Comparing Modal Sequent Systems

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ABSTRACT. This is an exploratory and expository paper, comparing display logic formulations of normal modal logics with labelled sequent systems. We provide a translation from display sequents into labelled sequents. The comparison between different systems gives us a different way to understand the difference between display systems and other sequent calculi as a difference between *local* and *global* views of consequence. The mapping between display and labelled systems also gives us a way to understand labelled systems as properly *structural* and not just as systems encoding modal logic into first-order logic.[1]

Keywords: Sequents, Display Logic, Labelled Proof System

Labelled systems and display systems are very different generalisations of the pure sequent calculus, giving what appear to be quite different accounts of modal deduction. Display sequents are equipped with a rich "structural" vocabulary, allowing us to directly express modal facts in the punctuation of a sequent. Labelled sequents encode into the proof theory the structure of a Kripke model. Formulas are equipped with labels (effectively replacing formulas by predicates of worlds) and the accessibility relation from the model makes its appearance in the syntax of the sequent. In this paper, I show how derivations in display logic may be converted into derivations in a labelled sequent system, lending some support to the claim that a labelled sequent system need be no *more* expressive than a display system. Using this result, we may we may simplify a labelled proof theory further, so that the labels disappear and we are left with a different, *structural* sequent system for modal logics.

1 Display Logic

In Belnap's *Display Logic* [1, 2], as in other sequent systems, we consider structured collections of formulas, *sequents*. Here, sequents are of the form $X \vdash Y$, where X and Y are *structures*, made up from formulas. Structures are made up of structure-connectives, constructing structures from smaller structures, in much the same way as formulas are constructed formula-connectives. Structures and their connectives have a polarity. They can be either *positive* or *negative* structures. In Belnap's original formulation

[1]This research is supported by the Australian Research Council, through grant DP0343388. See `http://consequently.org/writing/comparingmodal` for the latest version of the paper, to post comments and to read comments left by others.

of display logic, each structure-connective could appear both in negative and positive position — but connectives could be interpreted differently according to the positions in which they appeared. For example, Belnap's calculus features a binary structure-connective $\circ$, which is interpreted as conjunction-like in an negative position, and disjunction-like in a positive position. The unary structure-connective $*$ is negation-like in both positions. However, if $*X$ is in negative position, then the X underneath is positive position, and if $*X$ is in positive position, the X underneath is in negative position. If X is a negative structure, and Y is a positive structure, then $X \vdash Y$ is a sequent. The negative parts of the sequent are X and the proper negative parts of X and of Y. The positive parts of the sequent are Y and the proper positive parts of X and of Y.

The crucial feature of display logic is the *display property.* If $X \vdash Y$ is a sequent involving Z as a positive substructure, then $X \vdash Y$ can be transformed into $W \vdash Z$ using *display rules* alone, and W depends only on the position of Z in the original sequent. Similarly if Z is an negative substructure, then the sequent can be transformed into $Z \vdash V$ for some appropriate V. Here, the *display rules* are a particular class of rules involving only the structure-connectives alone. In Belnap's original formulation, the display rules were as follows:

$$\begin{gathered} X \circ Y \vdash Z \Longleftrightarrow X \vdash *Y \circ Z \\ X \vdash Y \circ Z \Longleftrightarrow X \circ *Y \vdash Z \Longleftrightarrow X \vdash Z \circ Y \\ X \vdash Y \Longleftrightarrow *Y \vdash *X \Longleftrightarrow **X \vdash Y \end{gathered}$$

In addition to rules that treat structure, there are rules that introduce connectives. Because of the display property, there is no loss of generality in assuming that the connective to be introduced can be either the entire antecedent or the entire consequent of the sequent. For example, these are Belnap's original rules for conjunction.

$$\frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \wedge B} \qquad \frac{A \circ B \vdash X}{A \wedge B \vdash X}$$

In these rules, we can see the way that a $\circ$ in negative position acts like conjunction. (Similarly, in positive position, it acts like disjunction.)

