,$$

where F is the smallest Δ-filter with $a \leftrightarrow b \in F$. By the characterization of Δ-filters (Lemma 11.7) $F = \Delta^{-1}[G]$, where G is the principal filter generated on $\Delta\mathbf{A}$ by $a \leftrightarrow b$. That is,

$$x \leftrightarrow y \in F \iff a \leftrightarrow b \leq \Delta(x \leftrightarrow y) \iff a \leftrightarrow b \leq x \leftrightarrow y.$$

If $a \leq b$ and $x \leq y$, then $a \leftrightarrow b = b \to a$ and

$$\begin{aligned} x \leftrightarrow y &= y \to x = (y \supset x) \wedge (-x \supset -y) \\ &= (-\Delta y \vee \Delta x) \wedge (-\Delta{-x} \vee \Delta{-y}) = (-\Delta y \vee \Delta x) \wedge (-\nabla y \vee \nabla x). \end{aligned}$$

The second claim now follows because Δy and ∇y both belong to the Boolean core of $\mathbf{A}$. □

Finally, we show that each De Morgan algebra with Δ can be represented as a subalgebra of a certain twist product of its Boolean core.

Definition 11.17 (Δ-twist products of Boolean algebras).
Let $\mathbf{A}$ be a Boolean algebra. We define the Δ-*twist product* of $\mathbf{A}$ as

$$\mathbf{A}^{\bowtie} := (A \times A, \wedge^{\bowtie}, \vee^{\bowtie}, \mathsf{t}^{\bowtie}, \mathsf{f}^{\bowtie}, -^{\bowtie}, \Delta^{\bowtie}),$$

where

$$\begin{aligned} \mathsf{t}^{\bowtie} &= \langle \mathsf{t}, \mathsf{f}\rangle, \\ \mathsf{f}^{\bowtie} &= \langle \mathsf{f}, \mathsf{t}\rangle, \\ -^{\bowtie}\langle a_+, a_-\rangle &= \langle a_-, a_+\rangle, \\ \Delta^{\bowtie}\langle a_+, a_-\rangle &= \langle a_+, \neg a_+\rangle, \\ \langle a_+, a_-\rangle \wedge^{\bowtie} \langle b_+, b_-\rangle &= \langle a_+ \wedge b_+, a_- \vee b_-\rangle, \\ \langle a_+, a_-\rangle \vee^{\bowtie} \langle b_+, b_-\rangle &= \langle a_+ \vee b_+, a_- \wedge b_-\rangle. \end{aligned}$$

The above definition yields the following ordering on $\mathbf{A}^{\bowtie}$:

$$\langle a_+, a_-\rangle \leq \langle b_+, b_-\rangle \iff a_+ \leq b_+ \text{ and } b_- \leq a_-.$$

Letting $\nabla^{\bowtie} x := -^{\bowtie}\Delta^{\bowtie}-^{\bowtie}x$, we get

$$\nabla^{\bowtie}\langle a_+, a_-\rangle = \langle \neg a_-, a_-\rangle.$$

The following facts are now easily verified.

Fact 11.18. $\mathbf{DM_4^{\Delta}} = \mathbf{B_2^{\bowtie}}$.

Fact 11.19. $\mathbf{A}^{\bowtie}$ *is a De Morgan algebra with* Δ *for each* $\mathbf{A} \in \mathsf{BA}$.

Theorem 11.20 (Twist representation of DMAΔs).
Each $\mathbf{A} \in \mathsf{DMA}\Delta$ *embeds into* $(\Delta\mathbf{A})^{\bowtie}$ *via the map* $\iota : a \mapsto \langle \Delta a, \Delta{-a} \rangle$.

Proof. The map ι is a injective by Lemma 11.9. To show that it is a homomorphism, observe that

$$\begin{aligned} \iota(-a) &= \langle \Delta{-a}, \Delta{--a} \rangle = \langle \Delta{-a}, \Delta a \rangle \\ &= -^{\bowtie}\langle \Delta a, \Delta{-a} \rangle = -^{\bowtie}\iota(a), \end{aligned}$$

and

$$\begin{aligned} \iota(\Delta a) &= \langle \Delta\Delta a, \Delta{-\Delta a} \rangle = \langle \Delta a, -\Delta a \rangle \\ &= \langle \Delta a, \neg\Delta a \rangle = \Delta^{\bowtie}\langle \Delta a, \Delta{-a} \rangle = \Delta^{\bowtie}\iota(a). \end{aligned}$$

Checking that $\iota(a \wedge b) = \iota(a) \wedge^{\bowtie} \iota(b)$ and $\iota(a \vee b) = \iota(a) \vee^{\bowtie} \iota(b)$ is straightforward. $\square$

11.2 Distributive lattices with $\overline{\Delta}$ and $\overline{\nabla}$

In this section, we provide an alternative presentation of De Morgan algebras with Δ. We shall replace the operators Δx and $-x$ by the operators

$$\overline{\Delta} a := -\Delta a \qquad \text{and} \qquad \overline{\nabla} a := -\nabla a.$$

The original operators Δx and ∇x may be reconstructed from these as

$$\Delta a := \overline{\overline{\Delta}\Delta} a \qquad \text{and} \qquad \nabla a := \overline{\overline{\nabla}\nabla} a.$$

Recovering the De Morgan negation from these operators is slightly more complicated. It can be defined as

$$-a := (\overline{\Delta} a \wedge \overline{\nabla} a) \vee (\overline{\nabla} a \wedge a) \vee (\overline{\Delta} a \wedge a).$$

We now provide an axiomatization of De Morgan algebras with Δ where we take $\overline{\Delta} x$ and $\overline{\nabla} x$ as primitive operators instead of Δx and $-x$.

