Nested Sequents

Habilitationsschrift

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Abstract

We see how *nested sequents*, a natural generalisation of hypersequents, allow us to develop a systematic proof theory for modal logics. As opposed to other prominent formalisms, such as the display calculus and labelled sequents, nested sequents stay inside the modal language and allow for proof systems which enjoy the subformula property in the literal sense.

In the first part we study a systematic set of nested sequent systems for all normal modal logics formed by some combination of the axioms for seriality, reflexivity, symmetry, transitivity and euclideanness. We establish soundness and completeness and some of their good properties, such as invertibility of all rules, admissibility of the structural rules, termination of proof-search, as well as syntactic cut-elimination.

In the second part we study the logic of common knowledge, a modal logic with a fixpoint modality. We look at two infinitary proof systems for this logic: an existing one based on ordinary sequents, for which no syntactic cut-elimination procedure is known, and a new one based on nested sequents. We see how nested sequents, in contrast to ordinary sequents, allow for syntactic cut-elimination and thus allow us to obtain an ordinal upper bound on the length of proofs.

Contents

1	Intro	oduction	1	
2	Syst	ems for Basic Normal Modal Logics	5	
	2.1	Modal Axioms as Logical Rules	6	
		2.1.1 The Sequent Systems	6	
		2.1.2 Soundness	12	
		2.1.3 Completeness	13	
		2.1.4 Syntactic Cut-Elimination	19	
	2.2	Modal Axioms as Structural Rules	28	
		2.2.1 The Sequent Systems	28	
		2.2.2 Syntactic Cut-Elimination	29	
	2.3	Relation to Deep Inference	38	
	2.4	Discussion	41	
3	Systems for Common Knowledge			
	3.1	The Shallow Sequent System	46	
		3.1.1 The Problem for Cut-Elimination	49	
	3.2	The Nested Sequent System	49	
	3.3	Cut-Elimination for the Nested System	52	
	3.4 Cut-Elimination for the Shallow System		56	
		3.4.1 Embedding Shallow into Deep	56	
		3.4.2 Embedding Deep into Shallow	57	
	3.5	An Upper Bound on the Depth of Proofs	64	
	3.6	Discussion	66	

Chapter 1

Introduction

The problem of the proof theory of modal logic. The proof theory of modal logic as developed in Gentzen's sequent calculus is widely recognised as unsatisfactory: it provides systems only for a few modal logics, and does so in a non-systematic way. To solve this problem, many extensions of the sequent calculus have been proposed. The survey by Wansing [53] discusses many of them. The three most prominent formalisms seem to be the *hypersequent calculus*, due to Avron [6], the *display calculus* due to Belnap [7, 52], and *labelled sequent systems*, which have been introduced and studied by many researchers. The book by Viganò [51] and the article by Negri [36] provide a recent account of labelled sequent systems where more references can be found.

Hypersequents, display calculus, and labelled sequents. The relationship between these formalisms might be summarised as follows. The hypersequent calculus is a comparatively gentle extension of the sequent calculus, in particular it allows for a subformula property in the literal sense. Both the display calculus and the labelled sequent calculus are departing further from the ordinary sequent calculus, in particular they only satisfy weaker forms of the subformula property. On the other hand, both the display calculus and labelled systems are more expressive than hypersequents. They are known to capture all the basic modal logics that we are going to consider here, which is not true for hypersequents. In fact, the only modal logic captured so far in the hypersequent calculus, that has not been captured in the ordinary sequent calculus, is the modal logic S5. In general, there seems to be a tension between the desire to have a formalism which is expressive and the desire to have a formalism in which cut-free proofs are simple objects with a true subformula property.

Staying inside the modal language. A hypersequent is a sequence of ordinary sequents and can be read as a formula of modal logic: it is a disjunction where all disjuncts are prefixed by a box modality. A display sequent generally does not correspond to a modal formula: it contains structural connectives which correspond to backward-looking modalities, so connectives of tense logic. Similarly, a labelled sequent does not correspond to a formula of modal logic: it contains variables and an accessibility relation, so notions from predicate logic. In this sense, hypersequents stay inside the modal language, while display calculus and labelled sequents do not. In this work, we develop a proof theory for modal logic which aims to be as systematic and expressive as the display calculus and labelled sequents, but stays within the modal language and allows for a true subformula property, like hypersequents.

Nested sequents. To that end, we use *nested sequents*, which are essentially trees of sequents. They naturally generalise both sequents (which are nested sequents of depth zero) and hypersequents (which essentially are nested sequents of depth one). The notion of nested sequent has been invented several times independently. Bull [15] gives a proof system based on nested sequents for a fragment of propositional dynamic logic with converse. Kashima [30] gives proof systems for some tense logics and attributes the idea to Sato [43]. Unaware of these works, the author introduced the same notion of nested sequent under the name *deep sequent* in [10]. Poggiolesi introduced again the same notion but with a rather different notation under the name *tree-hypersequent* [38]. Nested sequents are also used by Goré et al. to give a proof system for bi-intuitionistic logic which is suitable for proof-search [23].

Deep inference. Nested sequents are tree-like structures with formulas occurring deeply inside of them. The proof systems introduced in this work crucially rely on being able to apply inference rules to all formulas, including those deeply inside. The general idea of applying rules deeply has been proposed several times in different forms and for different purposes. Schutte already used it in the 1950s in order to obtain systems without contraction and weakening, which he considered more elegant [44]. Guglielmi developed a formalism which is centered around applying rules deeply and which replaces the traditional tree-format of sequent calculus proofs by a linear format [26]. This solved the problem of finding a proof-theoretic system for a certain substructural logic which cannot be captured in the sequent calculus. The name of this formalism used to be calculus of structures but is now simply deep inference. Deep inference systems then have also been developed for some modal logics [28, 46, 47, 25]. The design of the proof systems in this work is inspired by deep inference. We will see the precise connection between nested sequent systems and deep inference systems later.

The big picture. This work is a case study in designing proof-theoretic systems for non-classical logics. It is an instance of the widely-known phenomenon that the notion of sequent, so the structural level of the proof system, has to be extended in order to accommodate certain logics. Our methodology here is that the structural level is not extended by arbitrary structural connectives, but only by those from the logic. As we do this, sequents become nested structures and so more formula-like. It then turns out that we need to allow inference rules to apply inside of these nested structures in order to obtain complete cut-free proof systems. There are many other instances of this phenomenon. The *logic of bunched* implications by Pym [41] is a substructural logic which has both a multiplicative and an additive conjunction. The proof systems for this logic have two corresponding structural connectives, which can be nested. Logics with non-associative conjunction also naturally lead to sequents which are nested structures, for example the non-associative Lambek calculus which can be found in the handbook article by Moortgat [35]. Another example are the proof systems for logics with *adjoint modalities*, certain epistemic logics for reasoning about information in a multi-agent system, by Dyckhoff and Sadrzadeh [42].

The plan. In the following there are two chapters which are independent. In the first chapter we study nested sequent systems for all normal modal logics formed by some combination of the axioms for seriality, reflexivity, symmetry, transitivity and euclideanness. We establish soundness and completeness and some of their good properties, such as invertibility, admissibility of the structural rules, termination of proof-search, as well as syntactic cut-elimination. This chapter contains work from [10, 11] and also from [13] which is joint work with Lutz Straßburger.

In the second chapter we study the logic of common knowledge, a modal logic with a fixpoint modality. We look at two infinitary proof systems for this logic: an existing one based on ordinary sequents, for which no syntactic cutelimination procedure is known, and a new one based on nested sequents. We see how nested sequents, in contrast to ordinary sequents, allow for syntactic cut-elimination and thus allow us to obtain an ordinal upper bound on the length of proofs. This chapter contains work from [8, 13] which are joint work with Thomas Studer.

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Chapter 2

Systems for Basic Normal Modal Logics

In this chapter we consider modal logics formed from the least normal modal logic K by adding axioms from the set $\{d, t, b, 4, 5\}$ which is shown in Figure 2.1. This gives rise to the modal logics shown in Figure 2.2. In the first section we consider sequent systems in which modal axioms are turned into logical rules, namely rules for the \diamond -connective. For each modal logic we find a corresponding cut-free sequent system which is sound and complete for this logic. However, some modal logics can be axiomatised in different ways, for example S5 can be axiomatised as K + $\{t, b, 4\}$ and as K + $\{t, 5\}$. Without cut, some of these axiomatisations turn out to be incomplete. For those cut-free systems which are complete we give a syntactic cut-elimination procedure, in the course of which we discover certain structural modal rules. In the second section we then study sequent systems where modal axioms are formalised not by using logical rules, but by using the structural modal rules we just found. This turns out to yield cut-free systems where each possible way of axiomatising a modal logic is complete.

At the end of the chapter we discuss some related formalisms.

k: no condition	Т	$\Box(A \lor B) \supset (\Box A \lor \Diamond B)$
d: serial	$\forall s \exists t. \ s \to t$	$\Box A \supset \diamondsuit A$
t: reflexive	$\forall s. \ s \to s$	$A \supset \diamondsuit A$
b: symmetric	$\forall st. \ s \to t \ \supset \ t \to s$	$A \supset \Box \diamondsuit A$
4: transitive	$\forall stu. \ s \rightarrow t \ \land \ t \rightarrow u \ \supset \ s \rightarrow u$	$\Box A \supset \Box \Box A$
5: euclidean	$\forall stu. \ s \rightarrow t \ \land \ s \rightarrow u \ \supset t \rightarrow u$	$\Diamond A \supset \Box \Diamond A$

Figure 2.1: Frame conditions and modal axioms



Figure 2.2: The modal "cube" [21]

2.1 Modal Axioms as Logical Rules

The plan of this section is as follows: we first introduce the sequent systems and then we see that they are sound and complete for the respective Kripke semantics. After that we see the syntactic cut-elimination procedure.

2.1.1 The Sequent Systems

Formulas. Propositions p and their negations \bar{p} are *atoms*, with \bar{p} defined to be p. Atoms are denoted by a, b, c, d. Formulas, denoted by A, B, C, D are given by the grammar

$$A ::= p \mid \bar{p} \mid (A \lor A) \mid (A \land A) \mid \diamondsuit A \mid \Box A$$

Given a formula A, its *negation* \overline{A} is defined as usual using the De Morgan laws, $A \supset B$ is defined as $\overline{A} \lor B$ and \bot and \top are defined as $p \land \overline{p}$ and $p \lor \overline{p}$, respectively, for some proposition p. Binary connectives are left-associative: $A \lor B \lor C$ denotes $((A \lor B) \lor C)$, for example.

Nested sequents. The set of *nested sequents* is inductively defined as follows:

- 1. a finite multiset of formulas is a nested sequent,
- 2. the multiset union of two nested sequents is a nested sequent,
- 3. if Γ is a nested sequent then the singleton multiset containing $[\Gamma]$ is a nested sequent.

In the following a *sequent* is a nested sequent. Sequents are denoted by $\Gamma, \Delta, \Lambda, \Pi$ and Σ . We adopt the usual notational conventions for sequents, in particular the comma in the expression Γ, Δ is multiset union and there is no distinction between a singleton multiset and its element. A sequent of the form $[\Gamma]$ is also called a *boxed sequent*. Clearly, a sequent is always a multiset of formulas and boxed sequents, so it is of the form

$$A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]$$
.

We assume a fixed arbitrary linear order on formulas and another fixed arbitrary linear order on boxed sequents. The *corresponding formula* of a sequent Γ , denoted $\underline{\Gamma}_{\epsilon}$, is defined as follows: the corresponding formula of a sequent as given above is \perp if m = n = 0 and otherwise it is

$$A_1 \vee \ldots \vee A_m \vee \Box(\underline{\Delta}_{1_{\mathsf{F}}}) \vee \ldots \vee \Box(\underline{\Delta}_{n_{\mathsf{F}}})$$
,

where formulas and boxed sequents are list according to the fixed orders. Often we do not distinguish between a sequent and its corresponding formula, for example a model of a sequent is a model of its corresponding formula. A sequent Γ has a *corresponding tree*, denoted *tree*(Γ), whose nodes are marked with multisets of formulas. The corresponding tree of the above sequent is



Often we do not distinguish between a sequent and its corresponding tree, for example the *root* of a sequent is the root of its corresponding tree.

Sequent contexts, unary. Informally, a context is a sequent with holes. We will mostly encounter sequents with just one hole. To mark the place of a hole in a sequent we use the symbol $\{ \}$, called the *hole*. We inductively define the set of *unary contexts*:

- 1. the multiset containing a single hole is a unary context,
- 2. the multiset union of a sequent and a unary context is a unary context, and
- 3. given a unary context C, the multiset containing a single occurrence of [C] is a unary context.

Unary contexts are denoted by $\Gamma\{ \}, \Delta\{ \}$ and so on. The multiset containing a single hole is also called the *empty context*. Our conventions for writing sequents also apply to sequent contexts, in particular comma denotes multiset union. The *depth* of a unary context $\Gamma\{ \}$, denoted *depth*($\Gamma\{ \}$) is defined as follows:

- 1. $depth(\{ \}) = 0$
- 2. $depth(\Gamma, \Delta\{\}) = depth(\Delta\{\})$
- 3. $depth([\Delta\{\}]) = depth(\Delta\{\}) + 1$.

Given a unary context $\Gamma\{\ \}$ and a sequent Δ we can obtain the sequent $\Gamma\{\Delta\}$ by filling the hole in $\Gamma\{\ \}$ with Δ . Formally, $\Gamma\{\Delta\}$ is defined inductively as follows:

1. if
$$\Gamma\{ \} = \{ \}$$
 then $\Gamma\{\Delta\} = \Delta$,

2. if $\Gamma\{ \} = \Gamma_1, \Gamma_2\{ \}$ then $\Gamma\{\Delta\} = \Gamma_1, \Gamma_2\{\Delta\}$ and

3. if $\Gamma\{ \} = [\Gamma_1\{ \}]$ then $\Gamma\{\Delta\} = [\Gamma_1\{\Delta\}]$.

Example 2.1 Given the unary context $\Gamma\{\ \} = A, [[B], \{\ \}]$ and the sequent $\Delta = C, [D]$ we can obtain the sequent

$$\Gamma\{\Delta\} = A, [[B], C, [D]]$$

Sequent contexts, generally. We want to allow multiple holes in a context and we want to allow filling holes with contexts, not just sequents. This is conceptually straightforward and formally somewhat technical, so the reader is invited to skip to Example 2.2. To keep track of the order of holes we index them with a number i > 0 as in $\{ \}_i$. Later the indices will never be shown since holes in a context are of course naturally ordered when written down on paper. We inductively define the set of *precontexts*:

- 1. a multiset containing a single hole $\{ \}_i$ with i > 0 is a precontext,
- 2. a multiset containing a single formula is a precontext,
- 3. the multiset union of two precontexts is a precontext, and
- 4. given a precontext C, the multiset containing a single occurrence of [C] is a precontext.

The arity of a context is the number of holes occurring in it. A sequent context, or just context, is a precontext of arity n such that for each $i \leq n$ the hole $\{ \}_i$ occurs exactly once in it. Notice that sequents are exactly the contexts of arity zero and, disregarding the index on the hole, unary contexts are exactly the contexts of arity one. A context of arity n is denoted by

$$\Gamma\underbrace{\{\}\dots\{\}}_{n-\text{times}}$$

Given an *n*-ary context $\Gamma\{ \} \dots \{ \}$ and *n* contexts C_1, \dots, C_n we can obtain the context

$$\Gamma\{\mathcal{C}_1\}\ldots\{\mathcal{C}_n\}$$

by filling the holes in $\Gamma\{ \} \dots \{ \}$ with C_1, \dots, C_n . Formally, to define this we first need an auxiliary definition adjusting indices of holes. Given a precontext C, let C^{+j} be the precontext obtained from it by replacing each hole $\{ \}_i$ by $\{ \}_{i+j}$. Given a precontext C and contexts C_1, \dots, C_n we now inductively define $\{C_1\} \dots \{C_n\}$ as follows, where a_j is the arity of C_j :

1. if
$$C = \{ \}_i$$
 then $C\{C_1\} \dots \{C_n\} = C_i^{+\sum_{j < i} a_j}$,
2. if $C = C', C''$ then $C\{C_1\} \dots \{C_n\} = C'\{C_1\} \dots \{C_n\}, C''\{C_1\} \dots \{C_n\}$ and
3. if $C = [C']$ then $C\{C_1\} \dots \{C_n\} = [C'\{C_1\} \dots \{C_n\}]$.

Clearly, $C\{C_1\} \dots \{C_n\}$ is a context if C and C_1, \dots, C_n are contexts. We leave out replacements of holes by holes, so by convention we write $\Gamma\{C_1\}\{\}$ instead of $\Gamma\{C_1\}\{C_2\}$ if C_2 is a hole. **Example 2.2** Given the binary context $\Gamma\{ \}\{ \} = A, [[B], \{ \}], \{ \}$ and the unary context $\Delta\{ \} = C, [\{ \}]$ we can obtain the binary context

$$\Gamma\{\Delta\{\}\}\{\} = A, [[B], C, [\{\}]], \{\}$$

where we omitted the indices of holes since in all contexts the holes are ordered from left to right as shown.

Inference rules, derivations and proofs. In the following instance of an inference rule ρ

$$\rho \frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Delta}$$

we call $\Gamma_1 \ldots \Gamma_n$ its premises and Δ its conclusion. We write ρ^n to denote n instances of ρ and ρ^* to denote an unspecified number of instances of ρ . A system, denoted by S, is a set of inference rules. A derivation in a system S is a finite tree whose nodes are labelled with sequents and which is built according to the inference rules from S. The sequent at the root is the conclusion and the sequents at the leaves are the premises of the derivation. Derivations are denoted by \mathcal{D} . A derivation \mathcal{D} with conclusion Γ in system S is sometimes shown as



The depth of a derivation \mathcal{D} is denoted by $|\mathcal{D}|$. Note that the depth of a derivation, which is a tree, has nothing to do with the depth of the sequents in it, which are also trees. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ where all premises are instances of the axiom $\Gamma\{p, \bar{p}\}$. Proofs are denoted by \mathcal{P} .

The sequent systems. Figure 2.3 shows the set of rules from which we form our deductive systems. System K is the set of rules $\{\land, \lor, \Box, \diamondsuit, k_c\}$. We will look at extensions of System K with any combination of the rules $\diamondsuit d_c, \diamondsuit t_c, \diamondsuit b_c, \circlearrowright d_c, \diamondsuit f_c, \diamondsuit f_c, \cr f_$

The $\diamond 5_c$ -rule is a bit special since it uses a binary context. It can actually be decomposed into three rules that use unary contexts, as we will see. However, we prefer the presentation as a single rule. The rule is best understood as allowing to do the following: when going from premise to conclusion, take some formula $\diamond A$, which is not at the root, and copy it to any other place in the sequent.

Example 2.3 Here is an example of a proof in system K, namely of some instance

$$\begin{split} \Gamma\{p,\bar{p}\} & \wedge \frac{\Gamma\{A\} \ \Gamma\{B\}}{\Gamma\{A \land B\}} & \vee \frac{\Gamma\{A,B\}}{\Gamma\{A \lor B\}} \\ & \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} & \Diamond \mathsf{k}_{\mathsf{c}} \frac{\Gamma\{\Diamond A, [\Delta, A]\}}{\Gamma\{\Diamond A, [\Delta]\}} \\ & \diamond \mathsf{d}_{\mathsf{c}} \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} & \diamond \mathsf{t}_{\mathsf{c}} \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} & \diamond \mathsf{b}_{\mathsf{c}} \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \\ & \diamond \mathsf{d}_{\mathsf{c}} \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} & \diamond \mathsf{t}_{\mathsf{c}} \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} & \diamond \mathsf{b}_{\mathsf{c}} \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \\ & \diamond \mathsf{d}_{\mathsf{c}} \frac{\Gamma\{\Diamond A, [\Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} & \diamond \mathsf{f}_{\mathsf{c}} \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}} & depth(\Gamma\{\ \}\{\emptyset\}) > 0 \end{split}$$

Figure 2.3: System $K + \{ \Diamond d_c, \Diamond t_c, \Diamond b_c, \Diamond 4_c, \Diamond 5_c \}$

$$\operatorname{nec} \frac{\Gamma}{[\Gamma]} \qquad \operatorname{wk} \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \operatorname{ctr} \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \qquad \operatorname{cut} \frac{\Gamma\{A\}}{\Gamma\{\emptyset\}}$$

Figure 2.4: Necessitation, weakening, contraction and cut

of the k-axiom:

Г

$$^{\wedge} \frac{\diamond(\bar{a}\wedge\bar{b}),[a,\bar{a}],\diamond b}{\diamond^{\mathsf{k}_{\mathsf{c}}}} \frac{\diamond(\bar{a}\wedge\bar{b}),[a,\bar{b}],\diamond b}{\diamond(\bar{a}\wedge\bar{b}),[a,\bar{b}],\diamond b}}{\diamond^{\mathsf{k}_{\mathsf{c}}}} \frac{\diamond(\bar{a}\wedge\bar{b}),[a,\bar{a},\bar{b}],\diamond b}{\diamond(\bar{a}\wedge\bar{b}),[a],\diamond b}}{\overset{\diamond(\bar{a}\wedge\bar{b}),[a],\diamond b}{\diamond(\bar{a}\wedge\bar{b}),[a],\diamond b}}{\overset{\diamond(\bar{a}\wedge\bar{b}),[a],\diamond b}{\diamond(\bar{a}\wedge\bar{b}),\Box a,\diamond b}}} = \frac{\overset{\circ}{\frac{\diamond(\bar{a}\wedge\bar{b}),(\Box a\vee\diamond b)}{\diamond(\bar{a}\vee b)}}}{\overset{\circ(\bar{a}\wedge\bar{b})\vee(\Box a\vee\diamond b)}{\Box(a\vee b)\supset(\Box a\vee\diamond b)}}$$

Admissibility, derivability and invertibility. We write $S \vdash \Gamma$ if there is a proof of Γ in system S. An inference rule ρ is *(depth-preserving) admissible* for a system S if for each proof in $S \cup \{\rho\}$ there is a proof of in S with the same conclusion (and with at most the same depth). An inference rule ρ is *derivable* for a system S if for each instance of ρ there is a derivation D in S with the same conclusion and such that each premise of D is a premise of the given instance of ρ .

For each rule ρ there is its *inverse*, denoted by $\overline{\rho}$, which is obtained by exchanging premise and conclusion. The $\overline{\wedge}$ -rule allows both $\Gamma\{A\}$ and $\Gamma\{B\}$ as conclusions of $\Gamma\{A \land B\}$. An inference rule ρ is *(depth-preserving) invertible* for a system S if $\overline{\rho}$ is (depth-preserving) admissible for S.

The rules shown in Figure 2.4 turn out to be admissible. We will now show this for the first three rules, for the cut rule it will be shown later.

Lemma 2.4 (Admissibility of structural rules and invertibility) For each system

 $K + \Diamond X_c$ with $X \subseteq \{d, t, b, 4, 5\}$ the following hold:

(i) The rules necessitation, weakening and contraction are depth-preserving admissible.

(ii) All its rules are depth-preserving invertible.

Proof. The admissibility of necessitation and weakening follows from a routine induction on the depth of the proof. The same works for the invertibility of the \land,\lor and \Box -rules in (ii). The inverses of all other rules are just weakenings. For the admissibility of contraction we also proceed by induction on the depth of the proof tree, using depth-preserving invertibility of the rules. The cases are easy for the propositional rules and for the $\Box, \diamond d_c, \diamond t_c$ -rules. For the $\diamond k_c$ -rule we consider the formula $\diamond A$ from its conclusion $\Gamma\{\diamond A, [\Delta]\}$ and its position inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have the cases 1) $\diamond A$ is inside Σ or 2) $\diamond A$ is inside $\Lambda\{$ }. We have three subcases for case 1: 1.1) [Δ] inside $\Lambda\{$ }, 1.2) [Δ] inside Σ , 1.3) Σ, Σ inside [Δ]. There are two subcases of case 2: 2.1) [Δ] inside $\Lambda\{$ } and 2.2) [Δ] inside Σ . All cases are either simpler than or similar to case 1.2, which is as follows:

$$\stackrel{\diamond_{\mathsf{k}_{\mathsf{c}}}}{\underset{\mathsf{ctr}}{\frac{\Lambda'\{\diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta]\}}{\Lambda'\{\diamond A, \Sigma', [\Delta]\}}}}{\Lambda'\{\diamond A, \Sigma', [\Delta]\}} \quad \rightsquigarrow \quad \stackrel{\overline{\diamond_{\mathsf{k}_{\mathsf{c}}}}}{\underset{\mathsf{k}_{\mathsf{c}}}{\frac{\Lambda'\{\diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta]\}}{\Lambda'\{\diamond A, \Sigma', [\Delta, A]\}}}{\underset{\mathsf{k}_{\mathsf{k}_{\mathsf{c}}}}{\frac{\Lambda'\{\diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta, A]\}}{\Lambda'\{\diamond A, \Sigma', [\Delta]\}}} \quad$$

where the instance of $\overline{\diamond k_c}$ in the proof on the right is removed because it is depth-preserving admissible and the instance of contraction is removed by the induction hypothesis. The case for the $\diamond 4_c$ -rule works the same way.