In this paper, I will use slightly different rules for conjunction and disjunction, to facilitate comparison with Negri's labelled sequent system [8], the topic of our next section.

$$\frac{A \circ B \vdash Y}{A \wedge B \vdash Y}\,L\wedge \qquad \frac{A \vdash Y \quad B \vdash Y}{A \vee B \vdash Y}\,L\vee \qquad \frac{*A \vdash Y}{\neg A \vdash Y}\,L\neg$$

$$\frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}\,R\wedge \qquad \frac{X \vdash A \circ B}{X \vdash A \vee B}\,R\vee \qquad \frac{X \vdash *A}{X \vdash \neg A}\,R\neg$$

These modified rules are interderivable with Belnap's rules, given the presence of the structural rules of weakening and contraction for $\circ$, which we shall assume. In each case, the display property is satisfied and the introduced connective is the entire antecedent or the entire consequent of the sequent structure.

To model modal logics, we will need to consider another example of structure-connectives, and connective rules. Wansing [13, 14] extended Belnap's original work by adding a unary structure $\bullet$ (in both antecedent and consequent position) with display rules

$$\bullet X \vdash Y \Longleftrightarrow X \vdash \bullet Y$$

with corresponding modal rules

$$\frac{A \vdash Y}{\Box A \vdash \bullet Y}\,L\Box \qquad \frac{X \vdash \bullet A}{X \vdash \Box A}\,R\Box$$

These rules make clear that a $\bullet$ in positive position does the work of $\Box$. (A $\bullet$ in negative position does the work of a dual $\blacklozenge$ operator, looking *backwards* down the accessbility relation used by $\Box$.)

With these rules, we may derive valid sequents in the basic normal modal logic K. Here is an example:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \bullet A}\,L\Box}{\Box A \circ \Box B \vdash \bullet A}\,\mathit{weaken}}{\bullet(\Box A \circ \Box B) \vdash A}\,\mathit{display} \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{\Box B \vdash \bullet B}\,L\Box}{\Box A \circ \Box B \vdash \bullet B}\,\mathit{weaken}}{\bullet(\Box A \circ \Box B) \vdash B}\,\mathit{display}}{\dfrac{\dfrac{\dfrac{\bullet(\Box A \circ \Box B) \vdash A \wedge B}{\Box A \circ \Box B \vdash \bullet(A \wedge B)}\,\mathit{display}}{\Box A \circ \Box B \vdash \Box(A \wedge B)}\,R\Box}{\Box A \wedge \Box B \vdash \Box(A \wedge B)}\,L\wedge}\,R\wedge$$

The display rules are used to choreograph the deduction — they place the formula required as the main connective as the entire left or entire right side of the consecution.

Belnap gave an account of eight conditions sufficient to guarantee the admissibility of the rule *Cut* (from $X \vdash A$ and $A \vdash Y$ to infer $X \vdash Y$). That is, if there are deriviations of $X \vdash A$ and $A \vdash Y$, then there is also a derivation of $X \vdash Y$.

It is a straightforward result that every derivable sequent is valid on a model — once you define what it *is* for a sequent to be valid on a model. It generalises the notion for simple sequents $A \vdash B$, where we require that any world where A is true is a world where B is true. Given a frame $\langle W, R\rangle$ — consisting of a set W of worlds and a binary accessibility relation R on W — and a relation $\Vdash$ of truth at worlds, we may define for each structure X the conditions $P_w(X)$ (the structure X in *positive* position is true at world

w) and $N_w(X)$ (the structure X in *negative* position is true at world w). It is defined inductively:

	A	$*X$	$X \circ Y$	$\bullet X$
N_w	$w \Vdash A$	$\neg P_w(X)$	$N_w(X) \wedge N_w(Y)$	$(\exists v)(vRw \wedge N_v(X))$
P_w	$w \Vdash A$	$\neg N_w(X)$	$P_w(X) \vee P_w(Y)$	$(\forall v)(wRv \supset P_v(X))$

Then a sequent $X \vdash Y$ is said to be valid on a model if and only if, according to that model $(\forall w)(N_w(X) \supset P_w(Y))$. It is a straightforward induction on the construction of a derivation that all derivable sequents are valid. For example, the sequent $X \vdash \bullet Y$ is valid on a model if and only if on that model we have

$$(\forall w)(N_w(X) \supset (\forall v)(wRv \supset P_v(Y)))$$

a straightforward quantifier shift converts this to

$$(\forall v)((\exists w)(wRv \wedge N_w(X)) \supset P_v(Y))$$

which is $(\forall v)(N_v(\bullet X) \supset P_v(Y))$, the condition arising from the display equivalent sequent $\bullet X \vdash Y$.