Definition 11.21 (Distributive lattices with $\overline{\Delta}$ and $\overline{\nabla}$).
A *distributive lattice with* $\overline{\Delta}$ *and* $\overline{\nabla}$ is a bounded distributive lattice equipped with two unary operators $\overline{\Delta}$ and $\overline{\nabla}$ which satisfy the equations:

$$\begin{array}{cccc} \overline{\Delta}(x \wedge y) \approx \overline{\Delta} x \vee \overline{\Delta} y & \overline{\Delta}\mathsf{t} \approx \mathsf{f} & \overline{\nabla}(x \wedge y) \approx \overline{\nabla} x \vee \overline{\nabla} y & \overline{\nabla}\mathsf{t} \approx \mathsf{f} \\ \overline{\Delta}(x \vee y) \approx \overline{\Delta} x \wedge \overline{\Delta} y & \overline{\Delta}\mathsf{f} \approx \mathsf{t} & \overline{\nabla}(x \vee y) \approx \overline{\nabla} x \wedge \overline{\nabla} y & \overline{\nabla}\mathsf{f} \approx \mathsf{t} \\ \overline{\overline{\Delta}\Delta\Delta} x \approx \overline{\Delta} x & \overline{\Delta\nabla} x \approx \overline{\nabla\nabla} x & \overline{\overline{\nabla}\nabla\nabla} x \approx \overline{\nabla} x & \overline{\nabla\Delta} x \approx \overline{\Delta\Delta} x \end{array}$$

$$\begin{array}{ccc} x \wedge \overline{\Delta} x \leq y \vee \overline{\nabla} y & x \leq \overline{\nabla\nabla} x \vee \overline{\Delta\Delta} x & \overline{\Delta} x \wedge \overline{\Delta\Delta} x \leq \mathsf{f} \\ x \wedge \overline{\nabla} x \leq y \vee \overline{\Delta} y & \overline{\Delta\Delta} x \wedge \overline{\nabla\nabla} x \leq x & \overline{\nabla} x \wedge \overline{\nabla\nabla} x \leq \mathsf{f} \end{array}$$

The variety of distributive lattices with $\overline{\Delta}$ and $\overline{\nabla}$ will be denoted $\mathsf{DLat}\overline{\Delta}\overline{\nabla}$. The equations $\overline{\nabla}\overline{\nabla}\overline{\nabla}x \approx \overline{\nabla}x$ and $\overline{\nabla}x \wedge \overline{\nabla}\overline{\nabla}x \leq \mathsf{f}$ (or their analogues which feature $\overline{\Delta}$) are in fact redundant in this axiomatization but we include them for the sake of symmetry.

We now define a construction $\boldsymbol{\tau}$ which yields a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$ given a De Morgan algebra with Δ and an inverse construction $\boldsymbol{\rho}$ which yields a De Morgan algebra with Δ given a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$, showing that distributive lattices with $\overline{\Delta}$ and $\overline{\nabla}$ is nothing but De Morgan algebras with Δ in a slightly different presentation.

Let $\mathbf{A} = (A, \wedge, \vee, \mathsf{t}, \mathsf{f}, -, \Delta)$ be a De Morgan algebra with Δ. Then we define $\boldsymbol{\tau}(\mathbf{A}) := (A, \wedge, \vee, \mathsf{t}, \mathsf{f}, \overline{\Delta}, \overline{\nabla})$ where

$$\overline{\Delta}x := -\Delta x \qquad \text{and} \qquad \overline{\nabla}x := -\nabla x.$$

Conversely, let $\mathbf{B} = (B, \wedge, \vee, \mathsf{t}, \mathsf{f}, \overline{\Delta}, \overline{\nabla})$ is a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$. Then we define $\boldsymbol{\rho}(\mathbf{B}) := (B, \wedge, \vee, \mathsf{t}, \mathsf{f}, -, \Delta)$ where

$$-x := (\overline{\Delta}x \wedge \overline{\nabla}x) \vee (\overline{\nabla}x \wedge x) \vee (\overline{\Delta}x \wedge x) \qquad \text{and} \qquad \Delta x := \overline{\Delta}\overline{\Delta}x.$$

Theorem 11.22.
If $\mathbf{A}$ is a De Morgan algebra with Δ, then $\boldsymbol{\tau}(\mathbf{A})$ is a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$. Conversely, if $\mathbf{B}$ is a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$, then $\boldsymbol{\rho}(\mathbf{B})$ is a De Morgan algebra with Δ. Moreover, $\mathbf{A} = \boldsymbol{\rho}\boldsymbol{\tau}(\mathbf{A})$ for $\mathbf{A} \in \mathsf{DMA}\Delta$ and $\mathbf{B} = \boldsymbol{\tau}\boldsymbol{\rho}(\mathbf{B})$ for $\mathbf{B} \in \mathsf{DLat}\overline{\Delta}\overline{\nabla}$.

Proof. Proving that $\boldsymbol{\tau}(\mathbf{A}) \in \mathsf{DLat}\overline{\Delta}\overline{\nabla}$ whenever $\mathbf{A} \in \mathsf{DMA}\Delta$ is a straightforward task, which we leave to the interested reader. By the algebraic completeness theorem for $\mathsf{DMA}\Delta$ (Theorem 11.12) it suffices to verify that the translations of the axioms of $\mathsf{DLat}\overline{\Delta}\overline{\nabla}$ are valid in the algebra $\mathbf{DM_4^{\Delta}}$.