For the $\diamond b_c$ -rule we make a case analysis based on the position of $[\Delta, \diamond A]$ from its conclusion $\Gamma\{[\Delta, \diamond A]\}$ inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have three cases: 1) $[\Delta, \diamond A]$ inside $\Lambda\{\ \}, 2$) $[\Delta, \diamond A]$ in Σ and 3) Σ, Σ inside $[\Delta, \diamond A]$. Case 3 has two subcases: either $\diamond A \in \Sigma$ or not. All cases are trivial except for case 2 where invertibility of the $\diamond b_c$ -rule is used.

For the $\diamond 5_c$ rule we make a case analysis based on the positions of the sequent occurrences $\diamond A$ and \emptyset from its conclusion $\Gamma\{\diamond A\}\{\emptyset\}$ inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have two cases: 1) \emptyset inside $\Lambda\{\ \}, 2)$ \emptyset inside Σ . The first case is trivial, in the second we have two subcases: 1) $\diamond A$ inside $\Lambda\{\ \}$ and 2) $\diamond A$ inside Σ . Case 2.1 is similar to case 2.2 which is as follows:

$$\overset{\wedge 5_{c}}{\underset{ctr}{\underline{\Lambda\{\Sigma\{\diamond A\}\{\emptyset\}, \Sigma\{\diamond A\}\{\Diamond A\}\}}}{\Lambda\{\Sigma\{\diamond A\}\{\emptyset\}, \Sigma\{\diamond A\}\{\emptyset\}\}}} \sim \overset{\overline{\wedge 5_{c}}}{\underset{ctr}{\underline{\Lambda\{\Sigma\{\diamond A\}\{\diamond A\}, \Sigma\{\diamond A\}\{\diamond A\}\}}}}{\underset{\wedge 5_{c}}{\underline{\Lambda\{\Sigma\{\diamond A\}\{\diamond A\}, \Sigma\{\diamond A\}\{\diamond A\}\}}}{\Lambda\{\Sigma\{\diamond A\}\{\emptyset\}\}}} \sim \overset{\overline{\wedge 5_{c}}}{\underset{ctr}{\underline{\Lambda\{\Sigma\{\diamond A\}\{\diamond A\}, \Sigma\{\diamond A\}\{\diamond A\}\}}}}{_{\wedge 5_{c}}\underline{\Lambda\{\Sigma\{\diamond A\}\{\diamond A\}\}}}$$

By using weakening admissibility, we easily get the following proposition.

Proposition 2.5 (Relation between the \diamond -rules and the \diamond_c -rules) For each $\rho \in \{k, d, t, b, 4, 5\}$ we have that 

Figure 2.5: Diamond rules without built-in contraction

(i) the rule $\Diamond \rho$ is derivable for $\{\Diamond \rho_{c}, wk\}$ and admissible for system $\mathsf{K} + \Diamond \rho_{c}$, (ii) the rule $\Diamond \rho_{c}$ is derivable for $\{\Diamond \rho, \mathsf{ctr}\}$.

2.1.2 Soundness

To prove soundness, we first need some standard definitions for Kripke semantics.

Definition 2.6 (frames, models, validity) A frame is a pair (S, \rightarrow) of a nonempty set S of states and a binary relation \rightarrow on it. A model \mathcal{M} is a triple (S, \rightarrow, V) where (S, \rightarrow) is a frame and V is a a mapping which assigns a subset of S to each proposition, and which is called valuation. A model \mathcal{M} as given above induces a relation \models between states and formulas which is defined as usual. In particular we have $s \models p$ iff $s \in V(p)$, $s \models \bar{p}$ iff $s \notin V(p)$, $s \models A \lor B$ iff $s \models A$ or $s \models B$, $s \models A \land B$ iff $s \models A$ and $s \models B$, $s \models \Diamond A$ iff there is a state t such that $s \rightarrow t$ and $t \models A$, and $s \models \Box A$ iff for all t if $s \rightarrow t$ then $t \models A$. Further, a formula A is valid in a model \mathcal{M} , denoted $\mathcal{M} \models A$, if for all states s of \mathcal{M} we have $s \models A$. A formula A is valid in a frame (S, \rightarrow) , denoted $(S, \rightarrow) \models A$, if for all valuations V we have $(S, \rightarrow, V) \models A$. A formula is valid if it is valid in all frames. For a set of X of rule names or names of modal axioms we call a frame an X-frame if it satisfies all the frame conditions corresponding to the names in X. A formula is X-valid if it is valid in all X-frames.

The $\diamond 5_c$ -rule requires some care when proving its soundness because it is defined in terms of a binary context. We first show how it is derivable for three rules which, modulo built-in contraction, are special cases of the $\diamond 5_c$ -rule. The soundness of these rules is then easy to establish.

Lemma 2.7 (Decompose $\diamond 5_c$) The $\diamond 5_c$ -rule is derivable for $\{\diamond 5_1, \diamond 5_2, \diamond 5_3, ctr\}$, where $\diamond 5_1, \diamond 5_2, \diamond 5_3$ are the rules

$$\diamond \mathbf{5}_1 \frac{\Gamma\{[\Delta], \diamond A\}}{\Gamma\{[\Delta, \diamond A]\}} \quad , \quad \diamond \mathbf{5}_2 \frac{\Gamma\{[\Delta], [\Lambda, \diamond A]\}}{\Gamma\{[\Delta, \diamond A], [\Lambda]\}} \quad , \quad \diamond \mathbf{5}_3 \frac{\Gamma\{[\Delta, [\Lambda, \diamond A]]\}}{\Gamma\{[\Delta, \diamond A, [\Lambda]]\}}$$

Proof. Seen bottom-up, the $\diamond 5_c$ -rule allows to put a formula $\diamond A$ which occurs at a node different from the root into an arbitrary node. We can use contraction to duplicate $\diamond A$ and move one copy to the root and also to some child of the root by $\diamond 5_1$. By $\diamond 5_2$ we can move it to any child of the root and by $\diamond 5_3$ into any descendant of a child of the root.

Lemma 2.8 (Deep inference is sound) Let $X \subseteq \{d, t, b, 4, 5\}$, $\Gamma\{\}$ be a context and A, B be formulas. If the formula $A \supset B$ is X-valid then $\Gamma\{A\} \supset \Gamma\{B\}$ is X-valid.

Proof. By induction on the depth of $\Gamma\{\ \}$. We use the soundness of some Hilbert-style axiomatisation of K+X. To show the validity of

$$(\Gamma_1, [\Gamma_2\{A\}]) \supset (\Gamma_1, [\Gamma_2\{B\}])$$

we use the induction hypothesis to get $\Gamma_2\{A\} \supset \Gamma_2\{B\}$, necessitation to get $\Box(\Gamma_2\{A\} \supset \Gamma_2\{B\})$, the k-axiom to get $\Box(\Gamma_2\{A\}) \supset \Box(\Gamma_2\{B\})$, and finally propositional reasoning to get $\Gamma_1, [\Gamma_2\{A\}] \supset \Gamma_1, [\Gamma_2\{B\}]$.

Theorem 2.9 (Soundness) Let Γ, Δ and $\Gamma_1, \ldots, \Gamma_n$ be sequents. Then the following hold:

- (i) For any rule $\rho \in \mathsf{K}$ if $\rho \frac{\Gamma_1 \dots \Gamma_n}{\Delta}$ then $\Gamma_1 \wedge \dots \wedge \Gamma_n \supset \Delta$ is valid.
- (ii) For any rule $\rho \in \{d, t, b, 4, 5\}$ if $\Diamond \rho_c \frac{\Gamma}{\Delta}$ then $\Gamma \supset \Delta$ is $\{\rho\}$ -valid.
- (iii) For any $X \subseteq \{d, t, b, 4, 5\}$ if $K + \Diamond X_c \vdash \Gamma$ then Γ is X-valid.

Proof. The axiom is valid in all frames which follows from an induction on the depth of $\Gamma\{\ \}$ where necessitation is used in the induction step. Thus (i) and (ii) imply (iii). Most cases of (i) are trivial, for the \land -rule it follows from an induction on the context and uses the implication $\Box A \land \Box B \supset \Box (A \land B)$. Lemma 2.8 (Deep inference is sound) used together with the k-axiom yields that the premise of the $\diamond k_c$ -rule implies its conclusion. The cases from (ii) for the $\{\diamond d_c, \diamond t_c, \diamond b_c, \diamond d_c\}$ -rules are similar to the $\diamond k_c$ -rule, using the corresponding modal axiom.

For the soundness of the $\diamond 5_c$ -rule we use Lemma 2.7 (Decompose $\diamond 5_c$) and show soundness of the rules $\diamond 5_1, \diamond 5_2, \diamond 5_3$. For $\diamond 5_3$ we show that a euclidean countermodel for the conclusion is also a countermodel for the premise, the other cases are similar. A countermodel for $[\Delta, \diamond A, [\Lambda]]$ has to contain states $s \to t \to u$ such that $t \not\models \Delta, u \not\models \Lambda$ and $v \not\models A$ for any v with $t \to v$. We need to show that for any w with $u \to w$ we have $w \not\models A$. By euclideanness we obtain, in this order: $t \to t, u \to t, t \to w$. Thus $w \not\models A$.

2.1.3 Completeness

The current set of modal rules does not allow for a modular completeness result of the form "*if* Γ *is* X*-valid then* K + \diamond X_c \vdash Γ ". It is easy to check that some of our systems are incomplete.

Fact 2.10 (Incompleteness) For any propositional variable p we have that the formula $\Box p \supset \Box \Box p$ holds in any $\{t, 5\}$ -frame and the formula $\Diamond p \supset \Box \Diamond p$ holds in any $\{b, 4\}$ -frame, but:

- (i) $\mathsf{K} + \{ \diamondsuit \mathsf{t}_{\mathsf{c}}, \diamondsuit \mathsf{5}_{\mathsf{c}} \} \nvDash \Box p \supset \Box \Box p$ and
- (ii) $\mathsf{K} + \{ \diamondsuit \mathsf{b}_{\mathsf{c}}, \diamondsuit \mathsf{4}_{\mathsf{c}} \} \nvDash \diamondsuit p \supset \Box \diamondsuit p$

However, while not every combination of modal rules is sound and complete for the respective set of frames, we can define a condition on rule combinations which ensures that they are complete.

Definition 2.11 (45-closed) Let $X \subseteq \{d, t, b, 4, 5\}$. The set X is 45-closed if for $\rho \in \{4, 5\}$ we have that if all X-frames satisfy ρ then $\rho \in X$. Both of the sets $\{t, 5\}$ and $\{b, 4\}$ are not 45-closed, for example, while both $\{t, 4, 5\}$ and $\{b, 4, 5\}$ are. A set of modal rules is 45-closed if its underlying set of names of modal axioms is 45-closed.

The completeness result we are about to prove holds for 45-closed X. It is easy to check that for each set of frames which can be characterised by our five axioms there is a combination of modal rules which is 45-closed and thus is also sound and complete. In order to prove our completeness result, we first need some preliminary definitions which will help us to extract a tree-like Kripke model from a sequent.

Definition 2.12 (subtree of a sequent) A sequent Δ is an *immediate subtree* of a sequent Γ if there is a sequent Λ such that $\Gamma = \Lambda$, $[\Delta]$. It is a *proper subtree* if it is an immediate subtree either of Γ or of a proper subtree of Γ , and it is a *subtree* if it is either a proper subtree of Γ or $\Delta = \Gamma$. The set of all subtrees of Γ is denoted by $st(\Gamma)$. A formula A is *in* a sequent Γ if $A \in \Gamma$ and it is *inside* Γ if there is a subtree Δ of Γ such that $A \in \Delta$.

Our sequents are based on multisets. We need a way to stop proof search once their underlying sets remain the same, so we need the following notion:

Definition 2.13 (set sequent) The set sequent of the sequent

$$A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]$$

is the underlying set of

$$A_1,\ldots,A_m,[\Lambda_1],\ldots,[\Lambda_n]$$

where $\Lambda_1 \ldots \Lambda_n$ are the set sequents of $\Delta_1 \ldots \Delta_n$. Clearly the set sequent of a given sequent is again a sequent since a set is a multiset.

We will not directly prove completeness of the systems $K + \Diamond X_c$, but of different, equivalent systems $(K + \Diamond X_c)^\circ$ that we define now. For each rule ρ we define a rule ρ° which keeps the main formula from the conclusion. For most rules $\rho = \rho^\circ$ except for the following rules:

$$\wedge^{\circ} \frac{\Gamma\{A \wedge B, A\} \Gamma\{A \wedge B, B\}}{\Gamma\{A \wedge B\}} \qquad \vee^{\circ} \frac{\Gamma\{A \vee B, A, B\}}{\Gamma\{A \vee B\}}$$

 $\Box^{\circ} \ \frac{\Gamma\{\Box A, [A]\}}{\Gamma\{\Box A\}} \quad \text{where in the conclusion the node of the active formula} \\ \text{does not have a child node which contains } A$

$$\diamond d_{c}^{\circ} \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}}$$
 where in the conclusion the node of the active formula does not have a child node.

In addition, each rule ρ° carries the proviso that for all of its premises the set sequent is different from the set sequent of the conclusion. Given a system S the system S° is obtained by replacing each rule $\rho \in S$ by ρ° . Systems S and S° will turn out to be equivalent, as we will know after the completeness theorem. For now we just prove one direction of the equivalence.

Lemma 2.14 (S° into S) For all $X \subseteq \{d, t, b, 4, 5\}$ and for all sequents Γ we have that $(K + \Diamond X_c)^{\circ} \vdash \Gamma$ implies $K + \Diamond X_c \vdash \Gamma$.

Proof. By a standard induction on the proof tree, using contraction and weakening admissibility for $K + \Diamond X_c$.

In order to prove completeness we need some closures of relations.

Definition 2.15 (some closures of relations) Let \rightarrow be a binary relation on a set S. Then \leftarrow denotes its inverse, \leftrightarrow its symmetric closure, \rightarrow^+ its transitive closure and \rightarrow^* its reflexive-transitive closure. For $X \subseteq \{t, b, 4, 5\} \rightarrow^X$ denotes the smallest relation that includes \rightarrow and has the properties in X. The same conventions are used for different arrows that denote relations, such as \Rightarrow , the inverse of which is \Leftarrow , and so on.

We will see shortly that \rightarrow^{X} is well-defined. First we need to characterise the euclidean and the transitive-euclidean closure of a relation.

Definition 2.16 ((transitive-)euclidean connection) Let \rightarrow be a binary relation on a set S and let $s, t \in S$. A *euclidean connection* for \rightarrow from s to t is a nonempty sequence $s_1 \ldots s_n$ of elements of S such that we have

$$s \leftarrow s_1 \leftrightarrow s_2 \leftrightarrow \dots \leftrightarrow s_n \to t$$

A transitive-euclidean connection is defined likewise but such that

$$s = s_1 \leftrightarrow s_2 \leftrightarrow \dots \leftrightarrow s_n \to t$$

We write $s \rightarrow_{(4)5} t$ if there is a (transitive-)euclidean connection for \rightarrow from s to t.

Lemma 2.17 (\rightarrow^{X} is well-defined) Let \rightarrow be a binary relation on a set S. Then the following hold:

(i) For all $X \subseteq \{t, b, 4, 5\}$ the relation \rightarrow^X is well-defined.

(ii) The relation $\rightarrow \cup \rightarrow_5$ is the least euclidean relation that contains \rightarrow .

(iii) The relation \rightarrow_{45} is the least transitive and euclidean relation that contains \rightarrow .

Proof. (i) is easy to check except for the cases for $\{5\}$ and $\{4,5\}$, which follow from (ii) and (iii).

(ii) Euclideanness is easy to check. For leastness we show that any euclidean relation \Rightarrow that includes \rightarrow also includes \rightarrow_5 . If $s \rightarrow_5 t$ then $s \Rightarrow_5 t$. We show $s \Rightarrow_5 t$ for a euclidean connection of length n implies $s \Rightarrow t$ by induction on n. Assume there is an s_i in the euclidean connection such that $s_{i-1} \Rightarrow s_i \Leftarrow s_{i+1}$.

Then we have two smaller euclidean connections to which we apply the induction hypothesis and obtain $s \Rightarrow t$ by euclideanness. If there is no such s_i then the euclidean connection looks as follows:

$$s = s_0 \Leftarrow s_1 \Leftarrow \ldots \Leftarrow s_j \Rightarrow \ldots \Rightarrow s_n \Rightarrow s_{n+1} = t \quad ,$$

and by euclideanness we have $s_{j-1} \Rightarrow s_{j+1}$ and thus removing s_j yields a smaller euclidean connection from s to t which by induction hypothesis implies $s \Rightarrow t$.

(iii) Euclideanness and transitivity are easy to check. For leastness we show that any transitive-euclidean relation \Rightarrow that includes \rightarrow also includes \rightarrow_{45} . If $s \rightarrow_{45} t$ then $s \Rightarrow_{45} t$. If there is no s_i in the transitive-euclidean such that $s_i \Leftarrow s_{i+1}$, then $s \Rightarrow t$ follows by transitivity. Otherwise, choose the first such s_i . We have a euclidean connection from s_i to t, thus similarly to (ii) obtain $s_i \Rightarrow t$ and by transitivity $s \Rightarrow s_i$ and $s \Rightarrow t$.

Definition 2.18 (serial closure) Let \rightarrow be a binary relation on a set *S*. Its *serial closure*, denoted \rightarrow^{d} , is obtained from \rightarrow by adding $s \rightarrow s$ for each $s \in S$ which violates seriality. For $X \subseteq \{t, b, 4, 5\}$ the relation $\rightarrow^{X \cup \{d\}}$ is defined as $(\rightarrow^{X})^{d}$.

Lemma 2.19 (Serial closure preserves frame conditions) Let \rightarrow be a binary relation on a set S. If \rightarrow satisfies a frame condition in $\{t, b, 4, 5\}$ then \rightarrow^d also satisfies that frame condition.

Proof. For reflexivity this is clear since a reflexive relation is its own serial closure. For symmetry this is clear since only loops are added, which are their own inverses. For transitivity, assume that we have $s \rightarrow^{\mathsf{d}} t$ and $t \rightarrow^{\mathsf{d}} u$. If either s = t or t = u then we have $s \rightarrow^{\mathsf{d}} u$. So assume $s \neq t$ and $t \neq u$. Then $s \rightarrow t$ and $t \rightarrow u$ and by transitivity of \rightarrow we get $s \rightarrow u$ and thus $s \rightarrow^{\mathsf{d}} u$.

For euclideanness, assume that $s \to^{\mathsf{d}} t$ and $s \to^{\mathsf{d}} u$. We need to show that $t \to^{\mathsf{d}} u$. If s = t then we are done, so assume $s \neq t$ which implies $s \to t$. Since $s \to^{\mathsf{d}} u$ and since s does not violate seriality we have $s \to u$. By euclideanness of \to we obtain $t \to u$ and thus $t \to^{\mathsf{d}} u$.

Definition 2.20 (cyclic, finished, $prove(\Gamma, X)$) A leaf of a sequent is *cyclic* if there is an inner node in the sequent that carries the same set of formulas. A node in a sequent is *finished* for a system S if no rule from S applies to a formula in this node. A sequent is *finished* for a system S if all its nodes are either finished for S or cyclic. We define a procedure $prove(\Gamma, X)$, which takes a sequent Γ and a set $X \subseteq \{d, t, b, 4, 5\}$ and builds a derivation tree for Γ by applying rules from $(K + \Diamond X_c)^\circ$ to non-axiomatic and unfinished derivation leaves in a bottom-up fashion. It is shown in Figure 2.6. If $prove(\Gamma, X)$ terminates and all derivation leaves are axiomatic then it *succeeds* and if it terminates and there is a non-axiomatic derivation leaf then it *fails*.

Definition 2.21 (size of a sequent, $sf(\Gamma)$) The *size* of a sequent is the number of nodes of its corresponding tree. The set of subformulas of a sequent Γ , denoted $sf(\Gamma)$ is the set of all subformulas of all formulas which are element of some node of the sequent.

Repeat

(step 1) Apply the rules in $((K+\Diamond X_c)\setminus\{\Box,\Diamond d_c\})^\circ$ as long as possible.

(step 2) Wherever possible, apply the rules in $(\{\Box\} \cup (\Diamond X_c \cap \{\Diamond d_c\}))^\circ$ once.

Until each non-axiomatic derivation leaf is finished.

Figure 2.6: The algorithm $prove(\Gamma, X)$

Lemma 2.22 (Termination) For all sets $X \subseteq \{d, t, b, 4, 5\}$ and for all sequents Γ the procedure $prove(\Gamma, X)$ terminates after at most $2^{|sf(\Gamma)|}$ iterations (of the repeat-until-loop).

Proof. Consider a sequence of sequents along a given branch of the derivation starting from the root. A rule application in step 1 does not create new nodes in the sequent and causes the set of formulas at some node in the sequent to strictly grow. By the subformula property only finitely many formulas can occur in a node, so step 1 terminates. If after step 1 there is an unfinished leaf in a sequent then the size of the sequent strictly grows in step 2. Since there are only $2^{|sf(\Gamma)|}$ different sets of formulas that can occur each unfinished sequent leaf has to be cyclic before $2^{|sf(\Gamma)|}$ iterations. Then the sequent will be finished if it is not axiomatic, and thus the algorithm terminates.

Theorem 2.23 (Completeness) For all 45-closed sets $X \subseteq \{d, t, b, 4, 5\}$ and for all sequents Γ the following hold:

(i) If Γ is X-valid then $\mathsf{K} + \Diamond \mathsf{X}_{\mathsf{c}} \vdash \Gamma$.

(ii) If $prove(\Gamma, X)$ fails then there is a finite X-frame in which Γ is not valid.

Proof. The contrapositive of (i) follows from (ii): if $\mathsf{K} + \Diamond \mathsf{X}_{\mathsf{c}} \nvDash \Gamma$ then by Lemma 2.14 (\mathcal{S}° into \mathcal{S}) also ($\mathsf{K} + \Diamond \mathsf{X}_{\mathsf{c}}$) $\nvDash \Gamma$ and thus in particular $prove(\Gamma, \mathsf{X})$ cannot yield a proof and by Lemma 2.22 (Termination) has to fail. Thus by (ii) Γ is not X-valid. For (ii) we define a model \mathcal{M} on an X-frame for which we prove that it is a countermodel for Γ . Let Γ^* be the set sequent of the nonaxiomatic finished sequent obtained. Let Y be the set of all cyclic leaves in Γ^* . Let $S = st(\Gamma^*) \setminus Y$. Let $f: Y \to S$ be some function which maps a cyclic leaf to a sequent in S whose root carries the same set of formulas and extend f to $st(\Gamma^*)$ by the identity on S. Define a binary relation \to on S such that $\Delta \to \Lambda$ iff either 1) Λ is an immediate subtree of Δ or 2) Δ has an immediate subtree $\Sigma \in Y$ and $f(\Sigma) = \Lambda$. Let $V(p) = \{\Delta \in S \mid \bar{p} \in \Delta\}$. Let $\mathcal{M} = (S, \to^{\mathsf{X}}, V)$. We prove three claims about \mathcal{M} , each claim depending on the next. Since all rules seen top-down preserve countermodels Claim 1 implies that $\mathcal{M} \nvDash \Gamma$.

Claim 1 For each sequent $\Delta \in st(\Gamma^*)$ we have that $\mathcal{M}, f(\Delta) \not\models \Delta$.

By induction on the depth of Δ . For depth zero this follows from Claim 2 and the fact that a formula is in Δ iff it is in $f(\Delta)$. So let

 $\Delta = A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n] \quad \text{and} \quad n > 0 \quad .$

Then $f(\Delta) = \Delta$. We have $\mathcal{M}, f(\Delta) \not\models A_i$ for all $i \leq m$ by Claim 2 and $\mathcal{M}, \Delta \not\models [\Delta_i]$ because $\Delta \to f(\Delta_i)$ and by induction hypothesis $\mathcal{M}, f(\Delta_i) \not\models \Delta_i$. **Claim 2** For each sequent $\Delta \in S$ and for each formula $A \in \Delta$ we have that $\mathcal{M}, \Delta \not\models A$.