One may model the behaviour of many different modal logics by imposing new 'structural rules' governing the structural punctuation of $\bullet$, $\circ$ and $*$. For example the interchange of $\bullet *$ and $* \bullet$ underwrites the inference from A to $\Box\Diamond A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{*A \vdash *A}\,\textit{display}}{\neg A \vdash *A}\,L\neg}{\Box\neg A \vdash \bullet * A}\,L\Box}{\Box\neg A \vdash * \bullet A}\,\textit{sym}}{\bullet A \vdash * \Box\neg A}\,\textit{display}}{\bullet A \vdash \neg\Box\neg A}\,R\neg}{A \vdash \bullet\neg\Box\neg A}\,\textit{display}}{A \vdash \Box\neg\Box\neg A}\,R\Box$$

Very many more modal logics may be found by imposing purely structural rules governing $\bullet$, $\circ$ and $*$, preserving the admissibility of cut and the elegant properties of the display calculus.

Despite these pleasing features, display logic has not been widely used.[2] Part of this may be explained in terms of the unique features of display calculi: systems for modal logics are not merely *expansions* of classical Gentzen-style sequent systems, as proofs in the boolean fragment use the exotic machinery of $*$, and $\circ$ instead of the familiar sequent structure $X \vdash Y$

[2] This is not to disparage the work done in the area [4, 6, 10, 11, 14]. However, there is no doubt that the work in this area has been driven by a small number of researchers.

where X and Y are multisets (or lists) of formulas. This new structure does not *simplify* derivations: it *complicates* them with what seem to be *inessential* and *bureaucratic* choreography which does nothing to expose the essential deductive steps in a derivation.[3] The *essential* work of the display property seems to be to ensure that every position in a sequent is uniform, in that it is available for a *cut* or for a connective rule. A formula in a consecution may be displayed, and a displayed position is the site for a cut or for a connective step. Actually going to the trouble of *displaying* a formula in order to process it seems to indicate that we do not have the most perspicuous mode of formulating our proof theory.

We may highlight another, related, feature of display logic which is the cause of some dissatisfaction. The cut-elimination result for traditional sequent systems provides a number of important corollaries, such as the subformula property, and if we are lucky, decidability and interpolation. In this case, the decidability result does not give us the same fine degree of control as elsewhere, for even though we have the *subformula* property, we do not have a *substructure* property, and we have quite a rich structural vocabulary, instead of the slim vocabulary of the comma on the left and right in a traditional sequent system. To use a cut elimination argument to prove decidability of a display calculus is a difficult task [11]. No known *interpolation* result has been proved by means of a display calculus.

2 Labelled Sequents

Labelled sequents are a different solution to the issue of giving a proof theory for modal logics [8]. The core idea of a labelled sequent system is to internalise into the proof system the relational structure of the Kripke model. This allows us to construct derivations like this:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v : A \vdash v : A}{wRv, w : \Box A \vdash v : A}\,L\Box}{wRv, w : \Box A, w : \Box B \vdash w : A}\,\mathit{weaken} \qquad \dfrac{\dfrac{v : B \vdash v : B}{wRv, w : \Box B, \vdash v : B}\,L\Box}{wRv, w : \Box A, w : \Box B \vdash v : B}\,\mathit{weaken}}{wRv, w : \Box A, w : \Box B \vdash v : A \wedge B}\,R\wedge}{w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)}\,R\Box}{w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)}\,L\wedge
$$

This derivation is, at the one time, simpler than the corresponding display derivation of $\Box A \wedge \Box B \vdash \Box(A \wedge B)$, (it has fewer inference steps, leaving out each display step) and more complex (it introduces labels and the explicit relational symbol). It appears that have moved from a proof theory of modal formulas to a proof theory as a tool for reasoning about for modal models.