Conversely, let a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$ be given. We define

$$\Delta a := \overline{\Delta}\overline{\Delta}a \qquad \text{and} \qquad -a := (\overline{\Delta}a \wedge \overline{\nabla}a) \vee (\overline{\nabla}a \wedge a) \vee (\overline{\Delta}a \wedge a).$$

The equations

$$\Delta\Delta x \approx \Delta x$$

$$\begin{aligned} \Delta(x \wedge y) &\approx \Delta x \wedge \Delta y & \qquad \Delta\mathsf{t} &\approx \mathsf{t} \\ \Delta(x \vee y) &\approx \Delta x \vee \Delta y & \qquad \Delta\mathsf{f} &\approx \mathsf{f} \end{aligned}$$

are then satisfied by virtue of the equations

$$\overline{\Delta}\overline{\Delta}\overline{\Delta}x \approx \overline{\Delta}x$$

$$\begin{aligned} \overline{\Delta}(x \wedge y) &\approx \overline{\Delta}x \vee \overline{\Delta}y & \qquad \overline{\Delta}\mathsf{t} &\approx \mathsf{f} \\ \overline{\Delta}(x \vee y) &\approx \overline{\Delta}x \wedge \overline{\Delta}y & \qquad \overline{\Delta}\mathsf{f} &\approx \mathsf{t} \end{aligned}$$

To verify the other equations, we first compute the values of $-\overline{\Delta}a$, $-\overline{\nabla}a$, $\overline{\Delta}{-a}$, and $\overline{\nabla}{-a}$:

$$
\begin{aligned}
-\overline{\Delta}a &= (\overline{\Delta\Delta}a \wedge \overline{\nabla\Delta}a) \vee (\overline{\nabla\Delta}a \wedge \overline{\Delta}a) \vee (\overline{\Delta\Delta}a \wedge \overline{\Delta}a) \\
&= (\overline{\Delta\Delta}a \wedge \overline{\Delta\Delta}a) \vee (\overline{\Delta\Delta}a \wedge \overline{\Delta}a) \vee (\overline{\Delta\Delta}a \wedge \overline{\Delta}a) \\
&= (\overline{\Delta\Delta}a) \vee \mathsf{f} \vee \mathsf{f} \\
&= \overline{\Delta\Delta}a
\end{aligned}
$$

$$
\begin{aligned}
-\overline{\nabla}a &= (\overline{\Delta\nabla}a \wedge \overline{\nabla\nabla}a) \vee (\overline{\nabla\nabla}a \wedge \overline{\nabla}a) \vee (\overline{\Delta\nabla}a \wedge \overline{\nabla}a) \\
&= (\overline{\nabla\nabla}a \wedge \overline{\nabla\nabla}a) \vee (\overline{\nabla\nabla}a \wedge \overline{\nabla}a) \vee (\overline{\nabla\nabla}a \wedge \overline{\nabla}a) \\
&= \overline{\nabla\nabla}a \vee \mathsf{f} \vee \mathsf{f} \\
&= \overline{\nabla\nabla}a
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Delta}{-a} &= (\overline{\Delta\Delta}a \vee \overline{\Delta\nabla}a) \wedge (\overline{\Delta\nabla}a \vee \overline{\Delta}a) \wedge (\overline{\Delta\Delta}a \vee \overline{\Delta}a) \\
&= (\overline{\Delta\Delta}a \vee \overline{\nabla\nabla}a) \wedge (\overline{\nabla\nabla}a \vee \overline{\Delta}a) \wedge (\overline{\Delta\Delta}a \vee \overline{\Delta\Delta\Delta}a) \\
&= \big(\overline{\nabla\nabla}a \vee (\overline{\Delta\Delta}a \wedge \overline{\Delta}a)\big) \wedge \overline{\Delta}(\overline{\Delta}a \wedge \overline{\Delta\Delta}a) \\
&= (\overline{\nabla\nabla}a \vee \mathsf{f}) \wedge \overline{\Delta}\mathsf{f} \\
&= \overline{\nabla\nabla}a \wedge \mathsf{t} \\
&= \overline{\nabla\nabla}a
\end{aligned}
$$

$$
\begin{aligned}
\overline{\nabla}{-a} &= (\overline{\nabla\Delta}a \vee \overline{\nabla\nabla}a) \wedge (\overline{\nabla\nabla}a \vee \overline{\nabla}a) \wedge (\overline{\nabla\Delta}a \vee \overline{\nabla}a) \\
&= (\overline{\Delta\Delta}a \vee \overline{\nabla\nabla}a) \wedge (\overline{\nabla\nabla}a \vee \overline{\nabla\nabla\nabla}a) \wedge (\overline{\Delta\Delta}a \vee \overline{\nabla}a) \\
&= (\overline{\Delta\Delta}a \vee \overline{\nabla\nabla}a) \wedge (\overline{\Delta\Delta}a \vee \overline{\nabla}a) \wedge (\overline{\nabla\nabla}a \vee \overline{\nabla\nabla\nabla}a) \\
&= \big(\overline{\Delta\Delta}a \vee (\overline{\nabla\nabla}a \wedge \overline{\nabla})\big) \wedge \overline{\nabla}(\overline{\nabla}a \wedge \overline{\nabla\nabla}a) \\
&= (\overline{\Delta\Delta}a \vee \mathsf{f}) \wedge \overline{\nabla}\mathsf{f} \\
&= \overline{\Delta\Delta}a \wedge \mathsf{t} \\
&= \overline{\Delta\Delta}a
\end{aligned}
$$

We now verify that the De Morgan negation is antitone, i.e. that it satisfies the inequality $-(a \vee b) \leq -a$. We have

$$
\begin{aligned}
-(a \vee b) &= (\overline{\Delta}(a \vee b) \wedge \overline{\nabla}(a \vee b)) \vee (\overline{\nabla}(a \vee b) \wedge (a \vee b)) \vee (\overline{\Delta}(a \vee b) \wedge (a \vee b)) \\
&= (\overline{\Delta}a \wedge \overline{\Delta}b \wedge \overline{\nabla}a \wedge \overline{\nabla}b) \vee (\overline{\nabla}a \wedge \overline{\nabla}b \wedge a) \vee (\overline{\nabla}a \wedge \overline{\nabla}b \wedge b) \\
&\quad \vee (\overline{\Delta}a \wedge \overline{\Delta}b \wedge a) \vee (\overline{\Delta}a \wedge \overline{\Delta}b \wedge b)
\end{aligned}
$$

and

$$
\begin{aligned}
-a &= (\overline{\Delta}a \wedge \overline{\nabla}a) \vee (\overline{\nabla}a \wedge a) \vee (\overline{\Delta}a \wedge a) \\
&= (\overline{\Delta}a \vee a) \wedge (\overline{\nabla}a \vee a) \wedge (\overline{\Delta}a \vee \overline{\nabla}a).
\end{aligned}
$$

To prove that $-(a \vee b) \le -a$, it therefore suffices to prove the 12 inequalities relating the disjunctive form of $-(a \vee b)$ with the conjunctive form of $-a$. The only non-trivial ones are

$$\overline{\nabla} a \wedge \overline{\nabla} b \wedge b \le \overline{\Delta} a \vee a \qquad \text{and} \qquad \overline{\Delta} a \wedge \overline{\Delta} b \wedge b \le \overline{\nabla} a \vee a.$$

But these follow from the inequalities $b \wedge \overline{\nabla} b \le a \vee \overline{\Delta} a$ and $b \wedge \overline{\Delta} b \le a \vee \overline{\nabla} a$.