By induction on the depth of A. For atoms it is clear from the definition of \mathcal{M} and the fact that Γ^* is not axiomatic. For the propositional connectives it is clear from the shape of the \land, \lor -rules. If $A = \Box B$ then by the \Box -rule we have some $[\Lambda] \in \Delta$ with $B \in \Lambda$. By induction hypothesis we have $\mathcal{M}, \Lambda \not\models B$ and thus $\mathcal{M}, \Delta \not\models \Box B$. If $A = \Diamond B$ then by Claim 3 we have $B \in \Lambda$ for all Λ with $\Delta \rightarrow^{\mathsf{X}} \Lambda$, and thus $\mathcal{M}, \Lambda \not\models B$. Thus $\mathcal{M}, \Delta \not\models \Diamond B$.

Claim 3 For all sequents $\Delta, \Lambda \in S$ with $\Delta \to^{\mathsf{X}} \Lambda$ and for each formula A it holds that if $\diamond A \in \Delta$ then $A \in \Lambda$.

We make a case analysis on X. Note that each modal logic has exactly one 45-complete axiomatisation, with the exception of S5, which has two.

 $\mathbf{K} \mathsf{X} = \emptyset$: By the definition of \rightarrow there is an immediate subtree of Δ whose root node carries the same set of formulas as the root node of Λ . By the $\diamond \mathsf{k}_{\mathsf{c}}$ -rule we have A in (the root node of) all immediate subtrees of Δ .

 $\mathbf{T} \mathsf{X} = \{\mathsf{t}\} : \Delta \to {^{\{\mathsf{t}\}}} \Lambda \text{ iff } \Delta \to \Lambda \text{ or } \Delta = \Lambda.$ In the second case $A \in \Lambda$ follows from the $\diamond \mathsf{t}_{\mathsf{c}}$ -rule.

KB $X = \{b\}$: $\Delta \to \{b\}$ Λ iff $\Delta \to \Lambda$ or $\Lambda \to \Delta$. In the second case $A \in \Lambda$ follows by the $\diamond b_c$ -rule.

K4 X = {4}: $\Delta \rightarrow^{\{4\}} \Lambda$ iff there is a sequence

$$\Delta = \Delta_0 \to \Delta_1 \to \Delta_2 \to \dots \to \Delta_n = \Lambda ,$$

with $n \ge 1$. An induction on *i* gives us that $\Diamond A \in \Delta_i$ for $0 \le i \le n$ by using the $\Diamond 4_c$ -rule. By the $\Diamond k_c$ -rule it follows that $A \in \Delta_n$.

K5 X = {5}: By Lemma 2.17 (\rightarrow^{X} is well-defined) we have $\Delta \rightarrow^{\{5\}} \Lambda$ iff $\Delta \rightarrow \Lambda$ or there is a euclidean connection from Δ to Λ . In the second case there are sequents Π, Σ such that $\Delta \leftarrow \Pi$ and $\Sigma \rightarrow \Lambda$. Thus there is an immediate subtree Δ' of Π with the same formulas as Δ and an immediate subtree Λ' of Σ with the same formulas as Λ . Since $\Diamond A \in \Delta$ we have $\Diamond A \in \Delta'$ and since $\Delta' \neq \Gamma^*$ by the $\Diamond \mathsf{S}_{\mathsf{c}}$ -rule we have $\Diamond A \in \Sigma$. Thus by the $\Diamond \mathsf{k}_{\mathsf{c}}$ -rule we have A in Λ' and thus in Λ .

K45 X = {4,5}: By Lemma 2.17 (\rightarrow^{X} is well-defined) we have $\Delta \rightarrow^{\{4,5\}} \Lambda$ iff $\Delta \rightarrow \Lambda$ or there is a transitive-euclidean connection from Δ to Λ . In the second case there is a sequent Σ such that $\Sigma \rightarrow \Lambda$ and thus an immediate subtree Λ' of Σ with the same formulas as Λ . Since $\Diamond A \in \Delta$, by the $\Diamond 5_{c}$ - and $\Diamond 4_{c}$ -rules we have $\Diamond A$ in every subtree of Γ^* and thus also in Σ , and by the $\Diamond k_c$ -rule we have A in Λ' and thus in Λ . (It is sufficient to have the $\Diamond 5_{1c}$ -rule instead of the $\Diamond 5_{c}$ -rule for all X which contain 4.)

KB5 $X = \{b, 4, 5\}$: $\Delta \to {b,4,5} \Lambda$ iff $\Delta \leftrightarrow^+ \Lambda$. Thus there is a sequent Σ such that either $\Sigma \to \Lambda$ or $\Sigma \leftarrow \Lambda$. Rule 4, 5 imply that $\Diamond A$ is in every subtree of Γ^* and thus in particular in Σ . We have $A \in \Lambda$ in the first case by the $\Diamond k_c$ -rule and in the second case by the $\Diamond b_c$ -rule.

KTB $X = \{b, t\}$: $\Delta \to \{b, t\}$ Λ iff $\Delta \to \Lambda$ or $\Delta \leftarrow \Lambda$ or $\Delta = \Lambda$. In these cases

2.1. MODAL AXIOMS AS LOGICAL RULES

 $A \in \Lambda$ respectively follows from the $\Diamond k_{c}$ - or $\Diamond b_{c}$ - or $\Diamond t_{c}$ -rule.

S4 X = {t, 4}: $\Delta \rightarrow^{\{t,4\}} \Lambda$ iff $\Delta \rightarrow^+ \Lambda$ or $\Delta = \Lambda$. In the first case $A \in \Lambda$ follows from the rules $\diamond 4_c$ and $\diamond k_c$ and in the second case from the $\diamond t_c$ -rule.

S5(1) $X = \{t, 4, 5\}$: $\Delta \to {t, 4, 5}$ Λ iff $\Delta \leftrightarrow^* \Lambda$. We have $\Diamond A$ in all subtrees of Γ^* by the rules $\Diamond 4_c, \Diamond 5_c$ and thus also A by the $\Diamond t_c$ -rule.

S5(2) $X = \{d, b, 4, 5\}: \Delta \to {d, b, 4, 5} \Lambda$ iff $\Delta \leftrightarrow^* \Lambda$. We have $\Diamond A$ in all subtrees of Γ^* by the rules $\Diamond 4_c, \Diamond 5_c$ and thus also $\Diamond A \in \Lambda$. By the $\Diamond d_c$ -rule the root of Λ has a child node. By the $\Diamond 4_c$ -rule $\Diamond A$ is in this child node and by the $\Diamond b_c$ -rule $A \in \Lambda$.

KD,KDB,KD4,KD5,KD45 The argument for all these cases is similar to the same system without d. Take the corresponding X, then $\Delta \rightarrow^{X \cup \{d\}} \Lambda$ iff $\Delta \rightarrow^X \Lambda$ or ($\Delta = \Lambda$ and there is no Δ' with $\Delta \rightarrow^X \Delta'$). In the second case, due to the $\diamond d_c$ -rule, there is no formula $\diamond A$ in Δ and thus our claim is trivially true. \Box

Notice that each class of frames that can be characterised by our modal axioms can also be characterised by a 45-closed set of axioms. The restriction to 45-complete sets of rule names in the completeness theorem is thus irrelevant for the two following corollaries.

Corollary 2.24 (Finite Model Property) For all $X \subseteq \{d, t, b, 4, 5\}$ it holds that if a formula is not X-valid then there is a finite X-frame in which it is not valid.

Proof. Immediate from part (ii) of the completeness theorem. \Box

Corollary 2.25 (Decidability) For all $X\subseteq\{d,t,b,4,5\}$ it is decidable whether a formula is X-valid.

Proof. By the termination lemma and part (ii) of the completeness theorem. \Box

2.1.4 Syntactic Cut-Elimination

While cut admissibility is an easy corollary of the completeness theorem, it is still interesting to provide a nontrivial procedure which removes cuts from a proof. The existence of a step-by-step cut elimination procedure shows a certain symmetry, a certain good design of the inference rules. Also, it can serve as a starting point for a computational interpretation, maybe along the lines of [32].

We now see a cut-elimination procedure which follows the lines of the one for system G3 for first-order predicate logic, see for example [50]. The interesting twist is that the modalities require some form of multicut, similar to Gentzen's original procedure, even though contraction is admissible. We first need some standard definitions.

Definition 2.26 (depth of a formula) The *depth* of a formula A, denoted by depth(A), is defined as usual:

 $\begin{aligned} depth(p) &= depth(\bar{p}) = 0\\ depth(\Box A) &= depth(\Diamond A) = depth(A) + 1\\ depth(A \land B) &= depth(A \lor B) = max(depth(A), depth(B)) + 1 \quad . \end{aligned}$

Definition 2.27 (cut rank, cut-rank-preserving) Given an instance of the cut rule as shown in Figure 2.4, its *cut formula* is A and its *cut rank* is one plus the depth of its cut formula. For $r \ge 0$ we define the rule cut_r which is cut with at most rank r. The *cut rank* of a derivation is the supremum of the cut ranks of its cuts. A rule is *cut-rank (and depth-) preserving admissible* for a system S if for all $r \ge 0$ the rule is (depth-preserving) admissible for $S + \operatorname{cut}_r$. A rule is *cut-rank (and depth-) preserving invertible* for a system S if its inverse is *cut-rank (and depth-) preserving admissible* for S.

The problem with proving cut-elimination in the presence of the rules $\diamond 4_c$ and $\diamond 5_c$ is that these rules, seen upwards, do not decompose their main formula $\diamond A$. If that formula happens to be the cut formula, then we cannot form a new derivation by appealing to an induction hypothesis based on a lower rank. We thus generalise the cut-rule to incorporate instances of rules $\diamond 4_c$ and $\diamond 5_c$. This leads to the following definition.

Definition 2.28 (Y-cut) Let $\{\Delta\}^n$ denote $\underbrace{\{\Delta\} \dots \{\Delta\}}_{n-\text{times}}$. For $\mathsf{Y} \subseteq \{\mathsf{4},\mathsf{5}\}$ and $n \ge 0$ we define the rule

$$\operatorname{Y-cut} \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\diamond \bar{A}\}\{\diamond \bar{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

with the proviso that there is a derivation from $\Gamma\{\diamond\bar{A}\}\{\diamond\bar{A}\}^n$ to $\Gamma\{\diamond\bar{A}\}\{\emptyset\}^n$ in system Y.

Fact 2.29 (Properties of Y-cut) Consider an instance of Y-cut as above.

If $Y = \emptyset$ then it is an instance of cut, so n = 0.

If $\mathsf{Y} = \{\mathsf{4}\}$ then $\Gamma\{\}\{\}^n$ is of the form $\Gamma_1\{\{\},\Gamma_2\{\}^n\}$.

If $Y = \{5\}$ and n > 0 then the first hole is inside a box, so $depth(\Gamma\{ \}\{\emptyset\}^n) > 0$. (If $Y = \{4, 5\}$ then nothing can be said about the context since the proviso is trivially fulfilled.)

Structural modal rules. The rules which are shown in Figure 2.7 are called *structural modal rules*. They are structural in the sense of not affecting connectives of formulas. The modal rules $\diamond X_c$ are all \diamond -*rules*, in the sense that the active formula in the conclusion has \diamond as main connective. Given a set X of names of modal axioms, [X] is defined as $\{[\rho] \mid \rho \in X\}$. The structural modal rules have the obvious corresponding frame conditions.

We need the admissibility of these structural modal rules for our cut-elimination procedure. In some sense, they are the result of "reflecting" the corresponding diamond-rule at the cut. This comment will hopefully become more clear after the reduction lemma. The structural modal rules are cut-rank preserving admissible, as we will see.

The case of the seriality is a bit different from the other rules. The rule [d] is admissible, but we cannot show this in the presence of cut. Consider the problematic case where [d] cannot be pushed above cut:

$$\operatorname{cut} \frac{\Gamma\{[A]\} \quad \Gamma\{[\bar{A}]\}}{\left[\mathsf{d}\right] \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}}}$$

$$\begin{split} & [\mathsf{d}] \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \quad [\mathsf{t}] \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \quad [\mathsf{b}] \frac{\Gamma\{[\Delta, [\Sigma]]\}}{\Gamma\{[\Delta], \Sigma\}} \\ & [\mathsf{4}] \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}} \quad [\mathsf{5}] \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}} \quad depth(\Gamma\{\ \}\{\emptyset\}) > 0 \end{split}$$

Figure 2.7: Modal structural rules

So we cannot use [d]-admissibility in the cut-elimination proof. Our solution is to eliminate cut in the presence of [d] and only afterwards replace [d] by $\diamond d_c$. This means that in the following we always have to consider the possible presence of the [d]-rule.

Before we eliminate the cut we need to make sure that contraction and weakening can be eliminated without increasing the cut rank. We just strengthen Lemma 2.4 (Admissibility of structural rules and invertibility) accordingly to get the following lemma.

Lemma 2.30 (Cut-rank preserving admissibility of structural rules, invertibility) Let $X = \{d, t, b, 4, 5\}$. For each system K + Y with $Y \subseteq \Diamond X_c \cup \{[d]\}$ the following hold:

(i) The rules nec, wk and ctr are depth- and cut-rank preserving admissible.

(ii) All its rules are depth- and cut-rank preserving invertible.

Proof. The proof is just like the one for Lemma 2.4 (Admissibility of structural rules and invertibility) except that we also consider cut_r and [d]. In proving contraction admissibility there is one more case which is mildly interesting and which is handled as follows:

$$\begin{split} & \operatorname{cut}_{r} \frac{\Gamma\{\Delta\{A\}, \Delta\{\emptyset\}\} - \Gamma\{\Delta\{\bar{A}\}, \Delta\{\emptyset\}\}}{\operatorname{ctr} \frac{\Gamma\{\Delta\{\bar{A}\}, \Delta\{\emptyset\}\}}{\Gamma\{\Delta\{\emptyset\}\}}} & \sim \\ & \operatorname{wk} \frac{\Gamma\{\Delta\{A\}, \Delta\{\emptyset\}\}}{\Gamma\{\Delta\{A\}, \Delta\{A\}\}} - \operatorname{wk} \frac{\Gamma\{\Delta\{\bar{A}\}, \Delta\{\emptyset\}\}}{\Gamma\{\Delta\{\bar{A}\}, \Delta\{\bar{A}\}\}} & \underset{\operatorname{cut}_{r} \frac{\Gamma\{\Delta\{A\}\}}{\Gamma\{\Delta\{\bar{A}\}\}}}{\Gamma\{\Delta\{\bar{A}\}\}} & \cdot \\ \end{split}$$

Lemma 2.31 (Admissibility of the modal structural rules)

(i) Let X be a 45-closed subset of $\{t,b,4,5\}$ and let $\rho \in X$. Then the rule $[\rho]$ is cut-rank preserving admissible for system $K + \Diamond X_c$ and also for system $K + \Diamond X_c + [d]$.

(ii) Let X be a 45-closed subset of $\{d,t,b,4,5\}$ and let $d\in X.$ Then the rule [d] is admissible for system $K+\diamond X_c.$

Proof. For (i) the proof works by an outer induction on the number of instances of $[\rho]$ in a given proof, eliminating topmost instances first, and an inner induction on the depth of the proof above such a topmost instance. For each rule $[\rho]$ with $\rho \in X$ we make a case analysis on the rule σ above $[\rho]$. The induction base and the cases where σ is among the rules $\lor, \land, \Box, \operatorname{cut}_r, [d]$ and $\diamond t_c$ are trivial. We use cut-rank preserving admissibility of contraction and weakening provided by the previous lemma without explicitly mentioning it.

$$[\rho] = [t] :$$

$$\stackrel{\diamond k_{c}}{\underset{[t]}{\overset{\Gamma\{\diamond A, [A, \Delta]\}}{\Gamma\{\diamond A, \Delta\}}} \sim \stackrel{[t]}{\underset{\diamond t_{c}}{\overset{\Gamma\{\diamond A, [A, \Delta]\}}{\Gamma\{\diamond A, A, \Delta\}}} \sim \stackrel{[t]}{\underset{\diamond t_{c}}{\overset{\Gamma\{\diamond A, A, \Delta\}}{\Gamma\{\diamond A, A, \Delta\}}}$$

The case for $\sigma = \diamondsuit \mathsf{b}_{\mathsf{c}}$ is similar.

$$^{\diamond 4_{\mathsf{c}}} \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\underset{[\mathsf{t}]}{\Gamma\{\diamond A, [\Delta]\}}} \quad \sim \quad \overset{[\mathsf{t}]}{\underset{\mathsf{ctr}}{\Gamma\{\diamond A, [\diamond A, \Delta]\}}} \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\underset{\mathsf{ctr}}{\Gamma\{\diamond A, \diamond A, \Delta\}}}$$

For $\sigma = \diamond 5_c$ the case is trivial unless the diamond formula in its conclusion is at depth 1. Then there are two cases, either the $\diamond 5_c$ -rule moves the formula to somewhere outside the box that is removed by [t] or somewhere inside it. The second case is similar to the first, which is as follows, where ρ^* denotes several applications of ρ :

$$\overset{\diamond \mathbf{5_{c}}}{\underset{[\mathbf{t}]}{[\bigcirc A], \Delta, \Sigma\{\emptyset\}}{[\bigcirc A], \Delta, \Sigma\{\emptyset\}}} \sim \overset{[\mathbf{t}]}{\overset{[\bigcirc A], \Delta, \Sigma\{\Diamond A\}}{\diamond \mathbf{4_{c}}^{*}, \mathsf{wk}^{*}, \mathsf{ctr}^{*}}} \overset{[\Diamond A], \Delta, \Sigma\{\Diamond A\}}{\overset{\diamond A, \Delta, \Sigma\{\Diamond\}}{\diamond A, \Delta, \Sigma\{\emptyset\}}}$$

 $[\rho] = [\mathsf{b}]:$

For $\sigma = \diamond 5_c$ the case is trivial unless the diamond formula in its conclusion is at depth 2 and in the inner box in the premise of [b]. Then there are three similar

cases of which we just see the following one:

$$\overset{\diamond 5_{\mathsf{c}}}{\underset{[\mathsf{b}]}{\underbrace{[\Sigma, [\diamondsuit{A}, \Delta]], \Gamma\{\Diamond A\}}{\diamondsuit{A}, \Delta, [\Sigma], \Gamma\{\emptyset\}}} } \sim \overset{[\mathsf{b}]}{\overset{[\Sigma, [\diamondsuit{A}, \Delta]], \Gamma\{\Diamond A\}}{\diamondsuit{A}, \Delta, [\Sigma], \Gamma\{\Diamond\}} } \sim \overset{[\mathsf{b}]}{\overset{\diamond 4_{\mathsf{c}}^*, \mathsf{wk}^*, \mathsf{ctr}^*}{\underbrace{\diamondsuit{A}, \Delta, [\Sigma], \Gamma\{\Diamond A\}}{\diamondsuit{A}, \Delta, [\Sigma], \Gamma\{\emptyset\}}}$$

 $[\rho] = [4]:$

$$\overset{\wedge \mathsf{k}_{\mathsf{c}}}{\overset{\Gamma\{\diamond A, [A, \Delta], [\Sigma]\}}{[4]} \xrightarrow{\Gamma\{\diamond A, [\Delta], [\Sigma]\}}} \sim \overset{[4]}{\overset{\Gamma\{\diamond A, [A, \Delta], [\Sigma]\}}{[4]} \xrightarrow{\Gamma\{\diamond A, [[A, \Delta], \Sigma]\}}} \overset{(\mathsf{k}_{\mathsf{c}})}{\overset{\vee \mathsf{k}_{\mathsf{c}}}{[4]} \xrightarrow{\Gamma\{\diamond A, [[A, \Delta], \Sigma]\}}} \xrightarrow{\Gamma\{\diamond A, [[\Delta], \Sigma]\}}$$

The case for $\sigma = \diamondsuit 4_c$ is similar and the case for $\sigma = \diamondsuit 5_c$ is trivial.

$$\begin{split} & \stackrel{\diamond b_{c}}{\underset{[4]}{}} \frac{\Gamma\{A, [\diamond A, \Delta], [\Sigma]\}}{\Gamma\{[\diamond A, \Delta], [\Sigma]\}}}{\Gamma\{[[\diamond A, \Delta], \Sigma]\}} & \sim & \stackrel{[4]}{\underset{[4]}{}} \frac{\Gamma\{A, [\diamond A, \Delta], [\Sigma]\}}{\Gamma\{A, [[\diamond A, \Delta], \Sigma]\}}}{\Gamma\{A, [[\diamond A, \Delta], \Sigma]\}} \\ & \stackrel{[6]}{\underset{[5]}{}} \frac{\Gamma\{[\diamond A, [A, \Delta]\}\{\emptyset\}}{\Gamma\{\{\diamond A, [A, \Delta]\}\{\emptyset\}}} & \sim & \stackrel{[5]}{\underset{[5]}{}} \frac{\Gamma\{\diamond A, [A, \Delta]\}\{\emptyset\}}{\Gamma\{\diamond A\}\{[A, \Delta]\}}}{\Gamma\{\{\diamond A\}\{[\Delta]\}\}} \end{split}$$

The case for $\sigma = \Diamond 4_c$ is similar and the case for $\sigma = \Diamond 5_c$ is trivial. For $\sigma = \Diamond b_c$ we have:

$$\overset{\diamond_{\mathsf{b}_{\mathsf{c}}}}{\stackrel{[5]}{\xrightarrow{}} \frac{\Gamma\{[A, [\diamond A, \Delta], \Sigma]\}\{\emptyset\}}{\Gamma\{[\Sigma]\}\{[\diamond A, \Delta]\}}}{[\varsigma] \frac{\Gamma\{[A, [\diamond A, \Delta], \Sigma]\}\{\emptyset\}}{\Gamma\{[A, \Sigma]\}\{[\diamond A, \Delta]\}}} \sim \qquad \overset{[5]}{\stackrel{}{\xrightarrow{}} \frac{\Gamma\{[A, [\diamond A, \Delta], \Sigma]\}\{\emptyset\}}{\Gamma\{[A, \Sigma]\}\{[\diamond A, \Delta]\}}}{\frac{\Gamma\{[A, \Sigma]\}\{[\diamond A, \Delta]\}}{[\varsigma A, \Delta]\}}}{[\Gamma\{[\Sigma]\}\{[\diamond A, \Delta]\}}}$$

The proof for (ii) is similar to the one for (i), except that we exclude $\sigma = \operatorname{cut}_r$. The case $\sigma = \diamond b_c$ is trivial. $[\rho] = [\mathsf{d}]$:

$$\begin{array}{c} \diamond_{\mathsf{k}_{\mathsf{c}}} \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A, [\emptyset]\}} & \sim & \diamond_{\mathsf{d}_{\mathsf{c}}} \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}} \\ \\ \diamond_{\mathsf{d}_{\mathsf{c}}} \frac{\Gamma\{\diamond A, [\emptyset]\}}{\Gamma\{\diamond A\}} & \sim & \diamond_{\mathsf{d}_{\mathsf{c}}} \frac{\Gamma\{\diamond A, [\diamond A]\}}{\Gamma\{\diamond A\}} \\ \\ \overset{(\mathsf{d})}{\overset{}{=} \frac{\Gamma\{\diamond A, [\emptyset]\}}{\Gamma\{\diamond A\}}} & \sim & \diamond_{\mathsf{d}_{\mathsf{c}}} \frac{\Gamma\{\diamond A, [\diamond A]\}}{\overset{}{=} \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A, [A]\}}} \end{array}$$

$$\overset{\diamond 5_{c}}{[d]} \frac{\Gamma\{\Diamond A\}\{[\Diamond A]\}}{\Gamma\{\Diamond A\}\{[\emptyset]\}} \sim \overset{\mathsf{wk}^{2}}{\underset{\diamond 5_{c}}{\frac{\Gamma\{\Diamond A\}\{[\Diamond A]\}}{\Gamma\{\Diamond A\}\{\Diamond A, [\Diamond A, A]\}}}}{\overset{\diamond 6_{c}}{\frac{\Gamma\{\Diamond A\}\{\Diamond A, [\Diamond A, A]\}}{\left(\sum_{\diamond 5_{c}}{\frac{\Gamma\{\Diamond A\}\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}\{\Diamond A\}}}}}$$

To keep the cut-elimination procedure short and uniform, we define a structural rule which moves a box inside a sequent from one place to another. Notice that the conditions on the context in the proviso exactly match the conditions in the **Y-cut-**rule:

Definition 2.32 (Y-str-rule) For $Y \subseteq \{4, 5\}$ we define a rule

$$\mathsf{Y}\text{-}\mathsf{str}\,\frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}$$

with the proviso that:

24

if $Y = \emptyset$ then $\Gamma\{ \}\{ \}$ is of the form $\Gamma'\{\{ \}, \{ \}\},\$ if $Y = \{4\}$ then $\Gamma\{ \}\{ \}$ is of the form $\Gamma_1\{\{ \}, \Gamma_2\{ \}\},\$ and if $Y = \{5\}$ then $depth(\Gamma\{ \}\{\emptyset\}) > 0.$ (This means there is no proviso for the case $Y = \{4, 5\}.$)

Lemma 2.33 (Admissibility of Y-str) For 45-closed $X \subseteq \{[d], t, b, 4, 5\}$ and for $Y \subseteq \{4, 5\}$ the rule Y-str is cut-rank preserving admissible for system K + X if $Y \subseteq X$.