To make the point we do not need to look at all of the details of labelled proof theories. Detail may be found elsewhere [8, 12]. It suffices to

[3] A helpful characterisation of the costs and benefits of a display formulation of modal logics may be found in Phiniki Stouppa's Master's thesis [**?**].

understand a labelled system as one in which the sequents take the form $X \vdash Y$ where X and Y are multisets of labelled formulas (of the form $x : A$ where x is a label and A is a formula) together with relational statements (of the form xRy). Axiomatic sequents take the form $x : A \vdash x : A$.[4] For connective rules we have

$$\frac{x : A, x : B, X \vdash Y}{x : A \wedge B, X \vdash Y}\, L\wedge \qquad \frac{x : A, X \vdash Y \quad x : B, X \vdash Y}{x : A \vee B, X \vdash Y}\, L\vee \qquad \frac{X \vdash Y, x : A}{x : \neg A, X \vdash Y}\, L\neg$$

$$\frac{X \vdash Y, x : A \quad X \vdash Y, x : B}{X \vdash Y, x : A \wedge B}\, R\wedge \qquad \frac{X \vdash Y, x : A, x : B}{X \vdash Y, x, A \vee B}\, R\vee \qquad \frac{x : A, X \vdash Y}{X \vdash Y, x : \neg A}\, R\neg$$

$$\frac{x : A, X \vdash Y}{yRx, y : \Box A, X \vdash Y}\, L\Box \qquad \frac{xRy, X \vdash Y, y : A}{X \vdash Y, x : \Box A}\, R\Box$$

where the last rule has the side condition that y does not appear in $X \vdash Y$.

Notice that in these rules, relational statements appear only on the left of the sequent. We may without loss of deductive power, restrict our attention to sequents in $X \vdash Y$ which relational statements appear only in X and not in Y.

Just as with display logic, we may extend the system with rules governing the distinctive modal machinery (here R) to encode different modal systems. Negri [8] shows how different conditions on R may be added as rules without breaking the admissibility of cut (or indeed the admissibility of contraction and weakening in her G3-style system). We will not go through the detail of these conditions here.

Just as with display logic, we can understand what it is for a sequent to be valid on a model model. In this case, the translation is much simpler. To translate the sequent $X \vdash Y$ we replace each $x : A$ by $x \Vdash A$, we replace the multiset X by its conjunction; Y by its disjunction; the $\vdash$ by a conditional, and you universally quantify over all world labels. So, the sequent $xRy, x : A \vdash y : B, x : C$ is valid on a frame if and only if

$$(\forall x, y)((xRy \wedge x \Vdash A) \supset ((y \Vdash B) \vee (x \Vdash C)))$$

Notice that the rules here do not satisfy the *subformula* property if we take relational facts to be formulas, as xRy appears in the premise of $R\Box$ but not in the conclusion. We could repair this in two ways. One is to take relational facts to not be *formulas* properly so-called, or to take R the predicate to be present as a 'part' of the operator $\Box$, as it would be if we were to rewrite the modal rules as explicit special cases of the quantifier rules in first-order logic:

$$\frac{x : A, X \vdash Y}{yRx, (\forall z)(yRz \supset z : A), X \vdash Y}\, L\Box' \qquad \frac{xRy, X \vdash Y, y : A}{X \vdash Y, (\forall z)(xRz \supset z : A)}\, R\Box'$$

[4] One could take them instead to have the form $x : A, X \vdash Y, x : A$ if we wish to eliminate weakening as an explicit structural rule.

where we translate $x : \Box A$ by $(\forall z)(xRz \supset z : A)$. Then, clearly, a kind of subformula property is satisfied.

However, satisfying the subformula property in this way seems unsatisfactory. For one thing, we *still* have no genuine subformula property since we still have to deal with renaming variables. The formulas xRy and $y : A$ are not subformulas of the formula $(\forall x)(xRz \supset z : A)$ in any straightforward sense. Reworking our proof-theoretical analysis to deal with variables and quantification seems like a high price to play to deal with *modal* inference which does not explicitly mention such things. The alternative, of course, is to attempt to view the sequent system in such a way that the variables and relational statements occur as the *structure* of the sequent, and not as its *content*. After all, we have seen that we are not making use of the full power of first-order logic. Relational statements need only appear on the left side of the sequent. They are not compounded with other statements. They are used only to control the application of $L\Box$ and $R\Box$.