We also verify that De Morgan negation satisfies double negation introduction and elimination:

$$\begin{aligned}
--a &= (\overline{\Delta}{-a} \wedge \overline{\nabla}{-a}) \vee (\overline{\nabla}{-a} \wedge -a) \vee (\overline{\Delta}{-a} \wedge -a) \\
&= (\overline{\nabla\nabla} a \wedge \overline{\Delta\Delta} a) \vee (\overline{\Delta\Delta} a \wedge -a) \vee (\overline{\nabla\nabla} a \wedge -a) \\
&= (\overline{\nabla\nabla} a \wedge \overline{\Delta\Delta} a) \vee (\overline{\Delta\Delta} a \wedge \overline{\nabla} a \wedge a) \vee (\overline{\nabla\nabla} a \wedge \overline{\Delta} a \wedge a) \\
&= (a \vee \overline{\nabla\nabla} a) \wedge (a \vee \overline{\Delta\Delta} a) \\
&= a \wedge (\overline{\nabla\nabla} a \vee \overline{\Delta\Delta} a) \\
&= a
\end{aligned}$$

Here the fourth equality holds because $\overline{\nabla} a \vee \overline{\nabla\nabla} a = \mathsf{t}$ and $\overline{\Delta} a \vee \overline{\Delta\Delta} a = \mathsf{t}$ and the last equality holds by the axiom $a \le \overline{\nabla\nabla} a \vee \overline{\Delta\Delta} a$.

The inequality $-(a \vee b) \le -a$ and the equality $--a = a$ now imply that $-$ is indeed a De Morgan negation. By double negation elimination the equalities $-\overline{\Delta} a = \Delta a$ and $-\overline{\nabla} a = \nabla a$ imply that $\overline{\Delta} a = -\Delta a$ and $\overline{\nabla} = -\nabla a$. The last equations

$$\begin{aligned}
x \wedge \nabla y &\le y \vee \Delta x & \qquad \Delta x \wedge -\Delta x &\le \mathsf{f} \\
x \wedge \Delta y &\le y \vee \nabla x & \qquad \Delta x \wedge \nabla x &\le x
\end{aligned}$$

are now immediate consequences of the equations

$$\begin{aligned}
a \wedge \overline{\Delta} a &\le b \vee \overline{\nabla} b & \qquad \overline{\Delta} a \wedge \overline{\Delta\Delta} a &\le \mathsf{f} \\
a \wedge \overline{\nabla} a &\le b \vee \overline{\Delta} b & \qquad \overline{\nabla} a \wedge \overline{\nabla\nabla} a &\le \mathsf{f}
\end{aligned}$$

It remains to show that the two constructions are mutually inverse. To this end it suffices to verify that the equalities

$$\begin{aligned}
-a &= (-\Delta a \wedge -\nabla a) \vee (a \wedge -\nabla a) \vee (a \wedge -\Delta a) \\
\Delta a &= -\Delta{-\Delta a}
\end{aligned}$$

hold in $\mathsf{DMA}\Delta$ and that the equalities

$$\begin{aligned}
\overline{\Delta} a &= (\overline{\Delta\Delta\Delta} a \wedge \overline{\nabla\Delta\Delta} a) \vee (\overline{\nabla\Delta\Delta} \wedge \overline{\Delta\Delta} a) \vee (\overline{\Delta\Delta\Delta} a \wedge \overline{\Delta\Delta} a) \\
\overline{\nabla} a &= \overline{\Delta\Delta}((\overline{\Delta} a \wedge \overline{\nabla} a) \vee (a \wedge \overline{\nabla} a) \vee (a \wedge \overline{\Delta} a))
\end{aligned}$$

hold in $\mathsf{DLat}\overline{\Delta\nabla}$. The first pair of equalities may be verified directly in $\mathbf{DM}_{\mathbf{4}}^{\mathbf{\Delta}}$ by the algebraic completeness theorem for $\mathsf{DMA}\Delta$ (Theorem 11.12). Verifying the other pair of equalities is an easy task which we leave to the interested reader. □

Distributive lattices with $\overline{\Delta}$ and $\overline{\nabla}$ and De Morgan algebras with Δ are therefore simply two different presentations of the same variety. Subvarieties of $\mathsf{DMA}\Delta$ also have natural alternative presentations.

Proposition 11.23.
Let $\mathbf{A}$ be a distributive lattice with $\overline{\Delta}$ and $\overline{\nabla}$. Then $\boldsymbol{\rho}\mathbf{A}$ satisfies $\Delta x \leq x$ ($x \leq \Delta x$) if and only if $\mathbf{A}$ satisfies $\overline{\nabla}x \leq \overline{\Delta}x$ ($\overline{\Delta}x \leq \overline{\nabla}x$).