Proof. For $Y = \emptyset$ that is trivial. For $Y = \{4\}$ the rule is derivable as follows:

$$[4]^{*} \frac{\Gamma\{[\Delta], \Sigma\{\emptyset\}\}}{\Gamma\{[\ldots[\Delta], \ldots], \Sigma\{\emptyset\}\}} \frac{\Gamma\{\Sigma\{[\Delta]\}, \Sigma\{\emptyset\}\}}{\Gamma\{\Sigma\{[\Delta]\}, \Sigma\{[\Delta]\}\}} \frac{\Gamma\{\Sigma\{[\Delta]\}, \Sigma\{[\Delta]\}\}}{\Gamma\{\Sigma\{[\Delta]\}\}}$$

,

and thus admissible by Lemma 2.30 (Cut-rank preserving admissibility of structural rules) and Lemma 2.31 (Admissibility of the modal structural rules). For $Y = \{5\}$ the rule coincides with [5] and is thus admissible by Lemma 2.31. For $Y = \{4, 5\}$ an instance of the rule is either an instance of the Y-str-rule for $Y = \{4\}$ or $Y = \{5\}$ and thus admissible as in the previous two cases.

Lemma 2.34 (Reduction Lemma) Let X be a 45-closed subset of $\{t, b, 4, 5\}$, let Y be a subset of $\{4, 5\} \cap X$ and let either $Z = \Diamond X_c$ or $Z = \Diamond X_c + [d]$. Further, let r > 0 and $n \ge 0$.

(i) If there is a proof



with \mathcal{P}_1 and \mathcal{P}_2 in $\mathsf{K} + \mathsf{Z} + \mathsf{cut}_r$ then $\mathsf{K} + \mathsf{Z} + \mathsf{cut}_r \vdash \Gamma\{\emptyset\}$. (ii) If there is a proof



with \mathcal{P}_1 and \mathcal{P}_2 in $\mathsf{K} + \mathsf{Z} + \mathsf{cut}_r$ then $\mathsf{K} + \mathsf{Z} + \mathsf{cut}_r \vdash \Gamma\{\emptyset\}\{\emptyset\}^n$.

Proof. We prove (i) and (ii) simultaneously by induction on $|\mathcal{P}_1| + |\mathcal{P}_2|$. We perform a case analysis on the two lowermost rules in \mathcal{P}_1 and \mathcal{P}_2 . If one of the two rules is passive and an axiom then $\Gamma\{\emptyset\}$ is axiomatic as well. If one is active and an axiom then we have



If one rule is passive then we have



for case (i) and similarly for (ii). This leaves the case that both rules are active and not axioms. For (i) we have:



Notice that (i) is a special case of (ii) if A has a modality as its main connective. The remaining case is thus (ii) with both rules active and not axioms, and thus on one side the \Box -rule and on the other side either $\Diamond k_c, \Diamond t_c$ or $\Diamond b_c$ (the cases for $\Diamond 4_c$ and $\Diamond 5_c$ are trivial). The case for the $\Diamond k_c$ -rule is as follows:



where the Y-str-rule is applicable since its condition on the context matches the condition in the Y-cut-rule. The Y-str-rule can be removed by Lemma 2.33 (Admissibility of Y-str), weakening and contraction can be removed by Lemma 2.30 (Cut-rank preserving admissibility of structural rules) and the instance of Y-cut can be removed by induction hypothesis. The cases for $\diamond t_c$ and $\diamond b_c$ are as follows:



and

26



In general the Y-cut, seen upwards, introduces several diamond formulas. One of them is special in being in the same position as its dual cut formula in the other premise. In the transformations given above, the active formula of the diamond-rule above the cut is different from that special formula. That is not always the case, of course, but if the two coincide, then the transformations are simpler.

 \Box

Theorem 2.35 (Cut-Elimination) Let X be a 45-closed subset of $\{d, t, b, 4, 5\}$. Then we have:

If
$$\mathsf{K} + \Diamond \mathsf{X}_{\mathsf{c}} + \mathsf{cut} \vdash \Gamma$$
 then $\mathsf{K} + \Diamond \mathsf{X}_{\mathsf{c}} \vdash \Gamma$.

Proof. We first prove the theorem in case that $d \notin X$. Then it follows from a routine induction on the cut-rank of the given proof. The induction step follows by another induction, on the depth of the proof. It uses the reduction lemma in the case of a maximal-rank cut. In case $d \in X$ we first replace instances of the rule $\diamond d_c$ by instances of the rules $\diamond k_c$ and [d], then proceed as before, and finally apply Lemma 2.31 (Admissibility of modal structural rules) to replace [d] by $\diamond d_c$. □

This finishes the section of sequent systems where modal axioms are represented as logical rules. The systems cover the entire modal cube and are systematic in the sense that there is a one-to-one correspondence between the modal rules and the frame conditions. However, unlike Hilbert systems and labelled sequent systems, they are not modular in the sense that each combination of modal rules is complete for the corresponding class of frames. This forced us to resort to formulating the condition of 45-closed systems and proving completeness only for those. It is hard to see how to achieving modularity using these systems.

However, during the cut-elimination procedure we discovered the possibility of forming proof systems not using \diamond -rules but using the structural rules shown in Figure 2.7 on page 21.

In particular, the examples from Fact 2.10 (Incompleteness) which showed that systems $K + \{\diamond t_c, \diamond 5_c\}$ and $K + \{\diamond b_c, \diamond 4_c\}$ are incomplete are provable in systems $K + \{[t], [5]\}$ and $K + \{[b], [4]\}$, respectively:

$$\begin{array}{c} \mathbf{k} \underbrace{[\Diamond \bar{p}, [p, \bar{p}], [\emptyset]]}_{[5]} \\ \underbrace{[\Diamond \bar{p}, [p], [\emptyset]]}_{[t]} \\ \begin{bmatrix} [\phi \bar{p}, [p], [\emptyset]] \\ \vdots \\ \neg \bar{p}, [[p]] \\ \neg^{2} \underbrace{\langle \bar{p}, [[p]] \\ \Diamond \bar{p}, \Box \Box p \end{array} \end{array}$$
 and
$$\begin{array}{c} \mathbf{k} \underbrace{[[\bar{p}, p], \Diamond p]}_{[[\bar{p}], \Diamond p]} \\ \begin{bmatrix} [\mu], [\phi p] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \neg \bar{p}, \Box \Diamond p \end{array} \right)$$

$\Gamma\{p,\bar{p}\} \wedge \frac{\Gamma\{A\}}{\Gamma\{A \wedge B\}} \frac{\Gamma\{B\}}{\Gamma\{A \vee B\}} \vee \frac{\Gamma\{A,B\}}{\Gamma\{A \vee B\}}$
$\Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad k \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\diamondsuit A, [\Delta]\}} \qquad ctr \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}$
$[d] \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad [t] \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad [b] \frac{\Gamma\{[\Delta, [\Sigma]]\}}{\Gamma\{[\Delta], \Sigma\}}$
$[4] \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}} \qquad [5] \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}} depth(\Gamma\{\ \}\{\emptyset\}) > 0$

Figure 2.8: System $K_c + \{[d], [t], [b], [4], [5]\}$

We consider such proof systems in the next section.

2.2 Modal Axioms as Structural Rules

The plan of this section is as follows: we first introduce the sequent systems and state soundness, cut-elimination and completeness, which we prove by embedding a Hilbert system and using cut-elimination. The remainder of the section is devoted to proving cut-elimination. The cut-elimination proof is interesting: it relies on a decomposition of the contraction rule, similar to what has been observed in deep inference systems for propositional logic, where contraction is decomposed into an atomic version and a local *medial* rule [14].

2.2.1 The Sequent Systems

System $K_c + [X]$. Figure 2.8 shows the set of rules from which we form our deductive systems. *System* K_c is the set of rules $\{\land, \lor, \Box, k, ctr\}$. We will look at extensions of System K_c with the structural modal rules $[X] \subseteq \{[d], [t], [b], [4], [5]\}$ that we have encountered previously and that are shown in Figure 2.8 for convenience. Contrary to systems considered in the last section, the systems we consider now are not fully invertible and contraction is not admissible for the contraction-free systems. Of course it is easy to obtain equivalent systems which are fully invertible and for which contraction is admissible by using system K instead of K_c and by absorbing contraction into the modal structural rules. However, we choose not to do this because our cut-elimination technique, which relies on decomposing contraction, is more natural in a system with an explicit contraction rule.

Soundness of our systems is easily established similarly to soundness of the systems in the previous section.

Theorem 2.36 (Soundness) Let $X \subseteq \{d, t, b, 4, 5\}$. If a sequent is provable in

 $K_c + [X]$ then its corresponding formula is provable in a Hilbert system for the modal logic K extended by the axioms in X.

Our main result is cut-elimination, which we prove in the next subsection.

Theorem 2.37 (Cut-Elimination) Let $X \subseteq \{d, t, b, 4, 5\}$. If $K_c + [X] + cut \vdash \Gamma$ then $K_c + [X] \vdash \Gamma$.

By using cut-elimination we obtain the completeness theorem:

Theorem 2.38 (Completeness) Let $X \subseteq \{d,t,b,4,5\}$. If a formula is provable in a Hilbert system for the modal logic K extended by the modal axioms in X then it is provable in system $K_c + [X]$.

Proof. Given a proof in the Hilbert system we construct a proof in $K_c + [X] + cut$ as usual, and then apply Theorem 2.37 (Cut-elimination). We show proofs for the modal axioms:

$${}^{k^{2}} \frac{[\bar{A}, A]}{[d]} \xrightarrow{\langle \bar{A}, \Diamond A, [\emptyset]}{\langle \bar{A}, \Diamond A \rangle} = {}^{k} \frac{[A, \bar{A}]}{[\bar{A}], \Diamond A} \xrightarrow{k} \frac{[[A, \bar{A}]]}{[\bar{A}], \Diamond A]} \xrightarrow{k} \frac{[[A, \bar{A}]]}{\bar{A}, [\Box A]} \xrightarrow{k} \frac{[\bar{A}, A], [\emptyset]}{\bar{A}, [A], [\emptyset]} \xrightarrow{k} \frac{[\bar{A}, A]}{[\bar{A}], [A]} \xrightarrow{k} \frac{[\bar{A}, A]}{\bar{A}, \Box \land A} \xrightarrow{(A, \bar{A})} \xrightarrow{(A, \bar{$$

2.2.2 Syntactic Cut-Elimination

We first show that weakening and necessitation are admissible.

Lemma 2.39 (Weakening and necessitation admissibility) Let $X \subseteq \{d, t, b, 4, 5\}$. The wk-rule and the nec-rule are depth- and cut-rank-preserving admissible for $K_c + [X]$.

Proof. A routine induction shows that a single nec or wk-rule can be eliminated from a given proof, a second induction on the number of nec or wk-rules yields our lemma. \Box

Similarly to the $\diamond d_c$ -rule in the previous section, the [d]-rule is different from the other rules: it trivially permutes below the cut. So we can get it out of the way and then we need to prove cut-elimination only for the systems without it.

Lemma 2.40 (Push down seriality) Let $X \subseteq \{d, t, b, 4, 5\}$ and $d \in X$. For each proof as shown on the left there is a proof as shown on the right:



$m\square \frac{\Gamma\{[A,\ldots,A]\}}{\Gamma\{\square A\}} \qquad m^{\wedge} \frac{\Gamma\{A,\ldots,A\}}{\Gamma\{\square A\}}$	$ \begin{array}{c} [A, \dots, A] \Gamma\{B, \dots \\ \\ \Gamma\{A \land B\} \end{array} $	<i>,B</i>]}
$mcut \frac{\Gamma\{A, \dots, A\} \Gamma\{\bar{A}, \dots, \bar{A}\}}{\Gamma\{\emptyset\}}$	$\mathrm{med}\frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,\Sigma]\}}$	$\operatorname{fctr} \frac{\Gamma\{A,A\}}{\Gamma\{A\}}$

Figure 2.9: Multi-rules, medial, and formula contraction

Proof. By an easy permutation argument, making use of weakening admissibility. $\hfill \Box$

We also get contraction out of the way in order to eliminate the cut. First, we decompose contraction into the fctr-rule, which is contraction on formulas, and the med-rule, shown in Figure 2.9. We permute down the fctr-rule. It does not permute down below the rules cut, \Box and \land , so we generalise these rules as in Figure 2.9. We define a contraction-free system K_m as $K_m = K_c - ctr + \{med, m\Box, m\land\}$ and will show cut-elimination for that system. But first we develop the machinery to show that cut-elimination for K_m leads to cut-elimination for K_c (with any [X]).

Lemma 2.41 (Decompose contraction) The ctr-rule is derivable for {fctr, med}.

Proof. By induction the depth of a sequent which is contracted, we show the inductive step:

$$\operatorname{ctr} \frac{\Gamma\{A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n], A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n]\}}{\Gamma\{A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n]\}}$$

$$\sim \qquad \operatorname{med}^n \frac{\Gamma\{A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n], A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n]\}}{\operatorname{ctr}^n \frac{\Gamma\{A_1, \dots, A_m, A_1, \dots, A_m, [\Delta_1, \Delta_1], \dots, [\Delta_n, \Delta_n]\}}{\Gamma\{A_1, \dots, A_m, A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n]\}}$$

Lemma 2.42 (Weakening and necessitation admissibility for K_m) Let $X \subseteq \{d, t, b, 4, 5\}$. The wk-rule and the nec-rule are depth- and cut-rank-preserving admissible for $K_m + [X]$.

Lemma 2.43 (From mcut to cut) The rule $mcut_r$ is derivable for $\{cut_r, wk\}$.

Proof. We define the rule $mcut_r^{m,n}$ with m, n > 0 as

$$\frac{\Gamma\{\overbrace{A,\ldots,A}^{m-\text{times}}\}}{\Gamma\{\emptyset\}}$$
and show that rule derivable for $\{\mathsf{cut}_r, \mathsf{wk}\}$ by induction on m+n. The case for m = n = 1 is trivial, for m > 1 and n = 1 we replace

$$\operatorname{mcut}_{r}^{m,1} \frac{\Gamma\{A,\ldots,A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

by

$$\operatorname{mcut}_{r}^{m-1,1} \frac{\Gamma\{A,\ldots,A\}}{\operatorname{cut}_{r} \frac{\Gamma\{A\}}{\Gamma\{\bar{A},A\}}} \frac{\Gamma\{\bar{A}\}}{\Gamma\{\bar{A}\}}$$

and apply the induction hypothesis, and for m, n > 1 we replace

$$\mathrm{mcut}_r^{m,n} \frac{\Gamma\{A,\ldots,A\} \quad \Gamma\{\bar{A},\ldots,\bar{A}\}}{\Gamma\{\emptyset\}}$$

by

$$\max_{r} \frac{\Gamma\{A,\ldots,A\}}{\operatorname{cut}_{r}^{m-1,n}} \frac{\Gamma\{A,\ldots,A\}}{\Gamma\{\bar{A},\ldots,\bar{A},A\}} \xrightarrow[\operatorname{mcut}_{r}^{m,n-1}]{\operatorname{r}} \frac{\Gamma\{A,\ldots,A\}}{\Gamma\{\bar{A},\ldots,\bar{A},\bar{A}\}} \frac{\Gamma\{\bar{A},\ldots,\bar{A}\}}{\Gamma\{\bar{A},\ldots,\bar{A}\}}$$

and apply the induction hypothesis twice.

Lemma 2.44 (Push down contraction) Let $X \subseteq \{t, b, 4, 5\}$. Given a proof as shown on the left, with ρ a single-premise-rule from $K_m + [X] + wk$, there is a proof as shown on the right, with $|\mathcal{D}'| \leq |\mathcal{D}|$:



Proof. By induction on the length of \mathcal{D} and a case analysis on ρ . Most cases are trivial. We show the two interesting ones. For $\rho = \vee$ and $\rho = \mathsf{k}$ we apply the following transformations:

$$\operatorname{fctr} \frac{\Gamma\{A, A, B\}}{\overset{\vee}{\Gamma\{A, B\}}} \longrightarrow \operatorname{ctr} \frac{\Gamma\{A, A, B\}}{\Gamma\{A, B, A, B\}} \xrightarrow{\operatorname{wk}} \frac{\Gamma\{A, A, B\}}{\Gamma\{A, B, A, B\}} \operatorname{fctr} \frac{\Gamma\{A \lor B, A \lor B\}}{\Gamma\{A \lor B\}}$$

$$\int_{\mathsf{fctr}} \frac{\Gamma\{[A, A, \Delta]\}}{\Gamma\{[A, \Delta]\}} \sim \qquad \underset{\mathsf{fctr}}{\overset{\mathsf{k}^2}{\Gamma\{\Diamond A, [\Delta]\}}} \frac{\Gamma\{[A, A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

and in each case we apply the induction hypothesis twice.

32

Proposition 2.45 (Push down contraction) Given a proof as shown on the left, there is a proof as shown on the right:



Proof. We first prove the claim that for each proof as shown on the left there is a proof as shown on the right:



The proof of the claim is by induction on the depth of \mathcal{P}_1 , using Lemma 2.44 (Push down contraction). The proof of our proposition is as follows: by using Lemma 2.41 (Decompose contraction) we obtain a proof in $K_m + [X] + cut + fctr$, we apply our claim, then we use Lemma 2.43 (From mcut to cut), to replace mcut, starting with the top-most instances. Finally we remove weakening using weakening admissibility.

It turns out that during the proof of cut-elimination for some system $K_c + [X]$ some rules may be introduced that are not in [X] but that logically follow from X. These additional rule instances will then be removed from the proof after cut-elimination.

Definition 2.46 (X⁺) Given some $X \subseteq \{d, t, b, 4, 5\}$ we define

$$X^+ = \begin{cases} X \cup \{4\} & \mathrm{if} \ \{t, 5\} \subseteq X \ \mathrm{or} \ \{b, 5\} \subseteq X \\ X \cup \{5\} & \mathrm{if} \ \{b, 4\} \subseteq X \\ X & \mathrm{otherwise} \quad , \end{cases}$$

and likewise for $\diamond X$ and [X].

This definition matches the semantical notion of 45-closed that we defined earlier:

Fact 2.47 (X⁺ is 45-closure of X) If $X \subseteq \{d, t, b, 4, 5\}$ then X⁺ is the least set which contains X and is 45-closed.

The following lemma ensures that, after we have eliminated cut, we can indeed remove the additional rules in $X^+ - X$.

Lemma 2.48 (From X^+ to X)

(i) The [4]-rule is derivable for $\{[t], [5], nec\}$.

(ii) The [4]-rule is derivable for {[b], [5], nec}.

(iii) The [5]-rule is derivable for $\{[b], [4], wk\}$.

Proof. For (i) notice that the [4]-rule is a special case of the [5]-rule unless Γ { } has depth zero, and thus Γ { } Λ { }. In that case we have:

$$[4] \frac{\Lambda, [\Delta], [\Sigma]}{\Lambda, [[\Delta], \Sigma]} \quad \rightsquigarrow \quad \begin{bmatrix} nec \\ \overline{\Lambda, [\Delta], [\Sigma]} \\ [5] \\ \hline [\Lambda, [[\Delta], \Sigma] \\ [t] \\ \hline \overline{\Lambda, [[\Delta], \Sigma]} \\ \overline{\Lambda, [[\Delta], \Sigma]} \end{bmatrix}$$

.

For (ii) we again have to consider only the case where $\Gamma\{ \} = \Lambda, \{ \}$:

$$[4] \frac{\Lambda, [\Delta], [\Sigma]}{\Lambda, [[\Delta], \Sigma]} \quad \rightsquigarrow \quad \begin{bmatrix} nec^2 \\ \underline{\Lambda}, [\Delta], [\Sigma] \\ [5] \\ \underline{[[\Lambda, [\Delta], \Sigma]]} \\ [b] \\ \underline{[[\Lambda, [\Delta], \Sigma]]} \\ [b] \\ \underline{[[\Lambda], [\Delta], \Sigma]} \\ \Lambda, [[\Delta], \Sigma] \end{bmatrix}$$

For (iii) notice that a sequent has a tree structure and that, seen upwards, the [5]-rule allows to move a boxed sequent $[\Delta]$ to any position in that tree, but not to the root. To move a boxed sequent to any position in the tree it is enough if we are both able to move it a) from a given node the parent of this node and b) to move it from a given node to any child of that node. Point a) is just the [4]-rule and point b) is as follows:

$$\label{eq:wk} \begin{split} & \frac{\Gamma\{[\Lambda, [\Delta]]\}}{\Gamma\{[\Lambda, [\emptyset], [\Delta]]\}} \\ & \overset{[4]}{\overset{[4]}{\overset{}{\frac{\Gamma\{[\Lambda, [[\Delta]]]\}}{\Gamma\{[\Lambda], [\Delta]\}}}}} \\ & \overset{[b]}{\overset{}{\frac{\Gamma\{[\Lambda, [\Delta]]\}}{\Gamma\{[\Lambda], [\Delta]\}}}} \end{split} .$$

We are now preparing for the reduction lemma, which we prove as usual by pushing the cut rule upwards. In general we cannot push the cut above a modal structural rule, so we push it upwards together with the cut. The interesting case occurs once this conglomerate of cut and modal structural rules needs to be pushed above the \diamond k-rule. Then we have to permute the \diamond k-rule down through the modal structural rules to meet the cut. In the course of this permutation, the \diamond k-rule might turn into another \diamond -rule. The following two lemmas take care of these permutations.

Lemma 2.49 (Push down $\diamond 4$, $\diamond 5$) Let $X \subseteq \{t, b, 4, 5\}$ and $\rho \in (\diamond X \cap \{\diamond 4, \diamond 5\})$. Given a derivation as shown on the left, where ρ applies to $\diamond A$, there is a derivation as shown on the right, where all rules in \mathcal{D}_3 apply to the instance of $\diamond A$ shown, and where $|\mathcal{D}_2| \leq |\mathcal{D}_1|$:

$\Gamma(\triangle A)$		$\Gamma\{\diamondsuit A\}$
$\rho \frac{\Gamma(\langle A \rangle)}{\Gamma(\langle A \rangle)}$		$\mathcal{D}_2 \parallel [X] + med$
$1 \leq \langle X \rangle$	\sim	$\Gamma_2\{\diamondsuit A\}$
$\mathcal{D}_1 \parallel [X] + med$ $\Lambda \{ \diamondsuit A \}$		$\mathcal{D}_3 \parallel (\diamond X^+ \cap \{\diamond 4, \diamond 5\})$
<u>→</u> [> 21∫		$\Delta \{ \diamondsuit A \}$

.

Proof. The proof is by induction on the length of \mathcal{D}_1 . We permute the instance of ρ down and apply the induction hypothesis, possibly several times. We only show the non-trivial permutations.