Another shortcoming of the labelled system is that it is still not a straightforward extension of classical propositional logic. A classical propositional sequent with no modal operators is not derived with the traditional sequent derivation, but with a sequent derivation littered with redundant world labels. While the mismatch with traditional sequent systems is not as great, it is still there.

3 From Display Sequents to Labelled Sequents

Before attempting to resolve issues with either display logic or labelled sequent systems, we will attempt to understand the relationship between them. Consider the way that sequents in either system relate to models. A labelled sequent $Rxy, x : A \vdash y : B$ is valid on a model just when $(\forall x, y)((Rxy \wedge x \Vdash A) \supset y \Vdash B)$. This condition on a model is expressed by (at least) two different display sequents: $\bullet A \vdash B$ and $A \vdash \bullet B$. This suggests that we may find a way to translate display sequents into labelled sequents, and that this translation will be many-to-one. If we can do this, then it may point to a way to do away with the redundancy in a display system.

One way to go from a display sequent $X \vdash Y$ to a labelled sequent is to use the translation $N_w(X) \vdash P_w(Y)$. Unfortunately, N and P produce statements in the first-order theory of models, and not sequences of labelled formulas and relational facts. We will need to modify our tranlsation in order to produce a labelled sequent. This takes some care, the antecedent structure X might have substructures in *postive* position (those under an odd number of asterisks), and the result of translating these will quite possibly need to go into the *right* of the labelled sequent. So, the translation of $X \vdash Y$ is the sequent

$$n^l_w(X), p^l_w(Y) \vdash n^r_w(X), p^r_w(Y)$$

where n^l_w, p^l_w, n^r_w, p^r_w are ancillary translation functions, defined recursively. The functions n^l_w and n^r_w are used to give us the significance of a display structure appearing in *negative* position in the display sequent. $n^l_w(X)$ is the contribution X makes on the *left* side of the labelled sequent, considered as true at world w. $n^r_w(X)$ is the contribution that this structure makes to the *right* side of the labelled sequent, again taking the world of evaluation to be w. Similarly, $p^l_w(Y)$ and $p^r_w(Y)$ are the contributions of the structure Y, appearing in positive position in the display sequent, in the *left* and *right* of the labelled sequent, respectively. In each case, the output of each function is a multiset (possibly empty) of formulas and relation statements. Now we may present the definitions of the four functions:

	A	$*X$	$X \circ Y$	$\bullet X$
n^l_w	$w : A$	$p^l_w(X)$	$n^l_w(X), n^l_w(Y)$	$vRw, n^l_v(X)$
n^r_w	$-$	$p^r_w(X)$	$n^r_w(X), n^r_w(Y)$	$n^r_v(X)$
p^l_w	$-$	$n^l_w(X)$	$p^l_w(X), p^l_w(Y)$	$wRv, p^l_v(X)$
p^r_w	$w : A$	$n^r_w(X)$	$p^r_w(X), p^r_w(Y)$	$p^r_v(X)$

In the case for $\bullet X$, the world variable v is *fresh*.[5]

The display sequent $\bullet * (A \circ *\bullet B) \vdash *(D \circ E)$ is translated as follows:

$$n^l_w(\bullet * (A \circ *\bullet B)), p^l_w(*(D \circ E)) \vdash n^r_w(\bullet * (A \circ *\bullet B)), p^r_w(*(D \circ E))$$

after you trace through the inductive definitions, you get to:

$$vRw, uRv, u : B, w : D, w : E \vdash v : A$$

Notice that the structure $\bullet * (A \circ *\bullet B)$ deposits material on the right side of the turnstile ($v : A$) and the left side ($w : B$), and the w and v are related by a relational fact vRw.

Now we come to the first *fact*.

FACT 1. A display sequent is valid on a model if and only if its translation as a labelled sequent is also valid on that frame.