Proof. If $\Delta x \leq x$, then $\Delta x = \nabla\Delta x \leq \nabla x$, hence $-\nabla x \leq -\Delta x$, i.e. $\overline{\nabla}x \leq \overline{\Delta}y$. Conversely, if $\overline{\nabla}x \leq \overline{\Delta}x$, then $-\nabla x \leq -\Delta x$, hence $\Delta x \leq \nabla x$. But this implies $\Delta x \leq x$ by Lemma 11.9, since $\Delta\Delta x = \Delta x \leq \Delta x$ holds always and $\nabla\Delta x = \Delta x \leq \nabla x$. The proof of the other claim is analogous. □

11.3 The Belnap–Dunn logic with Δ

In this section, we consider the logic

$$\mathcal{BD}\Delta := \mathrm{Log}\langle \mathbf{DM}_{\mathbf{4}}^{\boldsymbol{\Delta}}, \{\mathsf{t}, \mathsf{b}\}\rangle.$$

We first connect this logic to the variety of De Morgan algebras with Δ and exploit this connection to describe all axiomatic extensions of $\mathcal{BD}\Delta$. Then we provide Hilbert calculi for $\mathcal{BD}\Delta$ and its axiomatic extensions.

Theorem 11.24 (Algebraizability of $\mathcal{BD}\Delta$).
The logic $\mathcal{BD}\Delta$ is algebraizable via the translations

$$\boldsymbol{\tau}(\varphi) := \Delta\varphi \approx \mathsf{t} \qquad \text{and} \qquad \boldsymbol{\rho}(t \approx u) := t \leftrightarrow u.$$

Its equivalent algebraic semantics is the variety $\mathsf{DMA}\Delta$.

Proof. To establish algebraizability, it suffices to verify that for $a, b \in \mathbf{DM}_{\mathbf{4}}^{\boldsymbol{\Delta}}$ we have $\Delta a = \mathsf{t}$ if and only if $a \in \{\mathsf{t}, \mathsf{b}\}$, and $a \leftrightarrow b \in \{\mathsf{t}, \mathsf{b}\}$ if and only if $a = b$. To establish that $\mathsf{DMA}\Delta$ is the equivalent algebraic semantics of $\mathcal{BD}\Delta$, it suffices to recall that the variety of De Morgan algebras with Δ is generated as a quasivariety by the finite algebra $\mathbf{DM}_{\mathbf{4}}^{\boldsymbol{\Delta}}$. □

We can now use this result to describe the axiomatic extensions of $\mathcal{BD}\Delta$. Let us define the logics

$$\begin{aligned}
\mathcal{CL}\Delta &:= \mathrm{Log}\langle \mathbf{B}_{\mathbf{2}}^{\boldsymbol{\Delta}}, \{\mathsf{t}\}\rangle, \\
\mathcal{K}\Delta &:= \mathrm{Log}\langle \mathbf{K}_{\mathbf{3}}^{\boldsymbol{\Delta}}, \{\mathsf{t}\}\rangle, \\
\mathcal{LP}\Delta &:= \mathrm{Log}\langle \mathbf{P}_{\mathbf{3}}^{\boldsymbol{\Delta}}, \{\mathsf{b}, \mathsf{t}\}\rangle,
\end{aligned}$$

with $\mathcal{KO}\Delta := \mathcal{K}\Delta \cap \mathcal{LP}\Delta$.

Note that completeness theorems for natural deduction calculi for $\mathcal{BD}\Delta$, $\mathcal{K}\Delta$, $\mathcal{LP}\Delta$, and $\mathcal{CL}\Delta$ were already proved by Sano and Omori [69, Prop 3.2].

Theorem 11.25 (Axiomatic Extensions of $\mathcal{BD}\Delta$).
The logic $\mathcal{BD}\Delta$ has exactly four non-trivial proper axiomatic extensions, namely the logics $\mathcal{KO}\Delta$, $\mathcal{K}\Delta$, $\mathcal{LP}\Delta$, and $\mathcal{CL}\Delta$. Moreover:

(i) $\mathcal{KO}\Delta$ is the extension of $\mathcal{BD}\Delta$ by the axiom $-\Delta p \vee -\Delta - p \vee q \vee -q$,

(ii) $\mathcal{K}\Delta$ is the extension of $\mathcal{BD}\Delta$ by the axiom $-\Delta p \vee -\Delta - p$,

(iii) $\mathcal{LP}\Delta$ is the extension of $\mathcal{BD}\Delta$ by the axiom $p \vee -p$,

(iv) $\mathcal{CL}\Delta$ is the extension of $\mathcal{BD}\Delta$ by the axioms $-\Delta p \vee -\Delta - p$ and $p \vee -p$.

Proof. Thanks to the algebraizability of $\mathcal{BD}\Delta$ (Theorem 11.24), the lattice of axiomatic extensions of $\mathcal{BD}\Delta$ is isomorphic to the lattice of subvarieties of $\mathsf{DMA}\Delta$ via an isomorphism which to subvariety of $\mathsf{DMA}\Delta$ axiomatized by the equations $t_i \approx u_i$ for $i \in I$ assigns the extension of $\mathcal{BD}\Delta$ by the axioms $\boldsymbol{\rho}(t_i \approx u_i)$. But the only non-trivial proper subvarieties of $\mathsf{DMA}\Delta$ are $\mathsf{BA}\Delta$, $\mathsf{KA}\Delta$, and $\mathsf{KPA}\Delta$ (Corollary 11.14).

Under this correspondence, the inequality $\Delta x \leq x$, i.e. $x \wedge \Delta x \approx \Delta x$, corresponds to the axiom $(p \wedge \Delta p) \leftrightarrow \Delta p$. Since $(p \wedge \Delta p) \rightarrow \Delta p$ is a theorem of $\mathcal{BD}\Delta$, this axiom is equivalent in $\mathcal{BD}\Delta$ to $\Delta p \rightarrow (p \wedge \Delta p)$, hence to $\Delta p \rightarrow p$ (it suffices to verify that $\Delta x \rightarrow (x \wedge \Delta x) \approx \Delta x \rightarrow x$ holds in $\mathbf{DM}_{\mathbf{4}}^{\boldsymbol{\Delta}}$). Since $\Delta p \supset p$ is a theorem of $\mathcal{BD}\Delta$, this is further equivalent to $-p \supset -\Delta p$, i.e. to $-\Delta p \vee -\Delta - p$.