$ \overset{\diamond 4}{\underset{med}{\overset{\Gamma\{[\diamondsuit{A},\Delta],[\Sigma]\}}{\frac{\Gamma\{\diamondsuit{A},[\Delta],[\Sigma]\}}{\Gamma\{\diamondsuit{A},[\Delta,\Sigma]\}}} } } $	\sim	$\underset{\diamond 4}{\operatorname{med}} \frac{\Gamma\{[\diamond A, \Delta], [\Sigma]\}}{\Gamma\{[\diamond A, \Delta, \Sigma]\}}$
$ \overset{\diamond 4}{\overset{\Gamma\{[\diamondsuit A,\Delta]\}}{\overset{[t]}{\frac{\Gamma\{\diamondsuit A,[\Delta]\}}{\Gamma\{\diamondsuit A,\Delta\}}}} } $	\sim	$[t] \frac{\Gamma\{[\diamondsuit A, \Delta]\}}{\Gamma\{\diamondsuit A, \Delta\}}$
$ \overset{\diamond 4}{\overset{\Gamma\{[\Delta, [\Diamond A, \Sigma]]\}}{\frac{\Gamma\{[\Diamond A, \Delta, [\Sigma]]\}}{\Gamma\{[\Diamond A, \Delta], \Sigma\}}} } $	\sim	$\overset{[\mathbf{b}]}{\underset{\diamond 5}{\overset{\Gamma\{[\Delta, [\Diamond A, \Sigma]]\}}{\frac{\Gamma\{\Diamond A, [\Delta], \Sigma\}}{\Gamma\{[\Diamond A, \Delta], \Sigma\}}}}$
$ \overset{\diamond 4}{\overset{\Gamma\{[\diamondsuit A, \Delta], [\Sigma]\}}{\frac{\Gamma\{\diamondsuit A, [\Delta], [\Sigma]\}}{\Gamma\{\diamondsuit A, [\Delta], [\Sigma]\}}} } $	\sim	$ \begin{array}{l} [4] \\ [4] \\ \uparrow \{ [\diamond A, \Delta], [\Sigma] \} \\ \diamond 4 \\ \uparrow \{ [[\diamond A, \Delta], \Sigma] \} \\ \diamond 4 \\ \hline \Gamma \{ [\diamond A, [\Delta], \Sigma] \} \\ \Gamma \{ \diamond A, [[\Delta], \Sigma] \} \end{array} $
$^{\diamond 4} \frac{\Gamma\{[\diamond A, \Delta]\}\{\emptyset\}}{\Gamma\{\diamond A, [\Delta]\}\{\emptyset\}} \frac{\Gamma\{\diamond A, [\Delta]\}\{\emptyset\}}{\Gamma\{\diamond A\}\{[\Delta]\}}$	\sim	$\overset{[5]}{\diamond 5} \frac{\Gamma\{[\diamond A, \Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\diamond A, \Delta]\}} \frac{\Gamma\{\emptyset\}\{[\diamond A, \Delta]\}}{\Gamma\{\diamond A\}\{[\Delta]\}}$

Permuting down the \diamond 5-rule is trivial except over the [t]-rule and the [b]-rule, and this is also trivial unless the restriction on the depth of the context in the \diamond 5-rule becomes relevant:

$$\overset{\diamond 5}{\underset{[t]}{\overset{[t]}{\xrightarrow{\Gamma_1, [\Delta], \Gamma_2\{\Diamond A\}}}}{\Gamma_1, [\Diamond A, \Delta], \Gamma_2\{\emptyset\}} } \qquad \sim \qquad \overset{[t]}{\overset{[t]}{\xrightarrow{\Gamma_1, [\Delta], \Gamma_2\{\Diamond A\}}}{\Gamma_1, \Delta, \Gamma_2\{\Diamond A\}}}{}_{\diamond 4^*} \overset{[t]}{\xrightarrow{\Gamma_1, \Delta, \Gamma_2\{\diamond A\}}}{}_{\Gamma_1, \diamond A, \Delta, \Gamma_2\{\emptyset\}}$$

$$\overset{\diamond 5}{\overset{[\Delta, [\Sigma]], \Gamma\{\diamond A\}}{[\Delta, [\Sigma, \diamond A]], \Gamma\{\emptyset\}}} \qquad \sim \qquad \overset{[b]}{\overset{[\Delta, [\Sigma]], \Gamma\{\diamond A\}}{[\Delta], \Sigma, \Diamond A, \Gamma\{\emptyset\}}} \qquad \qquad \qquad \qquad \overset{[b]}{\overset{[\Delta], \Sigma, \Gamma\{\diamond A\}}{[\Delta], \Sigma, \Diamond A, \Gamma\{\emptyset\}}}$$

Lemma 2.50 (Push down $\diamond k$, $\diamond t$, $\diamond b$) Let $X \subseteq \{t, b, 4, 5\}$ and let $\rho = \diamond k$ or $\rho \in (\diamond X \cap \{\diamond t, \diamond b\})$. Given a derivation as shown on the left, where ρ applies to $\diamond A$, there is a derivation as shown on the right, with $\sigma = \diamond k$ or $\sigma \in (\diamond X \cap \{\diamond t, \diamond b\})$, where all rules in \mathcal{D}_3 apply to the instance of $\diamond A$ shown, and where $|\mathcal{D}_2| \leq |\mathcal{D}_1|$:

Proof. The proof is by induction on the length of \mathcal{D}_1 . We permute the instance of ρ down and apply Lemma 2.49 (Push down $\diamond 4$, $\diamond 5$) and/or the induction hypothesis. We only show the non-trivial permutations.

$$\begin{array}{ccc} & & & & & & & & & & & & & \\ \uparrow \{[A, \Delta]\} \\ & & & & \uparrow \{[A, \Delta]\} \\ & & & & \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & \\ \uparrow \{[A, \Delta], [\Sigma]\} \\ & & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & & \\ \uparrow \{[A, \Delta]\} \{\emptyset\} \\ & & \\ \uparrow \{[A, A]\} \{\emptyset\} \\ & & \\ \downarrow \{[A, A]\}$$

The cases for $\rho = \diamondsuit t$ are trivial.

.

36

$ \overset{\diamond \mathbf{b}}{\overset{[\mathbf{t}]}{\overset{[\mathbf{t}]}{\frac{\Gamma\{[\Delta],A\}}{\Gamma\{[\Delta,\diamond A]\}}}} } $	\sim	$\stackrel{[t]}{\overset{[t]}{\to}} \frac{\Gamma\{[\Delta, A]\}}{\Gamma\{\Delta, A\}} \\ \stackrel{\diamond t}{\to} \frac{\Gamma\{\Delta, \diamond A\}}{\Gamma\{\Delta, \diamond A\}}$
$ \overset{\diamond \mathbf{b}}{\overset{[\mathbf{b}]}{\overset{[\mathbf{b}]}{\frac{\Gamma\{[\Sigma, [\Delta], A]\}}{\Gamma\{[\Sigma, [\Delta, \diamond A]]\}}}} } $	\sim	$\overset{[\mathbf{b}]}{\underset{\mathbf{k}}{\overset{[\mathbf{b}]}{\leftarrow}}} \frac{\Gamma\{[\Sigma, [\Delta], A]\}}{\Gamma\{[\Sigma, A], \Delta\}} \\ \overset{(\mathbf{b})}{\leftarrow} \frac{\Gamma\{[\Sigma, A], \Delta\}}{\Gamma\{[\Sigma], \Delta, \Diamond A\}}$
$ ^{\diamond b} \frac{\Gamma\{[\Delta], A, [\Sigma]\}}{\Gamma\{[\Delta, \diamond A], [\Sigma]\}} \frac{\Gamma\{[\Delta, \diamond A], [\Sigma]\}}{\Gamma\{[[\Delta, \diamond A], \Sigma]\}} $	\sim	$ \overset{[4]}{\overset{[4]}{\mapsto}} \frac{\Gamma\{[\Delta], A, [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma], A\}} \\ \overset{\diamond b}{\overset{\diamond 5}{\mapsto}} \frac{\Gamma\{[[\Delta], \Diamond A, \Sigma]\}}{\Gamma\{[[\Delta, \Diamond A], \Sigma]\}} $

For permuting down over the [5]-rule, in the only non-trivial case, notice that the context has to be of the form shown because of the restriction of context depth in the [5]-rule:

$$\stackrel{\diamond b}{\stackrel{[5]}{=}} \frac{\Gamma\{\emptyset\}\{[\Sigma, [\Delta], A]\}}{\Gamma\{[\Delta, \Diamond A]\}\{[\Sigma, \emptyset]\}} \sim \qquad \stackrel{[5]}{\longrightarrow} \frac{\Gamma\{\emptyset\}\{[\Sigma, [\Delta], A]\}}{\Gamma\{[\Delta]\}\{[A, \Sigma]\}} \\ \stackrel{\diamond k}{\longrightarrow} \frac{\Gamma\{[\Delta]\}\{[A, \Sigma]\}}{\Gamma\{[\Delta]\}\{[\Delta, [\Sigma, \emptyset]\}\}}$$

Once a \diamond -rule has been permuted down through the structural modal rules to meet the cut, we want to build a new derivation with a lower cut rank. This is not possible when this \diamond -rule is either $\diamond 4$ or $\diamond 5$ since these rules do not decrease the size of the main formula, when seen upwards. The solution is to "reflect" them at the cut and incorporate them in the structural rules that are pushed up together with the cut.

Lemma 2.51 (Reflect $\diamond 4$, $\diamond 5$) Let $X \subseteq \{4, 5\}$. Given a derivation as shown on the left, where all rules in \mathcal{D} apply to the instance of $\diamond A$ shown, then for each sequent Δ there is a derivation as shown on the right:

$\Gamma\{\diamondsuit A\}\{\emptyset\}$		$\Gamma\{\emptyset\}\{[\Delta]\}$	
$\mathcal{D} \land X$	\sim	$\mathcal{D}' \parallel [X]$	
$\Gamma\{\emptyset\}\{\diamondsuit A\}$		$\Gamma\{[\Delta]\}\{\emptyset\}$	

Proof. By induction on the length of \mathcal{D} .

We are now ready to prove the reduction lemma.

Lemma 2.52 (Reduction Lemma) Let $X \subseteq \{t, b, 4, 5\}$. Given a proof as shown on the left, with \mathcal{P}_1 and \mathcal{P}_2 in $K_m + [X] + cut_r$, then there is a proof \mathcal{P} in

 $K_m + [X]^+ + cut_r$ as shown on the right:



Proof. As usual, by an induction on $|\mathcal{P}_1| + |\mathcal{P}_2|$ and a case analysis on the lowermost rules in \mathcal{P}_1 and \mathcal{P}_2 . We only show the most complicated case, in which we cut a box introduced by the m \Box -rule against a diamond introduced by k-rule. All other cases are much simpler. We have

$$\begin{array}{c} \mathrm{m}\Box \frac{\Gamma_1\{[B,\ldots,B]\}}{\Gamma_1\{\Box B\}} & \mathrm{k}\frac{\Gamma_2'\{[\bar{B},\Delta]\}}{\Gamma_2'\{\diamond\bar{B},[\Delta]\}} \\ & \left\| [\mathrm{X}] + \mathrm{med} & \left\| [\mathrm{X}] + \mathrm{med} \right. \right. \\ & \mathrm{cut}_{r+1}\frac{\Gamma\{\Box B\}}{\Gamma\{\Diamond\}} \frac{\Gamma\{\diamond\bar{B}\}}{\Gamma\{\emptyset\}} \end{array}$$

In the left subderivation we permute down the instance of m \Box and on the right subderivation we apply Lemma 2.50 (Push ktb down) in order to obtain the following derivation, where $\Gamma\{ \} = \Gamma\{ \}\{\emptyset\}$. Note that the second hole in the binary context marks the position to which the $\Diamond \overline{B}$ is moved:

$$\begin{array}{c} \Gamma_2'\{[B,\Delta]\} \\ & \| \ [\mathrm{X}]+\mathrm{med} \\ \Gamma_1\{[B,\ldots,B]\} \\ & \| \ [\mathrm{X}]+\mathrm{med} \\ \Gamma_1\{\emptyset\}\{\Diamond B\} \\ \\ \Pi \square \frac{\Gamma\{[B,\ldots,B]\}\{\emptyset\}}{\Gamma\{\square B\}\{\emptyset\}} \\ \mathrm{cut}_{r+1} \frac{\Gamma\{\square B\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\emptyset\}} \\ \end{array} \begin{array}{c} \Gamma_1\{\emptyset\}\{\Diamond B\} \\ & \Pi \ [\mathrm{X}^+ \cap \{4,5\})^\diamond \\ \Gamma_2\{\emptyset\}\{\emptyset\} \\ \end{array} \end{array}$$

By using Lemma 2.51 (Reflect 45) we obtain a derivation \mathcal{D} and build:

$$\begin{array}{c} \Gamma_1\{[B,\ldots,B]\} \\ & \parallel [\mathtt{X}] + \mathtt{med} \\ \Gamma\{[B,\ldots,B]\}\{\emptyset\} & \Gamma'_2\{[\bar{B},\Delta]\} \\ & \mathcal{D} \parallel (\mathtt{X}^+ \cap \{4,5\})^{\cdot} & \parallel [\mathtt{X}] + \mathtt{med} \\ \\ & \mathbb{I}_{\mathsf{cut}_{r+1}} \frac{\Gamma\{\emptyset\}\{[B,\ldots,B]\}}{\Gamma\{\emptyset\}\{\Box B\}} & \sigma \frac{\Gamma_3\{\bar{B}\}}{\Gamma\{\emptyset\}\{\diamond \bar{B}\}} \\ & \Gamma\{\emptyset\}\{\emptyset\} \end{array}$$

.

We now consider the three possible cases for $\sigma \in \{k, \diamond t, \diamond b\}$ and apply one of the following transformations to the relevant part of the proof:

$$\underset{\mathsf{cut}_{r+1}}{\overset{\mathsf{D}\{[\bar{B},\ldots,\bar{B}],[\Delta]\}}{\underset{\mathsf{L}\{[\bar{\Delta}]\}}{\overset{\mathsf{L}\{[\bar{\Delta}]\}}}} \overset{\mathsf{k}}{\underset{\mathsf{D}\{\bar{\Delta}]\}}{\overset{\mathsf{L}\{[\bar{B},\Delta]\}}{\overset{\mathsf{L}\{\bar{B},\Delta]\}}} \xrightarrow{\mathsf{odd}} \underset{\mathsf{mcut}_{r}}{\overset{\mathsf{med}}{\overset{\mathsf{D}\{[\bar{B},\ldots,\bar{B}],[\Delta]\}}{\overset{\mathsf{L}\{[\bar{B},\Delta]\}}{\overset{\mathsf{L}\{[\bar{B},\Delta]\}}}} \Sigma\{[\bar{B},\Delta]\}$$

38

$$\underset{\mathsf{cut}_{r+1}}{\overset{\mathsf{m}\square}{\frac{\Sigma\{[B,\dots,B]\}}{\Sigma\{\squareB\}}}} \underset{\Sigma\{\emptyset\}}{\overset{\diamond\mathsf{t}}{\sum}\frac{\Sigma\{\bar{B}\}}{\Sigma\{\Diamond\bar{B}\}}} \sim \underset{\mathsf{mcut}_{r}}{\overset{[\mathsf{t}]}{\frac{\Sigma\{[B,\dots,B]\}}{\Sigma\{B,\dots,B\}}}} \underset{\Sigma\{\emptyset\}}{\overset{\mathsf{b}}{\sum}\frac{\Sigma\{\bar{B}\}}{\Sigma\{\emptyset\}}}$$

We then eliminate mcut by using Lemma 2.43 (From mcut to cut) and weakening admissibility. $\hfill \Box$

 $\begin{array}{l} \textbf{Proposition 2.53 (Cut-elimination for } K_m) \ \mathrm{Let} \ X \subseteq \{t,b,4,5\}. \ \mathrm{If} \ K_m + [X] + cut \vdash \\ \Gamma \ \mathrm{then} \ K_m + [X]^+ \vdash \Gamma. \end{array}$

Proof. We first prove the claim: If $K_m + [X] + \operatorname{cut}_{r+1} \vdash \Gamma$ then $K_m + [X]^+ + \operatorname{cut}_r \vdash \Gamma$. The claim is proved by induction on the depth of the given proof, using the reduction lemma. Our proposition then follows from an induction on the cut rank of the given proof, using the claim.

Finally, we can prove cut-elimination for the systems $K_c + [X]$.

Proof of Theorem 2.37 (Cut-elimination). We first prove the theorem for the cases where $d \notin X$. The transformation (i) is by Proposition 2.45 (Push down contraction), the transformation (ii) is Proposition 2.53 (Cut-elimination for K_m), and transformation (iii) is by Lemma 2.48 (From X^+ to X) and weakening and necessitation admissibility.



In the cases where $d \in X$ we first apply Lemma 2.40 (Push down seriality) and then proceed the same way with the upper part of the proof.

2.3 Relation to Deep Inference

Deep inference is a proof-theoretic formalism introduced by Guglielmi [26] where inference rules are term rewriting rules which work on formulas and where derivations are just reduction sequences from one formula to another. Some deep inference systems for modal logic have been studied by Hein, Stewart and Stouppa [28, 46, 47].

Stewart and Stouppa give certain deep inference rules for the modal axioms in their paper [46] and conjecture that all combinations yield cut-free systems that are complete for the corresponding frame conditions (Conjecture 11 in [46]). They prove their conjecture just for some modal logics, namely K, KD, KT, S4 and S5, and in all cases their method is embedding a cut-free (hyper-)sequent system. They do not provide cut-free deep inference systems for the other 10 logics of the cube. Also, their method does not extend to logics for which there is no known cut-free (hyper-)sequent system, such as KB and K5.

In this section we see cut-free deep inference systems for these modal logics. Nested sequent systems can be easily embedded into corresponding deep inference systems and via this embedding we get complete and cut-free deep inference systems for all the modal logics considered in this chapter. In fact, we get two sets, one based on the nested sequent systems with logical rules, and one based on the ones with structural rules. However, this does not settle Stewart and Stouppa's Conjecture 11, since our rules are different.

The embedding of nested sequent systems into corresponding deep inference systems is trivial: essentially, all derivations on nested sequents are special deep inference derivations where rules do not apply deeply with respect to all connectives, but only with respect to the comma (structural disjunction) and structural box. The reverse direction, embedding deep inference into nested sequent calculus is also easy, but requires cut.

In this section we extend our language of formulas by the constants t for true and f for *false*.

A deep inference rule is just a labelled rewrite rule as used in term rewriting. An example is the following *switch-down*-rule:

$$\mathfrak{s}\downarrow \frac{S\{A \land (B \lor C)\}}{S\{(A \land B) \lor C\}}$$

which in term rewriting would be written as

$$\mathsf{s} \downarrow : \quad (A \land B) \lor C \to A \land (B \lor C)$$

There is a notational difference: in the deep inference rule the context in which it can be applied is made explicit, in this case any formula context $S\{$ }. A *proof* of a formula is a rewriting sequence starting from the constant t and ending with that formula. For more explicit definitions and more discussion of deep inference systems, see [9].

A deep inference system for propositional logic is shown in Figure 2.10. This particular system is similar to the one given by Straßburger in [48] and slightly weaker than the one originally given in [9] because it replaces the equivalence rule by several explicit rules for for commutativity, associativity and units (which together are weaker than the equivalence rule). Let us call it system KS for the purpose of this section. Systems for modal logics can be obtained from it by adding rules from Figure 2.11. The *cut* in deep inference has the form

$$i\uparrow \frac{S\{A \land A\}}{S\{\mathsf{f}\}}$$

$\mathrm{asl} \frac{S\{A \lor (B \lor C)\}}{S\{(A \lor B) \lor A\}}$	$\frac{C}{C} \qquad \qquad \cot \frac{S\{A \lor B\}}{S\{B \lor A\}}$	$f \downarrow \frac{S\{A\}}{S\{A \lor f\}}$	$t \!\!\downarrow \! \frac{S\{A\}}{S\{A \wedge t\}}$
${}^{ai\downarrow}\frac{S\{t\}}{S\{a\vee\bar{a}\}}$	$\sup \frac{S\{A \land (B \lor C)\}}{S\{(A \land B) \lor C\}}$	$\operatorname{c} \downarrow \frac{S\{A \lor A\}}{S\{A\}}$	$w \downarrow \frac{S\{f\}}{S\{A\}}$

Figure 2.10: A deep inference system for propositional logic

Let an instance of $5\downarrow$ be an instance of either $5a\downarrow, 5b\downarrow$ or $5c\downarrow$. For a set X of rule names append the symbol \downarrow to each name to obtain $X\downarrow$. Let system KSk be system KS + {nec $\downarrow, k\downarrow, r\downarrow$ }.

Proposition 2.54 (Nested sequent calculus into deep inference)

 $\begin{array}{l} \mbox{For all } X \subseteq \{d,t,b,4,5\} \mbox{ and sequents } \Gamma \mbox{ we have that:} \\ \mbox{If } K + \Diamond X_c \ \vdash \ \Gamma \ \mbox{then } KSk + X \downarrow \ \vdash \ \underline{\Gamma}_F. \end{array}$

40

Proof. A routine induction on the depth of the proof and a straightforward extension of a corresponding embedding for the propositional system as given in [9]. Note that embedding the \wedge -rule requires the r \downarrow -rule.

Proposition 2.55 (Deep inference into nested sequent calculus)

For all $X \subseteq \{d, t, b, 4, 5\}$ and formulas A we have that: If $KSk + X \downarrow + i\uparrow \vdash A$ then $K + \Diamond X_c + cut \vdash A$.

Proof. A routine induction on the length of the proof and a straightforward extension of a corresponding embedding for the propositional system as given in [9]. \Box

These propositions, together with cut-elimination for our nested sequent systems, trivially yields cut-elimination for the corresponding deep inference systems. By the second proposition we translate a deep inference proof with cuts into a nested sequent calculus proof with cuts, eliminate the cuts, and translate back to deep inference by the first proposition.

Corollary 2.56 (Cut elimination for deep inference)

For all 45-closed $X \subseteq \{d, t, b, 4, 5\}$ we have that if a formula is provable in system $KSk + X \downarrow + i\uparrow$ then it is also provable in system $KSk + X \downarrow$.

A similar exercise will obtain cut-free and complete deep inference systems from the nested sequent systems with structural modal rules.

Remark 2.57 (for some systems the $r\downarrow$ -rule is admissible) Some of the deep inference systems are not minimal: for example in system KSk the $r\downarrow$ -rule is admissible for KSk $- r\downarrow$. This can be seen by embedding the usual sequent

$$\begin{split} & \operatorname{nec} \downarrow \frac{S\{\mathsf{t}\}}{S\{\Box\mathsf{t}\}} \quad \operatorname{k} \downarrow \frac{S\{\Box(A \lor B)\}}{S\{\Box A \lor \Diamond B\}} \quad \operatorname{r} \downarrow \frac{S\{\Box A \land \Box B\}}{S\{\Box(A \land B)\}} \\ & \operatorname{d} \downarrow \frac{S\{\Box A\}}{S\{\Diamond A\}} \quad \operatorname{t} \downarrow \frac{S\{A\}}{S\{\Diamond A\}} \quad \operatorname{b} \downarrow \frac{S\{A \lor \Box B\}}{S\{\Box(\Diamond A \lor B)\}} \quad \operatorname{d} \downarrow \frac{S\{\Box(A \lor \Diamond B)\}}{S\{\Box(\Diamond A \lor O)\}} \\ & \operatorname{s} \downarrow \frac{S\{\Diamond A \lor \Box B\}}{S\{\Box(\Diamond A \lor B)\}} \quad \operatorname{s} \downarrow \frac{S\{\Box B \lor \Box(\Diamond A \lor C)\}}{S\{\Box(\Diamond A \lor B) \lor \Box C\}} \quad \operatorname{s} \downarrow \frac{S\{\Box(A \lor \Box A \lor C)\}}{S\{\Box(A \lor \Diamond B \lor \Box C)\}} \end{split}$$

Figure 2.11: Deep inference rules for modal logic

system for K, of which we show the case for the \square -rule:

$$\begin{array}{c} & & & & & \\ & & & & \\ \square \underbrace{A, B_1, \dots, B_n} \\ \square \underbrace{A, \Diamond B_1, \dots, \Diamond B_n} \end{array} \xrightarrow{} & & & \\ & & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & & \\ & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ & & \\ \end{array} \xrightarrow{} \begin{array}{c} & & \\ \end{array} \xrightarrow{} \begin{array}{$$

where \mathcal{P}' is the translation of \mathcal{P} , $\Box \mathcal{P}'$ is obtained by adding a box to every formula in \mathcal{P}' and \mathbf{k}^n denotes n instances of the k-rule. For some systems, however, the $r\downarrow$ -rule is not admissible. For example in system $\mathsf{KSk} + \mathsf{b}\downarrow$ the formula $\Box(a \lor \diamond \diamond \bar{a}) \land (b \lor \diamond \diamond \bar{b}))$ is provable, but it is not provable without $r\downarrow$ -rule.

2.4 Discussion

We have seen how nested sequents allow us to give a systematic proof theory for the modal logics of the cube. In fact, we have seen two distinct prooftheories, one based on formalising modal axioms as logical rules and one based on formalising modal axioms as structural rules. The first option is closer to the ordinary sequent calculus and allows for a straightforward terminating proofsearch procedure, but fails to be modular: not every possible combination of rules yields a complete system for the corresponding logic. The second option yields a modular set of systems, but the presence of structural rules devalues the subformula property. In any case, we have seen that generalising hypersequents to nested sequents yields cut-free systems for more modal logics, so it leads to greater expressivity. It is particularly pleasant that this extra generality does not come at the cost of extra complexity, but in fact simplifies hypersequent systems: the two kinds of context in hypersequent inference rules (sequent context and hypersequent context) are merged into one. Our systems with logical rules enjoy invertibility of all rules. This property does not seem to be achievable in an ordinary sequent system for modal logic. In hypersequent systems it also does not seem to be achievable in a non-trivial way (although one could of course trivially make rules invertible by copying a component whenever a rule applies in it).