Proof. A straightforward induction on the construction of the translation. The simplest technique is to consider when a sequent fails in a model. For the display sequent $X \vdash Y$ itself, we need some w where $N_w(X) \wedge \neg P_w(Y)$. For the translation as a labelled sequent, we need a w where $n^l_w(X) \wedge \neg n^r_w(X) \wedge p^l_w(Y) \wedge \neg p^r_w(Y)$ is true. To prove the equivalence, it suffices to prove by induction that $\neg P_w(Y)$ holds if and only if $p^l_w(Y) \wedge \neg p^r_w(Y)$ and $N_w(X)$ if and only if $n^l_w(X) \wedge \neg n^r_w(X)$. (The free variables in the evaluation of n and p are to be treated as implicitly existentially quantified.) Proving the equivalence is a simple induction on the construction of X and of Y. ■

[5]Stating this condition precisely is not straightforward. We need, in fact, to keep a stack of world labels available for substitution throughout the translation, and pass the stack from one stage of the translation to another. I presume that the reader enough familiarity with translations into first-order logic to sidestep these fiddly details for the sake of ease of exposition.

Notice that the two display sequents $\bullet A \vdash B$ and $A \vdash \bullet B$ are translated as *very* similar labelled sequents:

$$vRw, v : A \vdash w : B \qquad wRv, w : A \vdash v : B$$

For the first translation, we evaluate B using our first world label w, and then step backwards to v to find the point of evaluation for A. For $A \vdash \bullet B$, on the other hand, we evaluate A at w, and move forward to a new world v to find B. The two display equivalent sequents are translated by labelled sequents differing only in the *identity* of the labels and not the structure of the sequent. If we think of the labelled sequent as 'starting' at w, then $\bullet A \vdash B$ 'says' that if we have world that can access this world, where A is true then B is true here. $A \vdash \bullet B$ says, on the other hand, that if A is true here then for any world accesible from here, B is true.

Both "facts" are unproblematic ways of stating the same thing. Considered as validities on a model, there is nothing to stand between them, as we did not pick out any particular point of evaluation. As a matter of fact, no world mentioned in a labelled sequent is "here." Labelled sequents take a global view of a model, not picking out any particular point as a starting point. Display sequents are no less general, but they express the validity of a deduction in a local manner by distinguishing (as sequents) the fact $\bullet A \vdash B$ (thinking of A worlds as ancestors of *this* B world) and $A \vdash \bullet B$ (thinking of B worlds as descendents of *this* A world). The position of the turnstile marks the location of the "you are here" marker in the modal model.

So, let us be a bit liberal concerning the identity of the labels in a labelled calculus. It is clear that the fact $vRw, v : A \vdash w : B$ as a modal sequent is no different to the fact $wRv, w : A \vdash v : B$, as both are valid if and only if they hold for each w and v. The world labels in labelled sequents are universally quantified[6] and the particular *labels* we use are no matter, provided that we keep different labels different.

Now, consider what happens when we translate each of the display rules in our system. The rules for conjunction and disjunction are quite trivial, and become, unproblematically, the rules for conjunction and negation in the labelled system. Negation might appear to be different, under translation it, too, becomes the labelled rule. Consider $L\neg$. $*A \vdash Y$ is translated into $n^l_w(*A), p^l_w(Y) \vdash n^r_w(*A), p^r_w(Y)$ which is $p^l_w(A), p^l_w(Y) \vdash p^r_w(A), p^r_w(Y)$. $p^l_w(A)$ disappears, but $p^r_w(A)$ is $w : A$, so the premise becomes

$$p^l_w(Y) \vdash p^r_w(Y), w : A$$

which looks just like the premise of the labelled $L\neg$ rule. The conclusion is more straightforward, and it becomes

$$w : \neg A, p^l_w(Y) \vdash p^r_w(Y)$$

[6]At least, if they are read as expressing *validity*. If you are pessimistic and look for *invalidity* you may think of them as existentially bound. The point is no difference

Now, the structures $p^l_w(Y)$ and $p^r_w(Y)$ are arbitrary (we can choose Y however we like, to put any material in the left and right of this sequent[7]) so this rule is just as general as the original labelled L¬ rule.