Likewise, the inequality $x \leq \Delta x$ corresponds to the axiom $(p \wedge \Delta p) \leftrightarrow p$, which is equivalent to $p \rightarrow \Delta p$, i.e. to $(p \supset \Delta p) \wedge (-\Delta p \supset -p)$. Since $p \supset \Delta p$ is a theorem, this is equivalent to $-\Delta p \supset -p$, hence to $\Delta p \vee -p$ and $p \vee -p$.

Finally, the inequality $x \wedge -x \leq y \vee -y$ similarly corresponds to the axiom $(p \wedge -p) \rightarrow (q \vee -q)$, i.e. to $((p \wedge -p) \supset (q \vee -q)) \wedge ((q \wedge -q) \supset (p \vee -p))$. Adding this formula as an axiom to $\mathcal{BD}\Delta$ is, by symmetry between p and q, equivalent to adding as an axiom the formula $(p \wedge -p) \supset (q \vee -q)$. But this formula is equivalent to $-\Delta p \vee -\Delta - p \vee q \vee -q$. □

We now provide a Hilbert-style axiomatization of the logic $\mathcal{BD}\Delta$. For the sake of simplicity, we shall in fact deal with the fragment of $\mathcal{BD}\Delta$ without the constants t and f, which can be defined as $p \supset p$ and $-(p \supset p)$.

This axiomatization of $\mathcal{BD}\Delta$ is based on an axiomatization of positive intuitionistic logic $\mathcal{IL}_+$, i.e. the fragment of intuitionistic logic in the signature $\{\wedge, \vee, \supset\}$. Unlike the axiomatization of $\mathcal{BD}$, whose only axioms involve the truth constants t and f, the axiomatization of $\mathcal{BD}\Delta$ will have many axioms but its only rule of inference will be Modus Ponens, i.e. the rule $p, p \supset q \vdash q$. The axioms for the implication fragment of $\mathcal{IL}_+$ are:

$$p \supset (q \supset p) \qquad (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$$

and the axioms for the other connectives are:

$$
\begin{array}{ll}
(p \wedge q) \supset p & p \supset (p \vee q) \\
(p \wedge q) \supset q & q \supset (p \vee q) \\
p \supset (q \supset (p \wedge q)) \qquad & (p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))
\end{array}
$$

Positive intuitionistic logic satisfies the deduction theorem in the form

$$\Gamma, \varphi \vdash_{\mathcal{IL}_+} \psi \iff \Gamma \vdash_{\mathcal{IL}_+} \varphi \supset \psi.$$

We now have two ways of extending this axiomatization of $\mathcal{IL}_+$ to an axiomatization of $\mathcal{BD}\Delta$, and each of them comes in two flavours depending on the status of the implication. Firstly, we can take the implication $a \supset b$ to be defined semantically as $-\Delta a \vee b$. If we then take $\varphi \supset \psi$ as a mere abbreviation for $-\Delta\varphi \vee \psi$, then our Hilbert calculus for $\mathcal{BD}\Delta$ will consist of the axioms of $\mathcal{IL}_+$ (with this reading of the implication) plus the axioms:

$$-\Delta p \supset (p \supset q)$$

$$
\begin{array}{cc}
\Delta p \supset p & p \supset \Delta p \\
--p \supset p & p \supset --p \\
-(p \vee q) \supset (-p \wedge -q) & (-p \wedge -q) \supset -(p \vee q) \\
-(p \wedge q) \supset (-p \vee -q) & (-p \vee -q) \supset -(p \wedge q)
\end{array}
$$

If we wish to treat $\varphi \supset \psi$ as a primitive connective instead of an abbreviation, it suffices to add axioms stating that $\varphi \supset \psi$ is equivalent to $-\Delta\varphi \vee \psi$ in all contexts. The axioms

$$(p \supset q) \supset (-\Delta p \vee q) \qquad -(p \supset q) \supset (p \wedge -q) \qquad (p \wedge -q) \supset -(p \supset q)$$

suffice for this purpose. These axioms imply, together with the others, that $\varphi \supset \psi$ and $-\Delta \vee \psi$ are equivalent in every context, or more precisely that substituting one for the other in any context yields an equivalent formula. We therefore only prove completeness for $\mathcal{BD}\Delta$ in the original language without treating implication as a primitive connective.

Theorem 11.26 (Completeness for $\mathcal{BD}\Delta$).
The Hilbert calculus defined above is sound and complete for $\mathcal{BD}\Delta$.

Proof. The strategy of the proof will be to appeal to the deduction theorem and the proof by cases property in order to reduce the completeness theorem for $\mathcal{BD}\Delta$ to the completeness theorem for $\mathcal{BD}$. The relation of provability in the Hilbert calculus will be denoted $\Gamma \vdash \varphi$. It will be convenient in this proof to use $\Gamma \vdash \Delta$ for finite Δ as an abbreviation for $\Gamma \vdash \bigvee \Delta$. The relation $\Gamma \vdash_{\mathcal{BD}\Delta} \Delta$ is defined similarly.

By a standard argument by induction over the length of proofs (which exactly copies the proof for classical and intuitionistic logic), the logic $\mathcal{BD}\Delta$ satisfies the deduction theorem in the form:

$$\Gamma, \varphi \vdash_{\mathcal{BD}\Delta} \Pi \iff \Gamma \vdash_{\mathcal{BD}\Delta} -\Delta\varphi, \Pi.$$

Using one of the axioms for disjunctions, it follows that it also satisfies the proof by cases property:

$$\Gamma, \varphi \vee \psi \vdash_{\mathcal{BD}\Delta} \Pi \iff \Gamma, \varphi \vdash_{\mathcal{BD}\Delta} \Pi \text{ and } \Gamma, \psi \vdash_{\mathcal{BD}\Delta} \Pi.$$

The rule of Modus Ponens is sound in $\mathcal{BD}\Delta$ and the axioms of the calculus are theorems of $\mathcal{BD}\Delta$. It therefore suffices to show that $\Gamma \vdash_{\mathcal{BD}\Delta} \Pi$ implies $\Gamma \vdash \Pi$. Since $\mathcal{BD}\Delta$ is a finitary logic by virtue of being defined by a finite matrix, we may without loss of generality restrict to finite Γ.