Relation to the display calculus. Nested sequents and display sequents share the idea of simply allowing the connective \Box as a structural connective. There are two crucial differences. First, display sequents also contain a structural connective for the backward-looking modality. This is crucial for the *display property* to hold, a central property of display calculi which allows to single out a formula in order to apply a logical rule to it. Since in our proof systems logical rules apply deeply inside nested sequents, there is no need for a display property, and thus no need for the backward-looking modality, so we can stay inside the modal language. The second difference is that in the display calculus one has to use structural rules called *display postulates* to move a formula to the top in order to apply a logical rule to it. In nested sequent systems one can apply the rule on the spot and thus has no need for such structural rules. Nested sequents thus allow for deductive systems with fewer rules and shorter derivations.

Relation to labelled systems. The main conceptual advantage of a nested sequent over a labelled sequent is that it can be read as a modal formula. Labelled sequents are more general than nested sequents: they can form an arbitrary graph, while nested sequents are always trees. A cut-free proof in nested sequents is thus in general a more restricted, simpler object than a cut-free proof in labelled sequents. I hope that this fact will help in using nested sequent systems for interpolation proofs, for which labelled systems do not seem to be well-suited. It should also be easy to embed cut-free nested sequent systems into corresponding cut-free labelled sequent systems, while the opposite is not true in general. I thus think of the completeness of a nested sequent system as a stronger result than the completeness of a corresponding labelled sequent system. To get this stronger result we had to work harder, for example in our completeness proof for the systems with logical rules: we had to establish certain properties of, say, the euclidean closure of a relation, which is not needed for labelled systems. There, that relation is part of the proof system and it is being closed under euclideanness by the appropriate inference rule. The extra work also shows in our cut-elimination procedure: we had to show admissibility of certain rules in order to push the cut over the rules for the frame properties. This, again, is not needed for labelled systems. There the rules for the frame conditions do not affect the cut-elimination procedure at all.

Relation to tableau systems. While the focus of tableau systems is on giving decision procedures, our focus is on giving proof systems which support proof-transformations, in particular cut-elimination. This is more easily and more commonly done with local rules, so in sequent systems instead of tableau systems. Nevertheless, there is correspondence between tableau systems and sequent systems. For an overview of modal tableau systems see the survey by Goré [22]. The tableau formalism which corresponds most closely to nested sequents is the prefixed tableau formalism, due to Fitting [19]. In particular, prefixes impose the same tree structure on formulas that is imposed in a nested sequent. However, prefixed tableaux are closer to the semantics. In particular they have rules which are parametrised by an accessibility relation, which is a marked difference from our inference rules.

Specific tableau rules which correspond to our inference rules have also been studied before, namely by Castilho et. al. [16]. Their systems are based on

2.4. DISCUSSION

graphs rather than trees, but they have structural rules which closely correspond to (some of) ours and *propagation rules* which correspond to our \diamond -rules. A difference is that propagation rules and structural rules are mixed in [16], while here we first treat systems purely consisting of propagation- or \diamond -rules in Chapter 2 and systems purely consisting of structural rules in Chapter 3.

Future work. Of course we would like to extend the range of logics for which there are cut-free nested sequent systems. Candidates are the set of modal logics formalised by so-called *primitive* axioms, which have been captured in the display calculus [52]. At the same time, it is interesting to generate such systems automatically, so it is our goal to devise 1) easily checkable criteria on rules, which guarantee cut-elimination, and 2) a procedure which turns modal axioms into rules which satisfy these criteria. Such a generic cut-elimination procedure exists already for the display calculus [52]. Recently, such a procedure has also been proposed by Ciabattoni et al. for certain hypersequent systems [17].

On the other hand, we would like to use nested sequent systems to obtain results which are harder or cannot be obtained with other proof-theoretic formalisms. Neither display calculus nor labelled sequent calculus seem to allow us to prove interpolation results, for example. Conservativity results are another interesting field. Here the property of staying inside the modal language is useful. The conservativity of tense logic over modal logic is an immediate consequence of the completeness of a cut-free nested sequent system for tense logic, as noted by Goré et al. [24]. This conservativity result is not an immediate consequence of cut-elimination in the display calculus, precisely because of the presence of (rules affecting) backward-looking structural connectives.

Another area to explore is the one of explicit modal logics [4]. Here the modality in modal logic which can be read as provability or as knowledge is replaced by specific terms which can be read as individual proofs or as pieces of evidence. Researchers study *realisation*-procedures which turn a proof in modal logic into a proof in explicit modal logic. Such procedures rely on cut-free systems for modal logics. Nested sequent systems may provide such realisation procedures. 44 CHAPTER 2. SYSTEMS FOR BASIC NORMAL MODAL LOGICS

Chapter 3

Systems for Common Knowledge

The notion of *common knowledge* is well-studied in epistemic logic, where modalities express knowledge of agents. Two standard textbooks on epistemic logic and common knowledge in particular, are [18] by Fagin, Halpern, Moses, and Vardi and [33] by Meyer and van der Hoek.

The fact that a proposition A is common knowledge can be expressed by the infinite conjunction "all agents know A and all agents know that all agents know A and so on". In order to express this in a finite way we can use fixpoints: common knowledge of A is then defined to be the greatest fixpoint of the function

 $X \mapsto$ everybody knows A and everybody knows X.

Such a definition was introduced by Halpern and Moses [27] and further studied in [18].

The traditional way to formalise common knowledge is to use a Hilbert-style axiom system. Such a system has a fixpoint axiom, which states that common knowledge is a fixpoint, and an induction rule, which states that this fixpoint is the greatest fixpoint. However, this approach does not work well for designing a Gentzen-style sequent calculus. In particular, Alberucci and Jäger show in [2] that a cut-free sequent system designed in this way is not complete.

To obtain a complete cut-free system Alberucci and Jäger replace the induction rule by an infinitary ω -rule. This results in a system in which proofs have transfinite depth and in which common knowledge is the greatest fixpoint of the function described above. Although this system has been further studied in [31, 29], no syntactic cut-elimination procedure has been found. Cut-elimination was proved only indirectly by showing completeness of the cut-free system. No non-trivial bound on the depth of proofs in this system is known.

In this chapter, we give a syntactic cut-elimination procedure for an infinitary system of common knowledge based on nested sequents. Since its inference rules apply deeply inside of the nested sequents we call this system "deep" while we call the system by Alberucci and Jäger "shallow". The deep system allows to straightforwardly apply the method of *predicative cut-elimination*, which is

a standard tool for the proof-theoretic analysis of systems of set theory and second order number theory, see Pohlers [39, 40] and Schütte [45]. Since the shallow and the deep system can be embedded into each other, this also yields a syntactic cut-elimination procedure for the shallow system. For both systems we thus obtain an upper bound of $\varphi_2 0$ on the depth of proofs, where φ is the Veblen function.

Please note that, like Alberucci and Jäger, our term *logic of common knowledge* refers to the least normal modal logic K, with an added fixpoint modality. Some people might prefer to call that the logic of common *belief*. The methods introduced here should transfer easily to cases where rules for the modal axioms are added that were studied in the previous chapter. The combination of the techniques presented here and the ones in the previous chapter should suffice to get cut-elimination for modal logics with additional modal axioms and common knowledge.

Several cut-free systems for logics with common knowledge exist already. The one that is closest to our system was introduced by Tanaka in [49] for predicate common knowledge logic and is based on Kashima's ideas. It essentially also uses nested sequents, but uses explicit labels to name the nodes of the tree. In fact, if one disregards the rather different notation and some choices in the formulation of rules, then one could say that our system is the propositional part of Tanaka's system. There are also finitary systems. Abate, Goré and Widmann, for example, introduce a cut-free tableau system for common knowledge in [1]. Cut-free system have also been studied in the context of *explicit modal logic* by Artemov [5] and by Antonakos [3].

However, we do not know of syntactic cut-elimination procedures for any of the systems mentioned. Typically, cut-elimination is established only indirectly. There are cut-elimination procedures for similar logics, for example by Pliuškevičius for an infinitary system for linear time temporal logic in [37]. For linear temporal logic there is no need for nested sequents. For this logic it is enough to use indexed formulas of the form A^i which denotes A at the *i*-th moment in time.

This chapter is organised as follows. We first review the shallow sequent system by Alberucci and Jäger and show the obstacle to cut-elimination. We then present our nested sequent system, prove the invertibility of its rules, the admissibility of the structural rules and finally cut-elimination. Then we embed the shallow system into the deep system and vice versa, thus establishing cutelimination for the shallow system. Then, by embedding the Hilbert system into our deep sequent system, we obtain an upper bound for the depth of proofs in both the shallow and the deep system. Some discussion about future work ends this chapter.

3.1 The Shallow Sequent System

Formulas and sequents. We are considering a language with h agents for some h > 0. Propositions p and their negations \bar{p} are *atoms*, with \bar{p} defined to be p.

$$\begin{array}{ccc} \Gamma,p,\bar{p} & \wedge \frac{\Gamma,A}{\Gamma,A\wedge B} & \vee \frac{\Gamma,A,B}{\Gamma,A\vee B} \\ & & \Box_i \frac{\Gamma, \otimes \Delta, A}{\diamondsuit_i \Gamma, \otimes \Delta, \Box_i A, \Sigma} \end{array}$$

$$\underset{\mathbb{H}}{\overset{\Gamma, \Box^k A \quad \text{for all } k \geq 1}{\Gamma, \boxtimes A} & \otimes \frac{\Gamma, \otimes A, \Diamond A}{\Gamma, \otimes A} \end{array}$$

Figure 3.1: System G_C

Formulas are denoted by A, B, C, D. They are given by the following grammar:

$$A ::= p \mid \bar{p} \mid (A \lor A) \mid (A \land A) \mid \diamondsuit_i A \mid \Box_i A \mid \circledast A \mid \circledast A$$

where $1 \leq i \leq h$. The formula $\Box_i A$ is read as "agent *i* knows *A*" and the formula $\blacksquare A$ is read as "*A* is common knowledge". The connectives \Box_i and \blacksquare have \diamondsuit_i and \circledast as their respective De Morgan duals. Binary connectives are left-associative: $A \lor B \lor C$ denotes $((A \lor B) \lor C)$, for example.

Given a formula A, its negation \overline{A} is defined as usual using the De Morgan laws, $A \supset B$ is defined as $\overline{A} \lor B$ and \bot is defined as $p \land \overline{p}$ for some proposition p. The formula $\Box A$ is an abbreviation for "everybody knows A":

$$\Box A = \Box_1 A \land \ldots \land \Box_h A \quad \text{and} \quad \Diamond A = \Diamond_1 A \lor \ldots \lor \Diamond_h A.$$

A sequence of $n \ge 0$ modal connectives can be abbreviated, for example

$$\Box^n A = \underbrace{\Box \dots \Box}_{n-\text{times}} A \quad .$$

A (shallow) sequent is a finite multiset of formulas. Sequents are denoted by $\Gamma, \Delta, \Lambda, \Pi, \Sigma$.

Inference rules. In an instance of the inference rule ρ

$$\rho \frac{\Gamma_1 \quad \Gamma_2 \quad \dots}{\Delta}$$

the sequents $\Gamma_1, \Gamma_2...$ are its *premises* and the sequent Δ is its *conclusion*. An *axiom* is a rule without premises. We will not distinguish between an axiom and its conclusion. A *system*, denoted by S, is a set of rules. Figure 3.1 shows system G_C , a shallow sequent calculus for the logic of common knowledge. Its only axiom is called *identity axiom*. Notice that the \mathbb{E} -rule has infinitely many premises. If Γ is a sequent then $\diamond_i \Gamma$ is obtained from Γ by prefixing the connective \diamond_i to each formula occurrence in Γ , and similarly for other connectives.

Derivations and proofs. In the following, a *tree* is a tree in the graph-theoretic sense, and may be infinite. A tree is *well-founded* if it does not have an infinite path. A *derivation* in a system S is a directed, rooted, ordered and well-founded tree whose nodes are labelled with sequents and which is built according to the

$$\mathsf{wk} \, \frac{\Gamma}{\Gamma, A} \qquad \mathsf{ctr} \, \frac{\Gamma, A, A}{\Gamma, A} \qquad \mathsf{cut} \, \frac{\Gamma, A \ \Delta, \bar{A}}{\Gamma, \Delta}$$

Figure 3.2: Weakening, contraction and cut for system Gc

inference rules from S. Derivations are visualised as upward-growing trees, so the root is at the bottom. The sequent at the root is the *conclusion* and the sequents at the leaves are the *premises* of the derivation. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ where all leaves are axioms. Proofs are denoted by \mathcal{P} . We write $S \vdash \Gamma$ if there is a proof of Γ in system S. Given a proof \mathcal{P} we denote its depth by $|\mathcal{P}|$. Notice that derivations here are in general infinitely branching, thus their depth can be infinite even though each branch has to be finite.

Formula rank. Notice that formulas in the premises of the \mathbb{E} -rule are generally larger than formulas in its conclusion. This is typically a problem for cutelimination, but we can easily solve this by defining an appropriate measure. For a formula A we define its rank rk(A) as follows:

$$\begin{split} rk(p) &= rk(\bar{p}) = 0\\ rk(A \wedge B) &= rk(A \vee B) = max(rk(A), rk(B)) + 1\\ rk(\Box_i A) &= rk(\diamondsuit_i A) = rk(A) + 1\\ rk(\textcircled{A}) &= rk(\diamondsuit A) = \omega + rk(A) \end{split}$$

Lemma 3.1 (Some properties of the rank) For all formulas A we have that (i) $rk(A) = rk(\overline{A})$, (ii) $rk(A) < \omega^2$,

(iii) for all $k < \omega$ we have $rk(\Box^k A) < rk(\blacksquare A)$.

Proof. Statements (i) and (ii) are immediate. For (iii), an induction on k yields that $rk(\Box^k A) = rk(A) + k \cdot h$. By (ii) it is then enough to check that for all k and all $\alpha < \omega^2$ we have $\alpha + k \cdot h < \omega + \alpha$.

Cut rank. The *cut rank* of an instance of **cut** as shown in Figure 3.2 is the rank of its *cut formula* A. For an ordinal γ we define the rule $\operatorname{cut}_{\gamma}$ which is cut with at most rank γ and the rule $\operatorname{cut}_{<\gamma}$ which is cut with a rank strictly smaller than γ . For a system S and ordinals α and γ and a sequent Γ we write $S \mid \frac{\alpha}{\gamma} \Gamma$ to say that there is a proof of Γ in system $S + \operatorname{cut}_{<\gamma}$ with depth bounded by α . We write $S \mid \frac{<\alpha}{\gamma} \Gamma$ to say that there is an ordinal $\alpha_0 < \alpha$ such that $S \mid \frac{\alpha_0}{\gamma} \Gamma$.

Admissibility and invertibility. An inference rule ρ is depth- and cut-rankpreserving admissible or, for short, perfectly admissible for a system S if for each instance of ρ with premises $\Gamma_1, \Gamma_2 \dots$ and conclusion Δ , whenever $S \mid_{\gamma}^{\alpha} \Gamma_i$ for each premise Γ_i then $S \mid_{\gamma}^{\alpha} \Delta$. For each rule ρ there is its *inverse*, denoted by $\overline{\rho}$, which has the conclusion of ρ as its only premise and any premise of ρ as its conclusion. An inference rule ρ is perfectly invertible for a system S if $\overline{\rho}$ is perfectly admissible for S.

We omit the proof of the following lemma, which is standard.

Lemma 3.2 (Admissibility of the structural rules and invertibility)

(i) The rules weakening and contraction from Figure 3.2 are perfectly admissible for system G_C .

(ii) All rules of G_C except for the \Box_i -rule are perfectly invertible for system G_C .

3.1.1 The Problem for Cut-Elimination

Let us look at the problem of cut-elimination in system $\mathsf{G}_\mathsf{C}.$ Consider the following proof:

$$\overbrace{\mathsf{cut}}^{\mathcal{P}_1} \underbrace{A, \Gamma, \circledast \bar{B}}_{\operatorname{cut}} \underbrace{\vdots \qquad \overset{\mathcal{P}_2k}{\square_i A, \diamondsuit_i \Gamma, \Sigma, \circledast \bar{B}}} \underset{\boxtimes}{\overset{\boxtimes}{\boxplus} B, \Delta}{\overset{\boxtimes}{\boxtimes} B, \Delta} \stackrel{\vdots}{\boxtimes} 1 \leq k < \omega$$

Here the inference rule above the cut on the left does not apply to the cut formula while the inference rule on the right does. The typical transformation would push the left rule instance below the cut, as follows:

$$\underset{\mathsf{cut}}{\underbrace{\frac{A,\Gamma, \circledast \bar{B}}{\bigcap_{i} \frac{A,\Gamma, \Delta}{\bigcap_{i} A, \Diamond_{i} \Gamma, \Sigma, \Diamond_{i} \Delta}}} } \overset{\textcircled{\mathcal{P}_{2k}}{=} \mathbb{E}}{\underbrace{\frac{P_{2k}}{\boxtimes B, \Delta}} \overset{\vdots}{=} 1 \leq k < \omega}$$

However, this transformation introduces the \diamond_i in $\diamond_i \Delta$, and thus it does not yield a proof of the original conclusion. This problem is caused by the context restriction in the \Box_i -rule.

Such a context restriction also occurs in the standard sequent calculus for the modal logic K. While it destroys invertibility, at least it does not cause any difficulties for syntactic cut-elimination for K. However, we see that the context restriction poses a genuine problem for logics with more modalities like in the logic of common knowledge. In the next section we will see how a more general format for sequents and inference rules solves the problem since it does not require context restrictions.

3.2 The Nested Sequent System

Nested sequents. A nested sequent is a finite multiset of formulas and boxed sequents. A boxed sequent is an expression $[\Gamma]_i$ where Γ is a nested sequent and $1 \leq i \leq h$. The letters $\Gamma, \Delta, \Lambda, \Pi, \Sigma$ from now on denote nested sequents and the word sequent from now on refers to nested sequent, except when it is clear

from the context that a sequent is shallow, such as a sequent appearing in a derivation in ${\sf G}_{\sf C}.$ A sequent is always of the form

$$A_1,\ldots,A_m,[\Delta_1]_{i_1},\ldots,[\Delta_n]_{i_n}$$

where the i_j denote agents and thus range from 1 to h. As usual, the comma denotes multiset union and there is no distinction between a singleton multiset and its element.

Fix an arbitrary linear order on formulas. Fix an arbitrary linear order on boxed sequents. The *corresponding formula* of a non-empty sequent Γ , denoted $\underline{\Gamma}_{\mathsf{F}}$, is defined as follows:

 $\underline{A_1,\ldots,A_m,[\Delta_1]_{i_1},\ldots,[\Delta_n]_{i_{n_{\mathsf{F}}}}} = \underline{A_1} \vee \ldots \vee \underline{A_m} \vee \Box_{i_1} \underline{\Delta_{1_{\mathsf{F}}}} \vee \ldots \vee \Box_{i_n} \underline{\Delta_{n_{\mathsf{F}}}},$

where formulas and boxed sequents are listed according to the fixed orders. The *corresponding formula* of the empty sequent is \perp . A sequent has a *corresponding tree* whose nodes are marked with multisets of formulas and whose edges are marked with agents. The corresponding tree of the above sequent is



where $tree(\Delta_1) \dots tree(\Delta_n)$ are the corresponding trees of $\Delta_1 \dots \Delta_n$. Often we do not distinguish between a sequent and its corresponding tree, for example the *root* of a sequent is the root of its corresponding tree.

Formula contexts and sequent contexts. A formula context is a formula with exactly one occurrence of the special atom $\{\ \}$, which is called the hole or the empty context. A sequent context is a sequent with exactly one occurrence of the hole, which does not occur inside formulas. Formula contexts are denoted by $A\{\ \}, B\{\ \}$, and so on. Sequent contexts are denoted by $\Gamma\{\ \}, \Delta\{\ \}$, and so on. Formally, sequent contexts are generated inductively as follows: if Δ is a sequent then Δ , $\{\ \}$ is a sequent context, and if Δ is a sequent and $\Gamma\{\ \}$ is a sequent context.

The formula $A\{B\}$ is obtained by replacing $\{ \}$ inside $A\{ \}$ by B and the sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{ \}$ inside $\Gamma\{ \}$ by Δ . For example, if $\Gamma\{ \} = A, [[B]_1, \{ \}]_2$ and $\Delta = C, [D]_3$ then

$$\Gamma\{\Delta\} = A, [[B]_1, C, [D]_3]_2$$

Formally, given a sequent Δ and a sequent context $\Gamma\{\ \}$ then $\Gamma\{\Delta\}$ is defined inductively as follows: if $\Gamma\{\ \} = \Gamma_1, \{\ \}$ then $\Gamma\{\Delta\} = \Gamma_1, \Delta$ and if $\Gamma\{\ \} = \Gamma_1, [\Gamma_2\{\ \}]$ then $\Gamma\{\Delta\} = \Gamma_1, [\Gamma_2\{\Delta\}]$.

The corresponding formula context of a sequent context $\Gamma\{ \}$, denoted $\underline{\Gamma\{ \}}_{r}$ is defined as follows:

$$\frac{\underline{\Gamma, \{\}}_{\mathsf{F}} = \underline{\Gamma}_{\mathsf{F}} \lor \{\}}{\underline{\Gamma, [\Delta\{\}]_{i_{\mathsf{F}}}} = \underline{\Gamma}_{\mathsf{F}} \lor \Box_{i} \underline{\Delta\{\}}_{\mathsf{F}}}$$

50

$$\begin{split} & \Gamma\{p,\bar{p}\} & \wedge \frac{\Gamma\{A\} \ \Gamma\{B\}}{\Gamma\{A \wedge B\}} & \vee \frac{\Gamma\{A,B\}}{\Gamma\{A \vee B\}} \\ & \Box_i \frac{\Gamma\{[A]_i\}}{\Gamma\{\Box_i A\}} & \diamondsuit_i \frac{\Gamma\{\diamondsuit_i A, [\Delta,A]_i\}}{\Gamma\{\diamondsuit_i A, [\Delta]_i\}} \\ & \\ & \boxplus \frac{\Gamma\{\Box^k A\} \ \text{for all } k \geq 1}{\Gamma\{\divideontimes A\}} & & \Leftrightarrow \frac{\Gamma\{\And A, \diamondsuit^k A\}}{\Gamma\{\And A\}} \end{split}$$

Figure 3.3: System D_C

$$\operatorname{nec} \frac{\Gamma}{[\Gamma]_i} \qquad \operatorname{wk} \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \operatorname{ctr} \frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \operatorname{cut} \frac{\Gamma\{A\}}{\Gamma\{\emptyset\}}$$

Figure 3.4: Necessitation, weakening, contraction and cut for system D_C

Figure 3.3 shows our nested sequent system D_C . Figure 3.4 shows the structural rules *necessitation*, *weakening* and *contraction* as well as the rule *cut*, which are associated to system D_C . Notice that the rules of system D_C and the associated rules are different from the corresponding rules in system G_C but have the same names. If we refer to a rule only by its name then it will be clear from the context which rule is meant. For example the cut in $G_C + cut$ is the one associated to system D_C .

Lemma 3.3 (Admissibility of the structural rules and invertibility)

(i) The rules necessitation, weakening and contraction from Figure 3.4 are perfectly admissible for system $\mathsf{D}_{\mathsf{C}}.$

(ii) All rules in D_C are perfectly invertible for D_C .

Proof. Admissibility of necessitation and weakening follow from a routine induction on the depth of the proof. The same works for the invertibility of the \land,\lor,\Box_i and \blacksquare -rules in (ii). The inverses of all other rules are just weakenings. For admissibility of contraction we also proceed by induction on the depth of the proof tree, using invertibility of the rules. The cases for the propositional rules and for the $\Box_i, \boxdot, \diamondsuit$ -rules are trivial. For the \diamondsuit_i -rule we consider the formula $\diamondsuit_i A$ from its conclusion $\Gamma\{\diamondsuit_i A, [\Delta]_i\}$ and its position inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have the cases 1) $\diamondsuit_i A$ is inside Σ or 2) $\diamondsuit_i A$ is inside $\Lambda\{\$. We have two subcases for case 1: 1.1) $[\Delta]_i$ inside $\Lambda\{\$, 1.2) $[\Delta]_i$ inside Σ . There are three subcases of case 2: 2.1) $[\Delta]_i$ inside $\Lambda\{\$ and 2.2) $[\Delta]_i$ inside $\Sigma, 2.3$ Σ, Σ inside $[\Delta]_i$. All cases are either simpler than or similar to case 2.2, which is as follows:

$$\stackrel{\wedge_{i}}{\underset{\operatorname{ctr}}{\frac{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta,A]_{i},\Sigma',[\Delta]_{i}\}}{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta]_{i},\Sigma',[\Delta]_{i}\}}}}{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta]_{i},\Sigma',[\Delta]_{i}\}} \quad \rightsquigarrow \quad \stackrel{\overline{\diamond_{i}}}{\underset{\operatorname{ctr}}{\frac{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta,A]_{i},\Sigma',[\Delta,A]_{i}\}}{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta,A]_{i},\Sigma',[\Delta,A]_{i}\}}}}{\stackrel{\wedge_{i}}{\underset{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta,A]_{i}\}}{\frac{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta,A]_{i}\}}{\Lambda'\{\diamond_{i}A,\Sigma',[\Delta,A]_{i}\}}}} ,$$

where the instance of $\overline{\diamond}_i$ in the proof on the right is removed because it is perfectly admissible and the instance of contraction is removed by the induction hypothesis.