The same thing happens with the modal rules. Consider the display rule $L\Box$. The premise $A \vdash Y$ is translated as

$$w : A, p^l_w(Y) \vdash p^r_w(Y)$$

the conclusion $\Box A \vdash \bullet Y$ becomes

$$w : \Box A, wRv, p^l_v(Y) \vdash p^r_v(Y)$$

which, is not *exactly* the labelled rule $L\Box$, but comes close. If we do not care about the identity of labels from premise to conclusion and hare happy to relabel the conclusion as $v : \Box A, vRw, p^l_w(Y) \vdash p^r_w(Y)$ then we would have *exactly* the rule required. Similarly, for $R\Box$. The premise $X \vdash \bullet A$ becomes

$$wRv, n^l_w(X) \vdash n^r_w(X), v : A$$

where v is not present in $n^l_w(X)$ and $n^r_w(X)$ (it is a label introuduced in the translation of $\bullet A$). The conclusion $X \vdash \Box A$ is

$$n^l_w(X) \vdash n^r_w(X), w : \Box A$$

which is precisely the conclusion of the $R\Box$ rule. In this explanation, we have gone quite some way to proving the following fact.

FACT 2. Any modal display derivation may be transformed step by step into a labelled derivation, using this translation, modulo some relabelling. 'Display' steps in a display inference are redundant in the labelled derivation.

To complete this proof we need to show that structural rules in the display calculus may be treated in the labelled sequent calculus. Different choices are available in each system, and provided that the effect of contraction and weakening is provided for in each system, this will work. The details are tedious and will be skipped here, as our topic is the significance of the choices of different sequent structure, and not the detail of

In the same vein, we shall not tarry to consider the effect of different modal rules in the display calculus and different sequent rules governing the accessibility relation. This is a rich and interesting area, as there are very many things one can do in display calculi [7].

4 Delabelling Labelled Sequents

We have seen that the labels in a labelled caclulus have their drawbacks. Not only do the labels break the subformula property, they also make translation from the display calculus less straightforward than they might be. In fact,

[7] Though note, we put R statements only on the left, never on the right.

the display calculus and the labelled system *both* suffer from bureaucracy at the same point. In display calculus, we have two ways of expressing the one *fact* about a model, $\bullet A \vdash B$ and $A \vdash \bullet B$. In the labelled system, we have many more: $wRv, w : A \vdash v : B$, $vRw, v : A \vdash w : B$, $vRx : v : A \vdash x : B$, and so on. It is limited only by the number of world labels available.

We could get around this needless multiplicitly by requiring that each labelled sequent be *canonically* labelled. Take a labelled sequent, replace the first label by w_0, everywhere it appears, the next one (other than w_0) by w_1, the third by w_2, etc. Then this modal fact is expressed in only one way: $w_0 R w_1, w_0 : A \vdash w_1 : B$.

An approach like this solves the multiplicity problem, but it does nothing for the subformula property. Instead, let us see the behaviour of R and the labels as a part of the *structural* furniture of a sequent rather than its *content*. For each labelled sequent of the form $\mathcal{R}, X \vdash Y$ where $\mathcal{R}$ is a collection of R statements, and X and Y are labelled formulas. The content of the sequent is given by the formulas in X and in Y. For each formula we have two pieces of information: the place on the network of "points" given by $\mathcal{R}$ and the *polarity*, given by its position, in X (in negative position) or Y (in positive position). Let us think of a different way of representing this information, without requiring labels. $\mathcal{R}$ is a directed graph with a node for each label occurring in the sequent, and an arc from w to v when wRv is in $\mathcal{R}$. Then, the label on a formula in X or in Y tells us where the formula can occur on the directed graph. If we have $w : A$ in X we put A in "antecedent" position at the w node in the graph. If $w : B$ is in Y we put $w : B$ in "consequent" position at the w node of the graph. How can we represent this? A straightforward way is to represent a *sequent* at the node, with antecedent formulas on the left and consequent formulas on the right. Once we have sequents at each node of the graph, we may rub out the labels. We have a directed graph of (traditional) sequents. For exmample, the labelled sequent $wRv, w : A \vdash v : B$ becomes the graph of sequents

$$A \vdash \longrightarrow \vdash B$$

The sequent $vRw, uRv, u : B, w : D, w : E \vdash v : A$ we saw before, coming from the display sequent $\bullet * (A \circ *\bullet B) \vdash *(D \circ E)$ becomes

$$B \vdash \longrightarrow D, E \vdash \longrightarrow \vdash A$$

This is not a new sequent structure, it is merely a new way of representing the labelled sequent structure, pushing the relational statements and labels into the syntax of the proof theory, leaving the formulas to remain as the content.[8] The derivation of the sequent $\Box A \wedge \Box B \vdash \Box(A \wedge B)$, rendered in