The axioms of the Hilbert calculus for $\mathcal{BD}\Delta$ ensure that each finite set of formulas is equivalent to a disjunction of conjunctions (as well as a conjunction of disjunctions) of formulas of the forms p, $-p$, and $-\Delta\psi$, where p is a propositional atom. Moreover, we may assume that transforming a formula into such an equivalent normal form does not increase the number of occurrences of the Δ operator.

Using the proof by cases property, it now suffices to prove that $\Gamma \vdash_{\mathcal{BD}\Delta} \Pi$ implies $\Gamma \vdash_{\mathcal{BD}\Delta} \Pi$ in case Γ and Π consist of formulas of the form p, $-p$, and $-\Delta\psi$. We prove this by induction over the number of occurrences of Δ in Γ and Π.

If Γ and φ contain no occurrences of Δ, then in fact $\Gamma \vdash_{\mathcal{BD}} \Pi$. By the completeness theorem for $\mathcal{BD}$ (Theorem 3.8) there is a proof of $\bigvee \Pi$ from Γ in the Hilbert calculus for $\mathcal{BD}$. To infer that $\Gamma \vdash \bigvee \Pi$ in the Hilbert calculus for $\mathcal{BD}\Delta$, it suffices to observe that the rules and axioms of the Hilbert calculus for $\mathcal{BD}$ are derivable in the Hilbert calculus for $\mathcal{BD}\Delta$.

Now suppose that Γ or Π contains a formula of the form $-\Delta\psi$. If $\Pi = \Lambda, -\Delta\psi$, then $\Gamma \vdash_{\mathcal{BD}\Delta} \Lambda, -\Delta\psi$ implies $\Gamma, \psi \vdash_{\mathcal{BD}\Delta} \Lambda$. But by the induction hypothesis $\Gamma, \psi \vdash \Lambda$, thus $\Gamma \vdash -\psi, \Lambda$ by the deduction theorem.

If $\Gamma = \Lambda, -\Delta\psi$, then $\Lambda, -\Delta\psi \vdash_{\mathcal{BD}\Delta} \Pi$, then $\Lambda \vdash_{\mathcal{BD}\Delta} -\Delta\psi, \Pi$ by the deduction theorem. But by the induction hypothesis $\Lambda \vdash -\Delta\psi, \Pi$ and $\Lambda, -\Delta\psi \vdash \Pi$ using the axiom $-\Delta p \supset (p \supset q)$. □

The second option is to interpret the implication $\varphi \supset \psi$ as $-\Delta\varphi \vee \Delta\psi$. If we treat the implication as an abbreviation, then the above calculus remains complete (by the same proof) even under this change of reading. If, on the other hand, we treat the implication as a primitive connective, then we need to replace the three additional axioms expressing the equivalence of $\varphi \supset \psi$ and $-\Delta\varphi \vee \psi$ by axioms expressing the equivalence of $\varphi \supset \psi$ and $-\Delta\varphi \vee \Delta\psi$. The following axioms suffice for this purpose:

$$\begin{array}{ll} -(p \supset q) \supset p & (p \supset q) \supset (-\Delta p \vee q) \\ -(p \supset q) \supset (q \supset r) & p \supset (q \vee -(p \supset q)) \end{array}$$

Finally, it remains to provide a calculus for the alternative presentation of $\mathcal{BD}\Delta$, called $\mathcal{DL}\overline{\Delta\nabla}$, whose primitive connectives are $\overline{\Delta}x$ and $\overline{\nabla}x$ rather than $-x$ and Δx. Again, for the sake of simplicity we in fact axiomatize the fragment without the constants t and f, definable as $p \supset p$ and $\overline{\Delta}(p \supset p)$.

The calculus for $\mathcal{DL}\overline{\Delta\nabla}$, will also build on the axiomatization $\mathcal{IL}_+$. This time, the implication $\varphi \supset \psi$ will be interpreted as an abbreviation for $\overline{\Delta}\varphi \vee \psi$. The calculus is then obtained from the axiomatization of $\mathcal{IL}_+$ by adding the axioms:

$$\overline{\Delta}p \supset (p \supset q)$$
$$\overline{\nabla\nabla}p \supset (\overline{\nabla}p \supset q)$$

$$\begin{array}{rr} p \supset \overline{\Delta\Delta}p & \overline{\Delta\Delta}p \supset p \\ p \supset \overline{\nabla\Delta}p & \overline{\nabla\Delta}p \supset p \\ \overline{\nabla\nabla}p \supset \overline{\Delta\nabla}p & \overline{\Delta\nabla}p \supset \overline{\nabla\nabla}p \\ \overline{\nabla}p \supset \overline{\nabla\nabla\nabla}p & \overline{\nabla\nabla\nabla}p \supset \overline{\nabla}p \end{array}$$

$$\begin{array}{rr} \overline{\Delta}(p \vee q) \supset (\overline{\Delta}p \wedge \overline{\Delta}q) & \overline{\nabla}(p \vee q) \supset (\overline{\nabla}p \wedge \overline{\nabla}q) \\ \overline{\Delta}(p \wedge q) \supset (\overline{\Delta}p \vee \overline{\Delta}q) & \overline{\nabla}(p \wedge q) \supset (\overline{\nabla}p \vee \overline{\nabla}q) \\ (\overline{\Delta}p \wedge \overline{\Delta}q) \supset \overline{\Delta}(p \vee q) & (\overline{\nabla}p \wedge \overline{\nabla}q) \supset \overline{\nabla}(p \vee q) \\ (\overline{\Delta}p \vee \overline{\Delta}q) \supset \overline{\Delta}(p \wedge q) & (\overline{\nabla}p \vee \overline{\nabla}q) \supset \overline{\nabla}(p \wedge q) \\ \overline{\nabla\nabla}(p \wedge q) \supset (\overline{\nabla\nabla}p \wedge \overline{\nabla\nabla}q) & (\overline{\nabla\nabla}p \wedge \overline{\nabla\nabla}q) \supset \overline{\nabla\nabla}(p \wedge q) \end{array}$$

Theorem 11.27 (Completeness for $\mathcal{DL}\overline{\Delta\nabla}$).
The Hilbert calculus defined above is sound and complete for $\mathcal{DL}\overline{\Delta\nabla}$.