Lemma 3.4 (Derivability of the general identity axiom) For all contexts $\Gamma\{ \}$ and all formulas A we have $D_{\mathsf{C}} \mid \frac{2 \cdot rk(A)}{0} \Gamma\{A, \bar{A}\}.$

Proof. We perform an induction on rk(A) and a case analysis on the main connective of A. The cases for atoms and for the propositional connectives are obvious. For $A = \Box_i B$ and $A = \mathbb{R}B$ we respectively have

$$\stackrel{\diamond_i}{\longrightarrow} \frac{\Gamma\{[B,\bar{B}]_i, \diamond_i\bar{B}\}}{\Gamma\{[B]_i, \diamond_i\bar{B}\}} \quad \text{and} \quad \underset{\mathbb{B}}{\longrightarrow} \frac{\Gamma\{\Box^k B, \diamond^k\bar{B}\}}{\Gamma\{\Box^k B, \circledast\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \diamond^k\bar{B}\}}{\Gamma\{\Box^k B, \Leftrightarrow^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \diamond^k\bar{B}\}}{\Gamma\{\Box^k B, \Leftrightarrow^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \diamond^k\bar{B}\}}{\Gamma\{\Box^k B, \diamondsuit^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \diamondsuit^k\bar{B}\}}{\Gamma\{\Box^k B, \diamondsuit^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \clubsuit^k\bar{B}\}}{\Gamma\{\Box^k B, \diamondsuit^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \clubsuit^k\bar{B}\}}{\Gamma\{\Box^k B, \diamondsuit^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k B, \clubsuit^k\bar{B}\}}{\Gamma\{\Box^k B, \clubsuit^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k\bar{B}, \clubsuit^k\bar{B}\}}{\Gamma\{\Box^k\bar{B}, \clubsuit^k\bar{B}\}} \stackrel{:}{\longrightarrow} \frac{\Gamma\{\Box^k\bar{B$$

On the left by induction hypothesis we get a proof of the premise of depth $2 \cdot rk(B)$ and thus a proof of the conclusion of depth $2 \cdot rk(B) + 2 = 2 \cdot (rk(B) + 1) = 2 \cdot rk(\Box_i B)$. On the right by Lemma 3.1 we can apply the induction hypothesis for each premise to get a proof of depth $2 \cdot rk(\Box^k B) = 2 \cdot (rk(B) + k \cdot h)$ and thus a proof of the conclusion of depth $2 \cdot (rk(B) + \omega) \le 2 \cdot (\omega + rk(B)) = 2 \cdot rk(\boxdot B)$. \Box

3.3 Cut-Elimination for the Nested System

We first need some notions concerning ordinals. For an introduction to ordinals we refer to Schütte [45]. We write $\alpha \# \beta$ for the *natural sum of* α and β which, in contrast to the ordinary ordinal sum, does not cancel additive components. In particular, the natural sum is commutative. To give names to the ordinals which measure our proofs we also need the following definition.

Definition 3.5 (Veblen function) The *binary Veblen function* φ is generated inductively as follows:

- 1. $\varphi_0\beta := \omega^\beta$,
- 2. if $\alpha > 0$, then $\varphi_{\alpha}\beta$ is the $(\beta + 1)$ th common fixpoint of the functions $\xi \mapsto \varphi_{\gamma}\xi$ for all $\gamma < \alpha$.

The Veblen function just generates an increasing sequence of ordinals, as follows: $\varphi_0 0 = \omega^0 = 1, \ \varphi_0 1 = \omega^1 = \omega, \ \dots, \ \varphi_0 \omega = \omega^{\omega}, \ \dots, \ \varphi_1 0 = \text{first fixpoint of the function } (\xi \mapsto \varphi_0 \xi = \xi \mapsto \omega^{\xi}) = \varepsilon_0, \ \dots,$

 $\varphi_2 0 =$ first fixpoint of the function $(\xi \mapsto \varphi_1 \xi = \xi \mapsto \varepsilon_{\xi}), \ldots$.

Here we will only need ordinals up to $\varphi_2 0$. In this subsection we write $\frac{\alpha}{\beta} \Gamma$ for $\mathsf{D}_{\mathsf{C}} \frac{\alpha}{\beta} \Gamma$. We now prove the central lemma.

52

Lemma 3.6 (Reduction Lemma) If there is a proof



with \mathcal{P}_1 and \mathcal{P}_2 in $\mathsf{D}_{\mathsf{C}} + \mathsf{cut}_{<\gamma}$ then $|\frac{|\mathcal{P}_1| \# |\mathcal{P}_2|}{\gamma} \Gamma\{\emptyset\}$.

Proof. By induction on $|\mathcal{P}_1| \# |\mathcal{P}_2|$. We perform a case analysis on the two lowermost rules in the given proofs. If one of the two rules is passive and an axiom then $\Gamma\{\emptyset\}$ is axiomatic as well. If one is active and an axiom then we have



and by contraction admissibility we have $\frac{|\mathcal{P}_2|}{0} \Gamma\{\bar{p}\}$ and thus $\frac{|\mathcal{P}_1| \# |\mathcal{P}_2|}{0} \Gamma\{\bar{p}\}$. If some rule ρ is passive then we have



where *i* ranges from 1 to the number of premises of ρ . By invertibility of ρ we get $\frac{|\mathcal{P}_1|}{\gamma} \Gamma_i\{A\}$, thus by induction hypothesis $\frac{|\mathcal{P}_1| \# |\mathcal{P}_{2i}|}{\gamma} \Gamma_i\{\emptyset\}$ for all *i* and by ρ we get $\frac{|\mathcal{P}_1| \# |\mathcal{P}_2|}{\gamma} \Gamma\{\emptyset\}$.

This leaves the case that both rules are active and neither is an axiom. We have:

 $(\land - \lor)$:



where by weakening admissibility we get $\frac{|\mathcal{P}_{12}|}{\gamma}$ $\Gamma\{\bar{B}, C\}$, and since $\sigma < \sigma + 1 = \gamma$ we get $\frac{|\alpha|}{\gamma}$ $\Gamma\{\emptyset\}$ for $\alpha = max(|\mathcal{P}_{11}|, max(|\mathcal{P}_{12}|, |\mathcal{P}_{21}|) + 1) + 1$. It is easy to check that $\alpha \leq |\mathcal{P}_1| \# |\mathcal{P}_2|$.

 $(\Box_i - \diamondsuit_i)$:



where the premises of the upper cut have been derived by use of weakening admissibility with depth $|\mathcal{P}_{11}| + 1$ and $|\mathcal{P}_{21}|$, the natural sum of which is smaller than $|\mathcal{P}_1| \# |\mathcal{P}_2|$. The induction hypothesis thus yields $\frac{(|\mathcal{P}_{11}|+1) \# |\mathcal{P}_{21}|}{\gamma}$ $\Gamma\{[\Delta, \bar{A}]_i\}$ and since $\sigma < \sigma + 1 = \gamma$ we get $\frac{|\mathcal{P}_1| \# |\mathcal{P}_2|}{\gamma} \Gamma\{[\Delta]_i\}$ by the lower cut. $(\mathbb{H} - \circledast)$:



54



where the induction hypothesis applied on the upper cut gives us $\frac{|\mathcal{P}_1| \# |\mathcal{P}_{21}|}{\gamma}$ $\Gamma\{\diamondsuit^j \bar{A}\}$ and since by Lemma 3.1 we have $\sigma + j \cdot h < \omega + \sigma = \gamma$ the lower cut yields $\frac{|\mathcal{P}_1| \# |\mathcal{P}_2|}{\gamma}$ $\Gamma\{\emptyset\}$.

From the reduction lemma we obtain the first and the second elimination lemma as usual, see for instance Pohlers [39, 40] or Schütte [45].

Lemma 3.7 (First Elimination Lemma) If $|_{\gamma+1}^{\alpha} \Gamma$ then $|_{\gamma}^{2^{\alpha}} \Gamma$.

Proof. By induction on α and a case analysis on the last rule applied. Most cases are trivial, in case of a cut with rank γ we apply the induction hypothesis to both proofs of the premises of the cut and then apply the reduction lemma to obtain $\left|\frac{2^{\alpha_0} \# 2^{\alpha_0}}{\gamma}\right| \Gamma$ for some $\alpha_0 < \alpha$ and thus $\left|\frac{2^{\alpha}}{\gamma}\right| \Gamma$.

Lemma 3.8 (Second Elimination Lemma) If $|\frac{\alpha}{\beta+\omega^{\gamma}} \Gamma$ then $|\frac{\varphi_{\gamma}\alpha}{\beta} \Gamma$.

Proof. By induction on γ with a subinduction on α . For $\gamma = 0$ this trivially follows from the first elimination lemma. Assume $\gamma > 0$. The non-trivial case is where the last rule in the given proof of Γ is a cut with a rank of β or greater. With $\Gamma = \Gamma\{\emptyset\}$ the proof is of the following form:



Let $\alpha_0 = max(|\mathcal{P}_1|, |\mathcal{P}_2|)$. We apply the subinduction hypothesis on the subproofs of the cut and obtain $|\frac{\varphi_{\gamma}(\alpha_0)}{\beta} \Gamma\{A\}$ and $|\frac{\varphi_{\gamma}(\alpha_0)}{\beta} \Gamma\{\bar{A}\}$. Since $rk(A) < \beta + \omega^{\gamma}$ a quick calculation by case analysis on γ yields the existence of σ with $\sigma < \gamma$ and of n such that $rk(A) < \beta + \omega^{\sigma} \cdot n$. Thus, by a cut we obtain $|\frac{\varphi_{\gamma}(\alpha_0)+1}{\beta+\omega^{\sigma} \cdot n} \Gamma$. We apply the induction hypothesis n times to obtain $|\frac{\varphi_{\sigma}^n(\varphi_{\gamma}(\alpha_0)+1)}{\beta} \Gamma$, where φ_{σ}^n means φ_{σ} applied n times. Since $\varphi_{\sigma}^n(\varphi_{\gamma}(\alpha_0)+1) < \varphi_{\gamma}(\alpha)$ we have $|\frac{\varphi_{\gamma}(\alpha)}{\beta} \Gamma$. \Box

The cut-elimination theorem follows by iterated application of the second elimination lemma.

Theorem 3.9 (Cut-elimination for the deep system) If $D_{\mathsf{C}} \frac{\alpha}{|\alpha|} \Gamma$ then $D_{\mathsf{C}} \frac{|\varphi_1^n(\alpha)|}{0} \Gamma$.



Figure 3.5: Overview of the various embeddings

3.4 Cut-Elimination for the Shallow System

In this section we give a cut-elimination procedure for the shallow system. To do so, we first embed the shallow system with cut into the deep system with cut, eliminate the cut there, and embed the cut-free deep system into the cut-free shallow system. Figure 3.5 gives an overview of the embeddings. We have seen the horizontal arrow on the right in the last section. Now we are going to see the vertical arrows. System H_C is a Hilbert system which we will see in the last section, together with the horizontal arrow on the left.

3.4.1 Embedding Shallow into Deep

This is the easy direction. We first define a notion of admissibility which is weaker than "depth-preserving": it allows the proof to grow by a finite amount.

Definition 3.10 A rule ρ is *finitely admissible* for a system S if for each instance of ρ with premises $\Gamma_1, \Gamma_2 \ldots$ and conclusion Δ there exists a finite ordinal n such that whenever $S \mid_{\gamma}^{\alpha} \Gamma_i$ for all i then $S \mid_{\gamma}^{\alpha+n} \Delta$.

Note that every perfectly admissible (that is, depth- and cut-rank-preserving admissible) rule is also finitely admissible: in that case the n in the above definition is zero. A finitary rule which is contained in a system is also finitely admissible for that system: in that case the n in the above definition is one. The cut rule, on the other hand, is generally not finitely admissible for (cut-free) infinitary systems.

Lemma 3.11 The rule $d \frac{\Gamma\{[\otimes A, \Delta]_i\}}{\Gamma\{\otimes A, [\Delta]_i\}}$ is finitely admissible for system D_{C} .

Proof. By induction on the depth of the proof of the premise. The only inter-

esting case is the one with a \circledast -rule:



where the instance of d shown on the right is removed by induction hypothesis. \Box

Theorem 3.12 (Shallow into deep) If $G_{\mathsf{C}} \frac{|\alpha|}{\gamma} \Gamma$ then $\mathsf{D}_{\mathsf{C}} \frac{|\omega \cdot \alpha|}{\gamma} \Gamma$.

Proof. By induction on α and a case analysis on the last rule in the proof. Each rule of G_{C} except for the \Box_i -rule is a special case of its respective rule in D_{C} . For the \Box_i -rule we have the following transformation:

$$\begin{array}{c} \overbrace{\Gamma, \circledast \Delta, A} \\ \square_i \underbrace{\Gamma, \circledast \Delta, A} \\ \bigcirc i_i \widehat{\diamond_i \Gamma, \circledast \Delta, \square_i A, \Sigma} \end{array} \xrightarrow{\sim} \begin{array}{c} \overbrace{\Gamma, \circledast \Delta, A} \\ wk^*, \diamond_i^* \underbrace{\overline{[\Gamma, \circledast \Delta, A]_i}} \\ \bigcirc i_i, wk \underbrace{\diamond_i \Gamma, [\circledast \Delta, A]_i} \\ \bigcirc i_i \Gamma, \circledast \Delta, \square_i A, \Sigma \end{array}$$

where \mathcal{P}' is obtained by induction hypothesis.

3.4.2 Embedding Deep into Shallow

This is the harder direction, since we need to simulate deep applicability of rules in the shallow system. We use the invertibility of rules in the shallow system in order to do so. The \Box_i -rule is the only rule in G_C which is not invertible. However, a somewhat weaker property than invertibility holds, which is sufficient for our purposes, and which is stated in the upcoming lemma.

Example 3.13 To motivate the following definition consider the following three provable sequents to which the \Box_i -rule cannot be applied (upwards) in an invertible way:

$$\Box_i(a \land b), \diamondsuit_i \bar{a} \lor \diamondsuit_i \bar{b} \qquad \Box_i(a \land b), \circledast \bar{a} \lor \circledast \bar{b} \qquad \Box_i a, \circledast \bar{a}$$

Definition 3.14 (hiding formula, \circledast-saturated sequent) A formula is *essentially* \diamond_i if 1) it is of the form $\diamond_i A$ for any formula A or 2) it is of the form $A \lor B, B \lor A, A \land B$ or $B \land A$ where A is any formula and B is a formula which is essentially

 \diamond_i . A formula is hiding \diamond_i in case 2). We define essentially \circledast and hiding \circledast formulas likewise. A formula is just hiding if it is either hiding \diamond_i for some i or hiding \circledast . A sequent Γ is \circledast -saturated if $\circledast A \in \Gamma$ implies $\diamond_i A \in \Gamma$, for each formula A and each i with $1 \leq i \leq h$.

Definition 3.15 (canonical \Box_i -instance) An instance of the rule

$$\Box_i \frac{\Gamma, \circledast \Delta, A}{\diamondsuit_i \Gamma, \circledast \Delta, \Box_i A, \Sigma}$$

is *canonical* if no formulas of the form $\diamondsuit_i B$ or $\circledast B$ are in Σ .

Lemma 3.16 (Quasi-invertibility of the \Box_i -**rule)** Let Γ be a \circledast -saturated sequent without hiding formulas and let there be a proof of the sequent $\Box_i A, \Gamma$ in G_C . Then there is a proof of the same depth in G_C either 1) of the sequent Γ or 2) of the sequent $\Box_i A, \Gamma$ where the last rule instance is a canonical instance of the \Box_i -rule applying to the shown formula $\Box_i A$.

Proof. By induction on the depth of the given proof and a case analysis on the last rule. If the endsequent is axiomatic then Γ is axiomatic and the first disjunct of our lemma applies. If the last rule is the \mathbb{F} -rule then the proof is of the form

$$\mathbb{B} \underbrace{ \begin{array}{c} & & & \\ & & & \\ \hline & & & \\ & & \\ & & & \\ \hline & & & \\ & & & \\ \hline & & & \\ & & & \\ \end{array}}_{iA,\Gamma_1,\boxtimes B} \stackrel{i}{\longrightarrow} B \stackrel{i}{\longrightarrow} 1 \leq k$$

We apply the induction hypothesis to each premise, with $\Gamma = \Gamma_1, \Box^k B$. Notice that Γ is \circledast -saturated and does not contain hiding formulas. There are two cases. First, if for all premises the first disjunct of the induction hypothesis is true then for each k we have a proof \mathcal{P}'_k such that the following shows the first disjunct of our lemma:

$$\mathbb{E} \frac{ \begin{array}{c} & & & \\ & & & \\ \hline & & & \\ & & \\ \hline & & & \\ \hline & & & \\ & & \\ \hline & & \\ & & \\ \end{array}} \frac{ \begin{array}{c} & & & \\ & & \\ & & \\ & & \\ \end{array}}{\Gamma_1, \mathbb{E} B} \xrightarrow{1 \leq k}$$

Second, if for some premise the second disjunct of the induction hypothesis is true then for some k we have a proof of the form



Notice that the \Box_i -rule can only introduce a formula of the form $\Box^k B$ in Σ , so

we can easily turn this into a proof



and we have shown the second disjunct of our lemma. The cases for \vee and \wedge are similar.

If the last rule is the \circledast -rule then the following transformation yields a shorter proof:

where by assumption of \circledast -saturation all the $\diamond_i B$ are in Γ_1 . To this proof we can now apply the induction hypothesis which yields our lemma.

If the last rule in the given proof is the \Box_j -rule, then we distinguish two cases. First, if $\Box_i A$ is the active formula then the second disjunct of our lemma is either immediate or obtained via weakening admissibility if the rule instance is not canonical.

Second, if $\Box_i A$ is not the active formula then the proof is of the form



where the formula $\Box_i A$ has been introduced inside Σ . We can thus change it into a proof

which shows the first disjunct of our lemma.

In order to translate a derivation with deep rule applications into a derivation where only shallow rules are allowed we need a way of simulating the deep applicability. It turns out that, for certain shallow rules, if they are admissible for the shallow system, then their "deep version" is also admissible.

Definition 3.17 (Make a shallow rule deep) Let $C\{ \}$ be a formula context. Given a rule ρ we define a rule rule $C\{\rho\}$ as follows: an instance of the rule ρ is shown on the left iff an instance of the rule $C\{\rho\}$ is shown on the right:

$$\rho \frac{\Gamma, A_1 \dots \Gamma, A_i \dots}{\Gamma, A} \qquad \qquad C\{\rho\} \frac{\Gamma, C\{A_1\} \dots \Gamma, C\{A_i\} \dots}{\Gamma, C\{A\}}$$

We define a *restricted context* as a formula context in which the hole is in the scope of at most the connectives from $\{\lor, \Box_1, \ldots, \Box_h\}$. Given a rule ρ we define the rule *rule* $\check{\rho}$ as follows: its set of instances is the union of all sets of instances of $C\{\rho\}$ where $C\{\}$ is a restricted context.

Lemma 3.18 (Deep applicability preserves finite admissibility) Let $C\{\}$ be a restricted context.

(i) There is an *n* such that for all Γ we have $\mathsf{G}_{\mathsf{C}} \left| \frac{n}{0} \right|_{\mathcal{T}} \Gamma, C\{p \lor \bar{p}\}$. (ii) If a rule ρ is finitely admissible for G_{C} then $C\{\rho\}$ is also finitely admissible for system G_C .

(iii) If a rule ρ is finitely admissible for G_C then $\check{\rho}$ is also finitely admissible for system G_C .

Proof. Statement (iii) is immediate from (ii). Both (i) and (ii) are proved by induction on $C\{\ \}$. The case with $C\{\ \} = C_1\{\ \} \lor C_2$ is of course analogous to the case with $C\{\ \} = C_1 \lor C_2\{\ \}$ and is omitted. We first prove (i). The case that $C\{\ \}$ is empty is handled by an application of the \lor -rule. If $C\{ \} = C_1 \vee C_2\{ \}$ or $C\{ \} = \Box_i C_1\{ \}$ then we obtain a proof respectively as follows:

$$\begin{array}{c|c} & & & & \\ & & & \\ & & & \\$$

where in both cases \mathcal{P} exists by induction hypothesis. For statement (ii) the case that $C\{\}$ is empty is clear, so we assume that it is non-empty. If $C\{\}$ $C_1 \vee C_2\{\}$ then the following transformation proves our claim:



If $C\{ \} = \Box_i C_1\{ \}$ then we have the following situation:

$$\Box_i C_1\{\rho\} \frac{\vdots \quad \Gamma, \Box_i C_1\{A_k\} \quad \vdots}{\Gamma, \Box_i C_1\{A\}}$$

In order to apply quasi-invertibility of \Box_i , Lemma 3.16, we first need to replace the shown instance of the rule $\Box_i C_1\{\rho\}$ by several instances of it which are applied in a context which is \circledast -saturated and free of hiding formulas. We apply conjunction invertibility, disjunction invertibility and weakening admissibility to each \mathcal{P}_k to obtain a sequence of proofs $\mathcal{P}_{k1} \dots \mathcal{P}_{km}$ such that for each k there is a proof of the form



where each Γ_j is \circledast -saturated and free of hiding formulas.

Fix some j. For all k apply quasi-invertibility of \Box_i , Lemma 3.16, to the proof \mathcal{P}_{kj} . Either this yields some proof \mathcal{P} of Γ_j or for each k it yields a proof \mathcal{P}'_{kj} of some sequent $\Gamma'_j, C_1\{A_k\}$. Then we can build either



where in the second case $C_1\{\rho\}$ is finitely admissible by induction hypothesis. Repeat this argument for each j with $1 \leq j \leq m$, which for each j yields a proof \mathcal{P}''_j in G_{C} . From those we build



which shows our lemma.

Lemma 3.19 (Some glue) The rules in Figure 3.6 are finitely admissible for system $\mathsf{G}_\mathsf{C}.$

Proof. The rules g_c, g_a and g_{ctr} are easily seen to be finitely admissible by using invertibility of the \lor -rule. For the g_{\diamondsuit} -rule we proceed by induction on the given proof of the premise and make a case analysis on the last rule in this proof.

$$\begin{split} \mathbf{g_c} & \frac{\Gamma, A \vee B}{\Gamma, B \vee A} \qquad \mathbf{g_a} \frac{\Gamma, (A \vee B) \vee C}{\Gamma, A \vee (B \vee C)} \\ \mathbf{g_{ctr}} & \frac{\Gamma, A \vee A}{\Gamma, A} \qquad \mathbf{g}_{\diamond} \frac{\Gamma, \Box_i (A \vee B)}{\Gamma, \diamond_i A, \Box_i B} \qquad \mathbf{g}_{\diamond} \frac{\Gamma, \diamondsuit^k A}{\Gamma, \diamond A} \text{ where } k \geq 1 \end{split}$$

Figure 3.6: Some glue

All cases are trivial except when this is the \Box_i -rule. We distinguish two cases: either 1) $\Box_i(A \lor B)$ is the active formula or 2) it is not. In the first case we have:

and in the second case we have the following:

$$\begin{array}{c} & \overbrace{C, \Gamma''}^{\mathcal{P}} & \\ & \underset{g \diamond}{\overset{\Box_i C, \Gamma', \Box_i (A \lor B)}{\Box_i C, \Gamma', \diamond_i A, \Box_i B}} & \\ \end{array} & \sim & \overbrace{C, \Gamma''}^{\mathcal{P}} \\ & \underset{i}{\overset{C}{\Box_i C, \Gamma', \diamond_i A, \Box_i B}} \end{array}$$

For the g_{\circledast} -rule we proceed by induction on k and a subinduction on the depth of the given proof of the premise. For k = 1 the g_{\circledast} -rule coincides with the \circledast -rule plus a weakening, so we assume that we have a proof of $\Gamma, \diamondsuit^{k+1}A$. By invertibility of the \lor -rule we obtain a proof



of the same depth. By induction on the depth of \mathcal{P} and a case analysis on the last rule in \mathcal{P} we now show that we have a proof of the same depth of $\Gamma, \otimes A$. All cases are trivial except when the last rule is \Box_i . Then the following transformation:

$$\begin{array}{c} & & & & & \\ & & & \\ & & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \\$$

proves our claim, where the instance of the g_{\otimes} -rule on the right is finitely admissible by the outer induction hypothesis.