[8] As far as I can tell, the detail here is new, and the claim that this structure is simply a de-labelling of labelled sequent structures. However, the idea of using a graph structure on sequents is not new [5, 3].

this format becomes

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \;\longrightarrow\; \vdash A}\,L\Box}{\Box A, \Box B \vdash \;\longrightarrow\; \vdash A}\,\textit{weaken} \qquad \dfrac{\dfrac{B \vdash B}{\Box B \vdash \;\longrightarrow\; \vdash B}\,L\Box}{\Box A, \Box B \vdash \;\longrightarrow\; \vdash B}\,\textit{weaken}}{\dfrac{\Box A, \Box B \vdash \;\longrightarrow\; \vdash A \wedge B}{\Box A, \Box B \vdash \Box(A \wedge B)}\,R\Box}\,R\wedge}{\Box A \wedge \Box B \vdash \Box(A \wedge B)}\,L\wedge$$

And the derivation using a structural rule becomes straightforward:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A}{x : \neg A, x : A \vdash}\,L\neg}{Ryx, y : \Box\neg A, x : A \vdash}\,L\Box}{Ryx, x : A \vdash y : \neg\Box\neg A}\,R\neg}{Rxy, Ryx, x : A \vdash y : \neg\Box\neg A}\,\textit{sym}}{x : A \vdash x : \Box\neg\Box\neg A}\,R\Box \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,L\neg}{\Box\neg A \vdash \;\longrightarrow\; A \vdash}\,L\Box}{\vdash \neg\Box\neg A \;\longrightarrow\; A \vdash}\,R\neg}{\vdash \neg\Box\neg A \;\rightleftarrows\; A \vdash}\,\textit{sym}}{A \vdash \Box\neg\Box\neg A}\,R\Box$$

The subformula property is now straightforward, and we no longer have a proliferation of ways to represent the one modal fact. The resulting simplification of the labelled sequent system (either by eliminating labels or by choosing a canonical labelling) is less bureaucratic than either display logic or the traditional labelled sequent system. We can acknowledge the behaviour of R as the behaviour of the *structure* of our modal deduction.

Another benefit of this approach is the relation with the nonmodal propositional sequent calculus. If we identify a sequent with the one-point graph of that sequent (with no arrow at all), then there is no difference between a proof of a classical nonmodal sequent in this sequent system when compared with a traditional classical sequent system with no modal features. We do not need to modify anything of the nonmodal system. The extensions model the new vocabulary and nothing else.

Furthermore, if we impose the constraints of reflexivity, symmetry and transitivity on R to find S5, the graph structure becomes trivial. Whenever a new node in our graph of sequents is introduced, it is implicitly and automatically taken to be related to all of the other nodes in the graph of sequents. Our graph is universal. Instead of a graph of sequents we need keep track of only a multiset of sequents. The result is the simple hypersequent calculus for S5. It is the simplest labelled calculus for S5, without requiring any relational facts and without all labels. All we need do is segregate the formulas in different nodes as required for the modal inference [9]. We have the following modal rules

$$\dfrac{A, X \vdash Y \;\mid\; S}{\Box A \vdash \;\mid\; X \vdash Y \;\mid\; S}\,L\Box \qquad \dfrac{X \vdash Y \;\mid\; \vdash A \;\mid\; S}{X \vdash Y, \Box A \;\mid\; S}\,R\Box$$

where S is a multiset of sequents. Our deductions become quite simple.

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \ \mid\ \vdash A}\,L\Box}{\Box A \vdash \ \mid\ \vdash \Box A}\,R\Box}{\Box A \vdash \Box\Box A}\,R\Box \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}\,L\neg}{A \vdash \ \mid\ \Box\neg A \vdash}\,L\Box}{A \vdash \ \mid\ \vdash \neg\Box\neg A}\,R\neg}{A \vdash \Box\neg\Box\neg A}\,R\Box$$

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