Proof. This proof is analogous to the proof of the completeness theorem for $\mathcal{BD}\Delta$ (Theorem 11.26). The only difference is in the normal forms involved. The axioms ensure that each finite set of formulas is equivalent in $\mathcal{DL}\overline{\Delta\nabla}$ to a disjunction of conjunctions (as well as a conjunction of disjunctions) of formulas of the forms p, $\overline{\Delta}p$, $\overline{\nabla}p$, and $\overline{\nabla\nabla}p$. For the purposes of this proof we call such formulas *literals*. Let us again use $\Gamma \vdash \varphi$ to denote provability in the Hilbert calculus for $\mathcal{DL}\overline{\Delta\nabla}$. We will again abbreviate $\Gamma \vdash \bigvee \Pi$ as $\Gamma \vdash \Pi$ for finite Π, and likewise for $\Gamma \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Pi$. By a standard argument, we again have the deduction theorem in the form

$$\Gamma, \varphi \vdash \Pi \iff \Gamma \vdash \overline{\Delta}\varphi, \Pi.$$

We prove the equivalence $\Gamma \vdash_{\mathcal{BD}\Delta} \Pi \iff \Gamma \vdash \Pi$ by induction over the number of literals of the forms $\overline{\Delta}p$ and $\overline{\nabla}p$ in Γ and Π. Suppose first that Γ and Π contain no such literals. If Γ and Π do not share any literal, then there is a valuation $v : \mathbf{Fm} \to \boldsymbol{\tau}(\mathbf{DM_4^\Delta})$ such that

$$v(p) := \mathsf{t} \text{ if } p \in \Gamma \text{ and } \overline{\nabla} p \in \Gamma,$$
$$v(p) := \mathsf{b} \text{ if } p \in \Gamma \text{ and } \overline{\nabla} p \notin \Gamma,$$
$$v(p) := \mathsf{f} \text{ if } p \notin \Gamma \text{ and } \overline{\nabla} p \notin \Gamma,$$
$$v(p) := \mathsf{n} \text{ if } p \notin \Gamma \text{ and } \overline{\nabla} p \in \Gamma.$$

This valuation witnesses that $\Gamma \nvdash_{\mathcal{BD}\Delta} \Pi$. Thus if $\Gamma \vdash_{\mathcal{BD}\Delta} \Pi$ and Γ and Π contain no literals of the form $\overline{\Delta} p$ or $\overline{\nabla} p$, then in fact $\Gamma \vdash \Pi$.

Now suppose that Γ or Π contains a literal of the form $\overline{\Delta} p$ or $\overline{\nabla} p$. If $\Gamma = \Lambda, \overline{\Delta} p$, then $\Lambda, \overline{\Delta} p \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Pi$ implies $\Lambda \vdash_{\mathcal{DL}\overline{\Delta\nabla}} p \vee \varphi$, hence by the induction hypothesis $\Lambda \vdash p \vee \varphi$, and $\Lambda, \overline{\Delta} p \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Pi$ by the axiom $\overline{\Delta} p \supset (p \supset q)$.

If $\Pi = \Lambda, \overline{\Delta} p$, then $\Gamma \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Lambda, \overline{\Delta} p$ implies $\Gamma, p \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Lambda$, hence by the induction hypothesis $\Gamma, p \vdash \Lambda$, and by the deduction theorem $\Gamma \vdash \overline{\Delta} p, \Lambda$.

If $\Gamma = \Lambda, \overline{\nabla} p$, then $\Lambda, \overline{\nabla} p \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Pi$ implies $\Lambda \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \overline{\nabla\nabla} p, \Pi$, hence by the induction hypothesis $\Lambda \vdash \overline{\nabla\nabla} p, \Pi$, and by the axiom $\overline{\nabla\nabla} p \supset (\overline{\nabla} p \supset q)$ we have $\Lambda, \overline{\nabla} p \vdash \Pi$.

If $\Pi = \Lambda, \overline{\nabla} p$, then $\Gamma \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Lambda, \overline{\nabla} p$ implies $\Gamma, \overline{\nabla\nabla} p \vdash_{\mathcal{DL}\overline{\Delta\nabla}} \Lambda$, hence by the induction hypothesis $\Gamma, \overline{\nabla\nabla} p \vdash \Lambda$, by the deduction theorem $\Gamma \vdash \Lambda, \overline{\Delta\nabla\nabla} p$, and $\Gamma \vdash \Lambda, \overline{\nabla} p$ by the axioms $\overline{\Delta\nabla} p \supset \overline{\nabla\nabla} p$ and $\overline{\nabla\nabla\nabla} p \supset \overline{\nabla} p$. $\square$

If we wish to treat $\varphi \supset \psi$ as a primitive connective, we again need to add axioms expressing the equivalence between $\varphi \supset \psi$ and $\overline{\Delta}\varphi \vee \psi$ in all contexts. The reader can easily verify that the following axioms will do:

$$\begin{array}{rl}
(\varphi \supset \psi) \supset (\overline{\Delta}\varphi \vee \psi) & \qquad (\overline{\Delta}\varphi \vee \psi) \supset (\varphi \supset \psi) \\
\overline{\Delta}(\varphi \supset \psi) \supset (\varphi \wedge \overline{\Delta}\psi) & \qquad (\varphi \wedge \overline{\Delta}\psi) \supset \overline{\Delta}(\varphi \supset \psi) \\
\overline{\nabla}(\varphi \supset \psi) \supset (\varphi \wedge \overline{\nabla}\psi) & \qquad (\varphi \wedge \overline{\nabla}\psi) \supset \overline{\nabla}(\varphi \supset \psi)
\end{array}$$

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