For our translation from deep into shallow we translate nested sequents into formulas and thus fix an arbitrary order and association among elements of a sequent. The arbitrariness of this translation gets in the way, and we work around it as follows: we write

$$\operatorname{ac} \frac{A}{B}$$

if the formula *B* can be derived from the formula *A* in $\{\check{\mathbf{g}}_{\mathsf{c}}, \check{\mathbf{g}}_{\mathsf{a}}\}$. Clearly, in that case *A* and *B* are equal modulo commutativity and associativity of disjunction. The converse is not the case. For example $\circledast(C \lor D)$ can not be derived from $\circledast(D \lor C)$ by ac , in general. Note that since $\check{\mathbf{g}}_{\mathsf{c}}$ and $\check{\mathbf{g}}_{\mathsf{a}}$ are finitely admissible for system G_{C} , so is the rule ac .

$\begin{array}{l} \textbf{Theorem 3.20 (Deep into shallow)} \\ \text{If } \mathsf{D}_{\mathsf{C}} \frac{\mid \alpha}{0} \ \Gamma \ \text{then we have } \ \mathsf{G}_{\mathsf{C}} \mid \frac{\omega \cdot (\alpha+1)}{0} \ \underline{\Gamma}_{\!\mathsf{F}} \ . \end{array}$

Proof. By induction on α . If the endsequent of the given proof is of the form $\Gamma\{p, \bar{p}\}$, then we have

$$\Gamma\{p,\bar{p}\} \sim \sim \sum_{\mathsf{ac}} \frac{\Gamma_{\mathsf{F}}\{p \lor \bar{p}\}}{\Gamma\{p,\bar{p}\}_{\mathsf{F}}}$$

where \mathcal{P} is of finite depth by Lemma 3.18 and **ac** is finitely admissible by Lemma 3.19 and Lemma 3.18. If the last rule is the \lor -rule then an application of **ac** proves our claim. The case of the \Box_i -rule is trivial since the corresponding formula for the premise is the corresponding formula of the conclusion. For the \blacksquare -rule we apply the following transformation, where the \mathcal{P}'_k are obtained by induction hypothesis:

Let the depth of the proof on the left be β with $\beta \leq \alpha$ and the depth of a proof \mathcal{P}_k be β_k . Note that the depth of the **ac**-derivations both below and above the infinitary rule is bounded by a finite ordinal m because the context $\Gamma\{ \}$ is finite. Then, by finite admissibility of the rule $\underline{\Gamma}_{\epsilon}\{ \Xi \}$ (Lemma 3.18) there is a finite ordinal n such that the proof on the right has the depth

$$\sup_{k} (|\mathcal{P}'_{k}| + m + 1) + n + m < \sup_{k} (|\mathcal{P}'_{k}|) + \omega$$

$$\leq \sup_{k} (\omega \cdot (\beta_{k} + 1)) + \omega = \omega \cdot \sup_{k} (\beta_{k} + 1) + \omega$$

$$= \omega \cdot \beta + \omega = \omega \cdot (\beta + 1) \leq \omega \cdot (\alpha + 1) .$$

The case for the \wedge -rule is similar. For the \diamond_i -rule we apply the following transformation, where \mathcal{P}' is obtained by induction hypothesis and the bound on the depth is easy to check:

$$\overset{\mathcal{P}}{\overset{\wedge}{}_{i}} \underbrace{\Gamma\{\Diamond_{i}A, [A, \Delta]_{i}\}}_{\Gamma\{\Diamond_{i}A, [\Delta]_{i}\}} \sim \underset{\underline{\Gamma}_{\mathsf{F}}\{\Diamond_{i}A \lor \varphi_{\mathsf{F}}\}}{\overset{\mathsf{ac}}{\overset{\underline{\Gamma}_{\mathsf{F}}\{\Diamond_{i}A \lor \Box_{i}(A \lor \underline{\Delta}_{\mathsf{F}})\}}{\underbrace{\Gamma_{\mathsf{F}}\{\Diamond_{i}A \lor (\Diamond_{i}A \lor \Box_{i}\Delta_{\mathsf{F}})\}}_{\mathbf{ac}}}_{\underbrace{\underline{\Gamma}_{\mathsf{F}}\{\Diamond_{i}A \lor (\Diamond_{i}A \lor \Box_{i}\Delta_{\mathsf{F}})\}}_{\mathbf{ac}\underbrace{\frac{\Gamma_{\mathsf{F}}\{\Diamond_{i}A \lor (\Diamond_{i}A \lor \Box_{i}\Delta_{\mathsf{F}})\}}{\underbrace{\Gamma_{\mathsf{F}}\{(\Diamond_{i}A \lor (\Box_{i}\Delta) \lor \Box_{i}\Delta_{\mathsf{F}})\}}}}_{\mathbf{ac}\underbrace{\frac{\Gamma_{\mathsf{F}}\{\Diamond_{i}A \lor \Box_{i}\Delta_{\mathsf{F}})\}}{\Gamma\{\Diamond_{i}A, [\Delta]_{i}\}_{\mathsf{F}}}}}$$

Note that here a rule like $C\{\rho \lor A\}$ means the rule ρ applied in the context $C\{\{ \} \lor A\}$, and is finitely admissible for G_{C} if is ρ is finitely admissible for G_{C} , by Lemma 3.18.

The case for the $\$ -rule is similar.

We can now state the cut-elimination theorem for the shallow system.

Theorem 3.21 (Cut-elimination for the shallow system) If $G_{\mathsf{C}} \mid_{\omega \cdot n}^{\alpha} \Gamma$ then $G_{\mathsf{C}} \mid_{0}^{\omega \cdot (\varphi_{1}^{n}(\omega \cdot \alpha)+1)} \Gamma$

3.5 An Upper Bound on the Depth of Proofs

The Hilbert system H_C is obtained from some Hilbert system for classical propositional logic by adding the axioms and rules shown in Figure 3.7. It is essentially the same as system K_h^C from the book [18], where also soundness and completeness are shown. We will now embed H_C into $D_C + cut$, keeping track of the proof depth and thus, via cut-elimination for D_C , establish an upper bound for proofs in D_C . Via the embedding of the deep system into the shallow system, this bound also holds for the shallow system.

Theorem 3.22 (H_C into D_C + cut) If H_C \vdash A then D_C $\mid \frac{\langle \omega^2}{\omega^2} A$.

Proof. The proof is by induction on the depth of the derivation in H_c. If A is a propositional axiom of H_c then there is a finite derivation of A in the propositional part of system D_c such that all premises are instances of the general identity axiom. Thus we obtain D_c $|\frac{\omega \cdot m}{0}|$ A for some $m < \omega$ by admissibility of the general identity axiom (Lemma 3.4).

64

(K)
$$\Box_i A \land \Box_i (A \supset B) \supset \Box_i B$$
 (CCL) $\circledast A \supset (\Box A \land \Box \circledast A)$
(IND) $\frac{B \supset (\Box A \land \Box B)}{B \supset \circledast A}$ (MP) $\frac{A \land A \supset B}{B}$ (NEC) $\frac{A}{\Box_i A}$

Figure 3.7: System H_C

If A is an instance of (K), then we obtain $D_{\mathsf{C}} \left| \frac{\omega \cdot m}{0} \right| A$ for some $m < \omega$ from the following derivation and admissibility of the general identity axiom to take care of the premises.

$$\diamond_{i} \frac{\diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), [B, A, \bar{A}]_{i}}{\diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), [B, A]_{i}} \diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), [B, \bar{B}]_{i}}{\diamond_{i}} \frac{\diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), [B, A \wedge \bar{B}]_{i}}{\diamond_{i}} \frac{\diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), [B, A \wedge \bar{B}]_{i}}{\diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), [B]_{i}}}{\overset{\Box_{i}}{\diamond_{i}\bar{A}, \diamond_{i}(A \wedge \bar{B}), \Box_{i}B}}_{\Box_{i}A \wedge \Box_{i}(A \supset B) \supset \Box_{i}B}}$$

If A is an instance of (CCL), then we obtain $D_{\mathsf{C}} \mid \frac{\omega \cdot m}{0} A$ for some $m < \omega$ from the following derivation and again admissibility of the general identity axiom to take care of the premises. An argument similar to the one used to derive the general identity axiom guarantees that all premises of the \mathbb{F} rule are derivable with depth smaller than $rk(\mathbb{E}A)$.

If the last rule in the derivation is an instance of (MP), then by the induction hypothesis there are $m_1, m_2, n_1, n_2 < \omega$ such that $\mathsf{D}_{\mathsf{C}} \left| \frac{\omega \cdot m_1}{\omega \cdot n_1} \right| A$ and $\mathsf{D}_{\mathsf{C}} \left| \frac{\omega \cdot m_2}{\omega \cdot n_2} \right| A \supset B$. Thus we get $\mathsf{D}_{\mathsf{C}} \left| \frac{\omega \cdot m_1}{\omega \cdot n_1} \right| A, B$ by weakening admissibility and $\mathsf{D}_{\mathsf{C}} \left| \frac{\omega \cdot m_2}{\omega \cdot n_2} \right| \overline{A}, B$ by invertibility. An application of cut yields $\mathsf{D}_{\mathsf{C}} \left| \frac{\omega \cdot m}{\omega \cdot n} \right| B$ for $m = max(m_1, m_2) + 1$ and $n = max(n_1, n_2, rk(A) + 1)$.

If the last rule in the derivation is an instance of (NEC), then the claim follows from the induction hypothesis, the fact that **nec** is cut-rank- and depthpreserving admissible, and an application of \Box_i .

If the last rule in the derivation is an instance of (IND), then by the induction

hypothesis there are $m_1, n_1 < \omega$ such that $\mathsf{D}_{\mathsf{C}} \vdash \frac{\omega \cdot m_1}{\omega \cdot n_1} B \supset (\Box A \land \Box B)$. Then by invertibility of the \wedge - and \vee -rules we obtain

1)
$$\mathsf{D}_{\mathsf{C}} \stackrel{\omega \cdot m_1}{\underset{\omega \cdot n_1}{\vdash}} \bar{B}, \Box B$$
 and 2) $\mathsf{D}_{\mathsf{C}} \stackrel{\omega \cdot m_1}{\underset{\omega \cdot n_1}{\vdash}} \bar{B}, \Box A$.

Let n_2 be such that $rk(\Box B) < \omega \cdot n_2$. We set $n = max(n_1, n_2)$. By induction on k we show that for all $k \ge 1$ there is an $m_2 < \omega$ such that $\mathsf{D}_{\mathsf{C}} \left| \frac{\omega \cdot m_1 + m_2}{\omega \cdot n} \right| \bar{B}, \Box^k A$. The case k = 1 is given by 2) and the induction step is as follows:

$$\operatorname{cut} \frac{\bar{B}, \Box^{k}A}{\bar{B}, \Box B} \xrightarrow{\wedge} \frac{\bar{B}, \Box^{k}A]_{i}}{\bar{\otimes}_{i}, \operatorname{wk}} \frac{\bar{B}, [\overline{D}^{k}A]_{i}}{\bar{\otimes}_{i}\bar{B}, [\overline{D}^{k}A]_{i}}}{\bar{\otimes}_{i}\bar{B}, \Box_{i}\Box^{k}A} \xrightarrow{:} 1 \leq i \leq h}$$

where the premise on the left is 1) and the premise on the right follows by induction hypothesis. The claim follows by applications of \mathbb{B} and \vee .

The embedding of the Hilbert system into the nested sequent system together with the cut-elimination theorem for the deep system gives us the following upper bounds on the depth of proofs in the cut-free systems.

Theorem 3.23 (Upper bounds) If A is a valid formula then (i) $D_{\mathsf{C}} \mid \frac{\langle \varphi_2 0}{0} A$, and (ii) $\mathsf{G}_{\mathsf{C}} \mid \frac{\langle \varphi_2 0}{0} A$.

Proof. If A is valid then by completeness of H_{C} we have $H_{\mathsf{C}} \vdash A$ and by the embedding of the Hilbert system into the nested sequent system there are nat-ural numbers m, n such that $D_{\mathsf{C}} \mid \frac{\omega \cdot m}{\omega \cdot n} A$. By the cut elimination theorem for the nested sequent system we obtain $D_{\mathsf{C}} \left[\frac{\varphi_1^n(\omega \cdot m)}{0} \right] A$. We know $\varphi_{\beta_1} \gamma_1 < \varphi_{\beta_2} \gamma_2$ if $\beta_1 < \beta_2$ and $\gamma_1 < \varphi_{\beta_2}\gamma_2$. Thus $\mathsf{D}_{\mathsf{C}} \mid \frac{<\varphi_2 0}{0} A$. For (ii) by the embedding of the deep system into the shallow system it suffices to check that for $\alpha < \varphi_2 0$ we have $\omega \cdot (\alpha + 1) < \varphi_2 0$. П

3.6 Discussion

We have introduced a nested sequent system for common knowledge which, in contrast to the ordinary sequent system by Alberucci and Jäger, admits a syntactic cut-elimination procedure. We have shown this cut-elimination procedure, and, via embedding the two systems into each other, have also provided a cut-elimination procedure for the shallow system. We embedded a Hilbert style system and obtained $\varphi_2 0$ as upper bound on the depth of cut-free proofs for both sequent systems.

Notice in particular how we used the nested sequent system as a tool in order to prove a result about an existing, ordinary sequent system. Designing some
3.6. DISCUSSION

kind of infinitary proof system with a syntactic cut-elimination for the logic of common knowledge was the less interesting part: we could have used both the display calculus and labelled sequents to do so. However, it is hard to imagine how a cut-free display calculus or cut-free labelled sequent calculus could have been translated back into our cut-free ordinary sequent calculus. The fact that nested sequents stay inside the modal language allowed us to do so.

Other modal logics. We have looked at common knowledge based on the least normal modal logic. In a sense, *common belief* would be a better name. Given the previous chapter, it seems that any other modal logic of the cube could be used instead and thus our approach is independent of the particular underlying axiomatisation of knowledge. The modal logic S5 is often proposed as an adequate logic for knowledge. As we have seen in the previous chapter, contrary to shallow sequents, nested sequents can easily handle S5. So it is easy to design a system for S5-based common knowledge. We just need to add a single rule to system D_C :

$$\mathsf{S5} \frac{\Gamma\{\diamondsuit A\}\{A\}}{\Gamma\{\diamondsuit A\}\{\emptyset\}}$$

However, hypersequents are already sufficient to capture S5 and a system for S5-based common knowledge based on the hypersequent system LS5 by Mints [34] seems to admit a cut-elimination procedure similar to the one given here for system D_c . So for S5-based common knowledge there is no need for nested sequents.

Future work. Of course there are also more speculative questions. What is the mathematical meaning of the upper bound on the depth of cut-free proofs? Is there a kind of boundedness lemma in modal logic similar to the one used in the analysis of set theories and second order arithmetic? What would be the equivalent of a well-ordering proof in modal logic? Is $\varphi_2 0$ the best possible upper bound on the depth of proofs? Can our analysis be extended to more powerful modal fixpoint logics? Work under preparation suggests that a cut-free infinitary nested sequent system for the μ -calculus can be bounded by $\varphi_{\omega} 0$.

A more interesting, and harder problem is the design of cut-free finitary sequent systems for modal fixpoint logics. While such systems exist, for example for temporal logics [20, 12], their rules are context-dependent in a way which makes it hard to study them proof-theoretically, in particular it seems hard to design a syntactic cut-elimination procedure.

68

Bibliography

- Pietro Abate, Rajeev Goré, and Florian Widmann. Cut-free single-pass tableaux for the logic of common knowledge. In Workshop on Agents and Deduction at TABLEAUX 2007, 2007.
- [2] Luca Alberucci and Gerhard Jäger. About cut elimination for logics of common knowledge. Annals of Pure and Applied Logic, 133:73–99, 2005.
- [3] Evangelia Antonakos. Justified and common knowledge: Limited conservativity. In Sergei Artemov and Anil Nerode, editors, *LFCS*, volume 4514 of *Lecture Notes in Computer Science*, pages 1–11. Springer, 2007.
- [4] Sergei Artemov. Logic of proofs. Annals of Pure and Applied Logic, 67:29– 59, 1994.
- [5] Sergei Artemov. Justified common knowledge. Theoretical Computer Science, 357(1):4–22, 2006.
- [6] Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In Wilfrid Hodges, Martin Hyland, Charles Steinhorn, and John Truss, editors, *Logic: from foundations to applications. Proc. Logic Colloquium, Keele, UK, 1993*, pages 1–32. Oxford University Press, New York, 1996.
- [7] Nuel D. Belnap, Jr. Display logic. Journal of Philosophical Logic, 11:375– 417, 1982.
- [8] Kai Brünnler and Thomas Studer. Syntactic cut-elimination for common knowledge. *Electronic Notes in Theoretical Computer Science*, 231:227 – 240, 2009. Proceedings of the 5th Workshop on Methods for Modalities (M4M5 2007).
- [9] Kai Brünnler. Deep Inference and Symmetry in Classical Proofs. PhD thesis, Technische Universität Dresden, September 2003.
- [10] Kai Brünnler. Deep sequent systems for modal logic. In Guido Governatori, Ian Hodkinson, and Yde Venema, editors, Advances in Modal Logic, volume 6, pages 107–119. College Publications, 2006.
- [11] Kai Brünnler. Deep sequent systems for modal logic. To appear in Archive for Mathematical Logic. Available from http://www.iam.unibe.ch/~kai/Papers/dsm.pdf, 2008.

- [12] Kai Brünnler and Martin Lange. Cut-free sequent systems for temporal logic. J. Log. Algebr. Program., 76(2):216–225, 2008.
- [13] Kai Brünnler and Thomas Studer. Syntactic cut-elimination for common knowledge. Annals of Pure and Applied Logic, 160(1):82 – 95, 2009.
- [14] Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.
- [15] Robert A. Bull. Cut elimination for propositional dynamic logic without *. Mathematische Logik und Grundlagen der Mathematik, 38:85–100, 1992.
- [16] Marcos A. Castilho, Luis Fariñas del Cerro, Olivier Gasquet, and Andreas Herzig. Modal tableaux with propagation rules and structural rules. *Fun*dam. Inf., 32(3-4):281–297, 1997.
- [17] Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. From axioms to analytic rules in nonclassical logics. In *Proceedings of LICS'08*, pages 229–240, 2008.
- [18] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about Knowledge*. The MIT Press, Boston, 1995.
- [19] Melvin Fitting. Proof methods for modal and intuitionistic logics. Synthese library. D. Reidel, Dordrecht, Holland, 1983.
- [20] Joxe Gaintzarain, Montserrat Hermo, Paqui Lucio, Marisa Navarro, and Fernando Orejas. A cut-free and invariant-free sequent calculus for PLTL. In Jacques Duparc and Thomas A. Henzinger, editors, *CSL*, volume 4646 of *Lecture Notes in Computer Science*, pages 481–495. Springer, 2007.
- [21] Jim Garson. Modal logic. In Edward N. Zalta, editor, The Stanford Encyclopedia of Philosophy. Stanford University, Spring 2008. http://plato.stanford.edu/archives/spr2008/entries/logic-modal/.
- [22] Rajeev Goré. Tableau methods for modal and temporal logics. In M. D'Agostino, D. Gabbay, R. Haehnle, and Posegga J., editors, *Handbook* of *Tableau Methods*, pages 297–396. Kluwer Academic Publishers, 1999.
- [23] Rajeev Goré, Linda Postniece, and Alwen Tiu. Cut-elimination and proofsearch for bi-intuitionistic logic using nested sequents. In Advances in Modal Logic, pages 43–66. College Publications, 2008.
- [24] Rajeev Goré, Linda Postniece, and Alwen Tiu. Taming displayed tense logics using nested sequents with deep inference. Accepted at Tableaux 2009, 2009.
- [25] Rajeev Goré and Alwen Tiu. Classical modal display logic in the calculus of structures and minimal cut-free deep inference calculi for S5. Journal of Logic and Computation, 17(4):767–794, 2007.
- [26] Alessio Guglielmi. A system of interaction and structure. ACM Transactions on Computational Logic, 8(1):1–64, 2007.

- [27] Joseph Y. Halpern and Yoram Moses. Knowledge and common knowledge in a distributed environment. *Journal of the ACM*, 37(3):549–587, 1990.
- [28] Robert Hein and Charles Stewart. Purity through unravelling. In Paola Bruscoli, François Lamarche, and Charles Stewart, editors, *Structures and Deduction*, pages 126–143. Technische Universität Dresden, 2005.
- [29] Gerhard Jäger, Mathis Kretz, and Thomas Studer. Cut-free common knowledge. J. Applied Logic, 5(4):681–689, 2007.
- [30] Ryo Kashima. Cut-free sequent calculi for some tense logics. Studia Logica, 53:119–135, 1994.
- [31] Mathis Kretz and Thomas Studer. Deduction chains for common knowledge. Journal of Applied Logic, 4:331–357, 2006.
- [32] Simone Martini and Andrea Masini. A computational interpretation of modal proofs. In H. Wansing, editor, *Proof theory of modal logic*, volume 2 of *Applied logic series*, pages 213–241. Kluwer, 1996.
- [33] John-Jules Meyer and Wiebe van der Hoek. *Epistemic Logic for AI and Computer Science*. Cambridge University Press, 1995.
- [34] Grigorii E. Mints. Lewis' systems and system T. In Selected papers in proof-theory (1965–1973). Bibliopolis, 1992.
- [35] Michael Moortgat. Categorial type logics. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 93–177. Elsevier, 1997.
- [36] Sara Negri. Proof analysis in modal logic. Journal of Philosophical Logic, 34(5-6):507-544, 2005.
- [37] Regimantas Pliuskevicius. Investigation of finitary calculus for a discrete linear time logic by means of infinitary calculus. In *Baltic Computer Sci*ence, Selected Papers, pages 504–528, London, UK, 1991. Springer-Verlag.
- [38] Francesca Poggiolesi. The tree-hypersequent method for modal propositional logic. In Jacek Malinowski David Makinson and Heinrich Wansing, editors, *Towards Mathematical Philosophy*, Trends in Logic, pages 9–30. Springer, 2009.
- [39] Wolfram Pohlers. Proof Theory An Introduction. Springer, 1989.
- [40] Wolfram Pohlers. Subsystems of set theory and second order number theory. In Sam Buss, editor, *Handbook of Proof Theory*, pages 209–335. Elsevier, 1998.
- [41] David J. Pym. The Semantics and Proof Theory of the Logic of Bunched Implications, volume 26 of Applied Logic Series. Kluwer Academic Publishers, 2002.
- [42] Mehrnoosh Sadrzadeh and Roy Dyckhoff. Positive logic with adjoint modalities: Proof theory, semantics and reasoning about information. ENTCS, MFPS, University of Oxford, April 2009., 2009.

- [43] Masahiko Sato. A study of Kripke-type models for some modal logics by Gentzen's sequential method. Publications of the Research Institute for Mathematical Sciences, Kyoto University, 13:381–468, 1977.
- [44] Kurt Schütte. Schlussweisen-Kalküle der Prädikatenlogik. Mathematische Annalen, 122:47–65, 1950.
- [45] Kurt Schütte. Proof Theory. Springer-Verlag, 1977.
- [46] Charles Stewart and Phiniki Stouppa. A systematic proof theory for several modal logics. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds, and Heinrich Wansing, editors, Advances in Modal Logic, volume 5 of King's College Publications, pages 309–333, 2005.
- [47] Phiniki Stouppa. A deep inference system for the modal logic S5. Studia Logica, 85(2):199–214, 2007.
- [48] Lutz Straßburger. From deep inference to proof nets via cut elimination. Journal of Logic and Computation, 2009. In press. http://www.lix.polytechnique.fr/~lutz/papers/deepnet.pdf.
- [49] Yoshihito Tanaka. Some proof systems for predicate common knowledge logic. Reports on mathematical logic, 37:79–100, 2003.
- [50] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.
- [51] Luca Viganò. Labelled Non-Classical Logics. Kluwer Academic Publishers, Dordrecht, 2000.
- [52] Heinrich Wansing. Displaying Modal Logic, volume 3 of Trends in Logic Series. Kluwer Academic Publishers, Dordrecht, 1998.
- [53] Heinrich Wansing. Sequent systems for modal logics. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, 2nd edition, volume 8, pages 61–145. Kluwer, Dordrecht, 2002.